Jurnal Matematika UNAND
Vol. **12** No. **1** Hal. 65 – 77
Edisi Januari 2023
ISSN : 2303–291X
e-ISSN : 2721–9410

A COMPUTATION PERSPECTIVE FOR THE EIGENVALUES OF CIRCULANT MATRICES INVOLVING GEOMETRIC PROGRESSION

SISWANDI, SUGI GURITMAN, NUR ALIATININGTYAS, TEDUH WULANDARI*
Division of Pure Mathematics,
Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
IPB University,
email: {siswandi,sugigu,nural,teduhma}@apps.ipb.ac.id

Diterima 23 Oktober 2022 Direvisi 23 Desember 2022 Dipublikasikan 30 Januari 2023

Abstract. In this article, the eigenvalues and inverse of circulant matrices with entries in the first row having the form of a geometric sequence are formulated explicitly in a simple form in one theorem. The method for deriving the formulation of the determinant and inverse is simply using elementary row or column operations. For the eigenvalues, the known formulation of the previous results is simplified by considering the specialty of the sequence and using cyclic group properties of unit circles in the complex plane. Then, the algorithm of eigenvalues formulation is constructed, and it shows as a better computation method.
Keywords: Circulant matrix, Eigenvalues, Inverse, Cyclic group, Geometric sequence

1. Introduction

Circulant matrices have applications in many problems of mathematics: numerical analysis, linear differential equations, operator theory, and many others; hence these are also connected to computer science and engineering. Those take advantage of the nice structure of the circulant matrix so that the calculation of eigenvalues, eigenvectors, determinants, and inverse of the matrices can be formulated explicitly and computed efficiently.

The above problems with various specializations have been studied by many researchers in the last few decades. Without intending to exclude any of them whose similar topic to this article but missed from our consideration, in the following we refer to some of those. Bueno [15] formulated the determinants and inverse of circulant matrices with geometric progression. Shen et al. [16] gave conditions for the invertibility of circulant matrices with entry of the Fibonacci Lucas numbers, the formulations of the determinant and inverse are derived as well. Jiang et al. [14]

*Corresponding Author

generalized those works by defining circulant matrices with the k-Fibonacci and k-Lucas numbers. Jiang and Li [13] continued those results by applying to the left circulant and $\mathcal{G}$-circulant matrices. In the same year, the explicit determinants of circulant and left circulant matrices with Tribonacci numbers or generalized Lucas numbers investigated by Li et al. in [12].

Study the explicit inverse matrices continued, but now of entry Tribonacci with the matrix structure is skew circulant, performed by Jiang and Hong in [10]. A computational approach using a symbolic algorithm for computing the determinant and inverse of general bordered tridiagonal matrices presented by Jia and Li in [11]. Then, Radicic [9] followed the study of k-circulant matrices with geometric sequence, while Bozkurt and Tam [8] were interested in r-circulant matrices associated with any number sequence. Most recently, similar problems can be seen in [7], [6], [4], [5], [3], and [2].

In this article, we review that the determinant and inverse of the circulant matrices with geometric progression are formulated explicitly in a much simpler than in the formulation for the general case. In this study, the method for determining the formulations is based on a series of elementary row or column operations to get a simpler equivalent matrix. Here we have the same problem with the topic in Bueno [15], but our approach and the method of the formulation are slightly different. Our main result is the eigenvalue formulation. The previous formulation from the general case is simplified by considering the specialty of the geometric sequence and using cyclic group properties of the unit circle in the complex plane. Then, we combine the method with the result of Bueno [15] to get a better result. Below is the outline of this paper.

In Section 2, we review the general circulant matrix notion and the previous results associated with its eigenvalues, determinant, and inverse; we also review the geometric sequence associated with the definition of its circulant matrix. In Section 3, we prove a theorem containing a simple formulation of the determinant and inverse of the matrix as defined in Section 2. The main result in this article is the eigenvalues formulation which shows efficient and fast computation, presented in Section 4. We close the paper with a concluding remark in Section 5.

2. Circulant Matrix and Geometric Progression

The first subsection of this section talks about the notion of the general circulant matrix and the previous results associated with the formulation of the eigenvalues, determinants, and inverse. For the last subsection, we review the notion of geometric progression and discuss some of its properties which are connected to the subsequent sections.

2.1. *Circulant Matrix*

For any sequence of numbers $c_0, c_1, \cdots, c_{n-2}, c_{n-1}$, the $n \times n$ *circulant matrix* is defined as:

$$Circ\,(c_0, c_1, \cdots, c_{n-2}, c_{n-1}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

Let $C = Circ\,(c_0, c_1, \cdots, c_{n-2}, c_{n-1})$, λ_k be the eigenvalues, and $\boldsymbol{\mu}_k$ be the corresponding eigenvectors of C, for $k = 0, 1, 2, \cdots, n-1$. The well-known formulation (see in [[20], [17], [1]]) for λ_k and $\boldsymbol{\mu}_k$ are:

$$\lambda_k = \sum_{j=0}^{n-1} c_j \omega^{jk} \text{ and } \boldsymbol{\mu}_k = \left(1, \omega^k, \omega^{2k}, \cdots, \omega^{(n-2)k}, \omega^{(n-1)k}\right), \tag{2.1}$$

where $\omega = e^{\frac{2\pi}{n}} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$ and $i = \sqrt{-1}$. In fact, the set $\mathcal{S} = \left\{1, \omega, \omega^2, \cdots, \omega^{n-1}\right\}$ is a cyclic subgroup in the multiplication group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of complex numbers. In this case, ω is one of the primitive roots (generators) of $\mathcal{S}$. All elements in $\mathcal{S}$ are n-th roots of unity over $\mathbb{C}$ and that means as the solutions of $x^n - 1 = 0$. For simplification, Equation (2.1) will be written as:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{(n-1)2} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix}. \tag{2.2}$$

The above notions will be very important when we discuss the formulation of the eigenvalues in the last section.

Again, we refer to [20] and [17] that the direct consequences of the Equation 2.1 is the formulation for the determinate and inverse of C:

$$\det(C) = \prod_{k=0}^{n-1} \sum_{j=0}^{n-1} c_j \omega^{jk} \text{ and } C^{-1} = Circ\,(b_0, b_1, \cdots, b_{n-2}, b_{n-1}), \tag{2.3}$$

where $b_j = \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k \omega^{-jk}$ for $j = 0, 1, \cdots, n-1$. Even though these formulations seem very simple to be calculated, but when n is getting larger, those determinant and inverse formulas are computationally not very efficient to be implemented. This is especially because of involving complex number arithmetic even if the entries of the matrix are real numbers.

However, when the sequence of $c_0, c_1, \cdots, c_{n-2}, c_{n-1}$ follows a nice pattern, such as coming from recurrence relation, then there is a great opportunity to simplify the explicit forms of the determinant, the inverse, even the eigenvalues, of C. These kinds of studies become more interesting research topics over the last decades which mostly focus on the determinant and the inverse. Thus, one of the main results of this paper is also observing the simplification of the eigenvalues formula. In this occasion, we use $c_0, c_1, \cdots, c_{n-1}$ as a *geometric sequence.*

2.2. *Geometric Progression*

For the basic theory in this subsection, we refer to [18]. When a first-order linear homogeneous recurrence relation with constant coefficients is defined recursively as:

$$u_n = ru_{n-1}, \text{ for all integer } n \geq 1, \text{ with } r \neq 0 \text{ and } r \neq 1 \text{ is a constant,}$$

then it will get a sequence on the form $u_0, ru_0, r^2u_0, r^3u_0, \cdots, r^ju_0, \cdots$ which is a well-known sequence called a *geometric sequence* (progression). If we set the initial condition of the relation $u_0 = A$, then the *unique solution* is $u_n = Ar^{n-1}$. It is easy to see that $r = \frac{u_n}{u_{n-1}}$ and we call it as the *common ratio* of the sequence. The following proposition is easy to prove by mathematical induction, and the next it will be referred to in the proof of the formulation of the eigenvalues.

Proposition 2.1. *For any positive integer n, the sum of the first n terms in the geometric sequence is formulated as:*

$$S_n = \sum_{j=0}^{n-1} u_0 r^j = \frac{u_0\left(1-r^n\right)}{1-r}. \tag{2.4}$$

For the case of n is even, the sum of the first n alternating terms in the geometric sequence is formulated as:

$$T_n = \sum_{j=0}^{n-1} (-r)^j\, u_0 = \frac{u_0\left(1-r^n\right)}{1+r}. \tag{2.5}$$

We close this section by defining the matrix that will become the main object of this topic in this paper.

Definition 2.2. *For any integer $n \geq 2$ and constant values A and r, the $n \times n$ circulant matrix with entry of the geometric sequence $\left\{Ar^{j-1}\right\}_{j=1}^{n}$, is the matrix:*

$$G_{n,r,A} = Circ\left(A, Ar, Ar^2, \cdots, Ar^{n-2}, Ar^{n-1}\right).$$

3. Inverse Formulation

For the proof of the following theorem, the basic theory we refer to [19].

Theorem 3.1. *For any integer $n \geq 2$, let $G_{n,r,1} = Circ\left(1, r, r^2, \cdots, r^{n-2}, r^{n-1}\right)$ be the matrix defined in Definition 2.2, then we have:*

$$\det\left(G_{n,r,1}\right) = \left(1-r^n\right)^{n-1} \text{ and } G_{n,r,1}^{-1} = \frac{Circ\left(1, -r, 0, \cdots, 0, 0\right)}{1-r^n}.$$

Proof. According to Definition 2.2,

$$G_{n,r,1} = \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-3} & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & \cdots & r^{n-4} & r^{n-3} & r^{n-2} \\ r^{n-2} & r^{n-1} & 1 & \cdots & r^{n-5} & r^{n-4} & r^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ r^3 & r^4 & r^5 & \cdots & 1 & r & r^2 \\ r^2 & r^3 & r^4 & \cdots & r^{n-1} & 1 & r \\ r & r^2 & r^3 & \cdots & r^{n-2} & r^{n-1} & 1 \end{pmatrix},$$

Let E_1 be a series of elementary row operations on $G_{n,r,1}$: by substituting the i-th row with the resulting operation of the i-th row is subtracted by the result of r multiplied by the $(i+1)$-th row, for $i = 2, 3, \cdots (n-1)$; and then, n-th row is substituted by the n-th row subtracted by the first row. The result is $G_{n,r,1} \sim D_1 = E_1\left(G_{n,r,1}\right)$, that is

$$D_1 = \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-3} & r^{n-2} & r^{n-1} \\ 0 & 1-r^n & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1-r^n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-r^n & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1-r^n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1-r^n \end{pmatrix}.$$

Since the value of $\det\left(G_{n,r,1}\right)$ will not change when we apply a series of elementary row operations of the third type on $G_{n,r,1}$, and D_1 having a type of upper triangular matrix, then we obtain our formula that $\det\left(G_{n,r,1}\right) = \left(1-r^n\right)^{n-1}$. Besides, in that operation of E_1, there exists a unique nonsingular matrix $P = E_1\left(I_n\right)$ such that $PG_{n,r,1} = D_1$ where:

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -r & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -r & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -r & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -r \\ -r & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Next, let K_1 be a series of elementary column operations on D_1 by substituting the j-th column with the resulting operations of the j-th column subtracted by the result of the first column multiplied by r^{j-1} for $j = 2, 3, \cdots, n$. Then, we have $G_{n,r,1} \sim D = K\left(D_1\right)$, that is:

$$D = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1-r^n & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1-r^n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-r^n & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1-r^n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1-r^n \end{pmatrix},$$

and in this case, there exists a unique non-singular matrix R such that $PG_{n,r,1}R =$

D, where

$$R = K\left(I_n\right) = \begin{pmatrix} 1 & -r & -r^2 & \cdots & -r^{n-3} & -r^{n-2} & -r^{n-1} \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Now, consider that $G_{n,r,1} = P^{-1}DR^{-1} \Leftrightarrow$

$$G^{-1}_{n,r,1} = RD^{-1}P \text{ where } D^{-1} = diag\left[1, \frac{1}{1-r^n}, \cdots, \frac{1}{1-r^n}\right],$$

and then $G^{-1}_{n,r,1} = \left(RD^{-1}\right)P =$

$$\begin{pmatrix} 1 & \frac{-r}{1-r^n} & \frac{-r^2}{1-r^n} & \cdots & \frac{-r^{n-3}}{1-r^n} & \frac{-r^{n-2}}{1-r^n} & \frac{-r^{n-1}}{1-r^n} \\ 0 & \frac{1}{1-r^n} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{1-r^n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{1-r^n} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{1-r^n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{1-r^n} \end{pmatrix} P =$$

$$\begin{pmatrix} \frac{1}{1-r^n} & \frac{-r}{1-r^n} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{1-r^n} & \frac{-r}{1-r^n} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{1-r^n} & \frac{-r}{1-r^n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{1-r^n} & \frac{-r}{1-r^n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{1-r^n} & \frac{-r}{1-r^n} \\ \frac{-r}{1-r^n} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{1-r^n} \end{pmatrix} =$$

$$\frac{1}{1-r^n} \times \begin{pmatrix} 1 & -r & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -r & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -r & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -r & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -r \\ -r & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \Leftrightarrow$$

$$G^{-1}_{n,r,1} = \frac{Circ\left(1, -r, 0, \cdots, 0, 0\right)}{1-r^n}. \qquad \square$$

The determinant formulation for the general case $\det\left(G_{n,r,A}\right)$ can be considered as a corollary of Theorem 3.1, whose proof is so simple based on the elementary properties of the determinant matrix.

Corollary 3.2. *Given any integer $n \geq 2$ and constant values A and r, where $A \neq 0$, $r \neq 0$ and $r \neq 1$, then we have:*

$$\det(G_{n,r,A}) = A^n (1 - r^n)^{n-1}.$$

Proof. Since $G_{n,r,A} = A.G_{n,r,1}$, then $\det(G_{n,r,A}) = A^n \cdot \det(G_{n,r,1})$. □

From Corollary 3.2, it is so clear to have the following corollary.

Corollary 3.3. *Given any integer $n \geq 2$ and constant values A and r, where $A \neq 0$, $r \neq 0$ and $r \neq 1$, then we have the matrix $G_{n,r,A}$ is invertible.*

The inverse formulation for the general case of $G_{n,r,A}$ can also be considered as a corollary of Theorem 3.1 that its proof is so simple based on the elementary properties of the matrix operations and the inverse.

Corollary 3.4. *Given any integer $n \geq 2$ and constant values A and r, where $A \neq 0$, $r \neq 0$ and $r \neq 1$, then we have $G^{-1}_{n,r,A} = \dfrac{Circ(1, -r, 0, \cdots, 0, 0)}{A(1 - r^n)}$.*

Proof. Consider that $G_{n,d,A} = G_{n,d,1} \times D$ where $D = diag[A, A, \cdots, A]$ and we have:

$$G^{-1}_{n,d,A} = D^{-1} \times G^{-1}_{n,d,1} \text{ where } D^{-1} = diag\left[\frac{1}{A}, \frac{1}{A}, \cdots, \frac{1}{A}\right] \Leftrightarrow$$
$$G^{-1}_{n,d,A} = \frac{1}{A} \times G^{-1}_{n,d,1} = \frac{Circ(1, -r, 0, \cdots, 0, 0)}{A(1 - r^n)}.$$
□

Example 3.5. For the matrix $G_{5,3,2}$, we have $n = 5$, $A = 2$, and $r = 3$. Thus,

$$\det(G_{5,3,2}) = 2^5 \times \left(1 - 3^5\right)^4 = 109\ 751\,747\,072,$$
$$G^{-1}_{5,3,2} = \frac{Circ(1, -2, 0, \cdots, 0, 0)}{2(1 - 3^5)} = \frac{Circ(1, -2, 0, \cdots, 0, 0)}{-484}.$$

4. Eigenvalues Formulation

Recall the cyclic group $\mathcal{S} = \left\{1, \omega, \omega^2, \cdots, \omega^{n-1}\right\}$ in Section 2. Geometrically, all n elements of S occupy the unit circle in the complex plane and divide the circle into n equal parts. Then, for $l = 1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$, we have:

$$\omega^l + \omega^{n-l} = \omega^l + \omega^{-l} = 2\cos(l\theta) \text{ and } \omega^l - \omega^{n-l} = \omega^l - \omega^{-l} = 2i\sin(l\theta), \quad (4.1)$$

where $\theta = \frac{2\pi}{n}$. Equation (4.1) will be used in the proof of the following theorem.

Theorem 4.1. *Given any integer $n \geq 2$ and constant values A and r, where $A \neq 0$, $r \neq 0$ and $r \neq 1$, let $G_{n,r,A}$ be the matrix defined in Definition 2.2, and for $j = 0, 1, 2, \cdots, n-1$, let λ_j be the eigenvalues of $G_{n,r,A}$. If $\theta = \frac{2\pi}{n}$ and $m = \left\lfloor \frac{n-1}{2} \right\rfloor$, then $\lambda_0 = \frac{A(1-r^n)}{1-r}$ and for $k = 1, 2, \cdots, m$,*

$$\lambda_k = R_k + C_k i \text{ and } \lambda_{n-k} = \overline{\lambda_k} = R_k - C_k i \text{ where}$$

$$R_k = A\left(1+\sum_{t=1}^{m}\left(r^t + r^{n-t}\right)\cos\left(tk\theta\right)\right) \text{ and } C_k = A\sum_{t=1}^{m}\left(r^t - r^{n-t}\right)\sin\left(tk\theta\right).$$

For the case of n *is even, we add* $\lambda_{m+1} = \frac{A(1-r^n)}{1+r}$ *and changing* R_k *to:*

$$R_k = A\left(1+(-1)^k r^{m+1} + \sum_{t=1}^{m}\left(r^t + r^{n-t}\right)\cos\left(tk\theta\right)\right).$$

Proof. In this proof, we exploit the fact that $\mathcal{S}$ is a cyclic group. Based on Equation (2.2) in Section 2, in the context of matrix $G_{n,r,A}$, here we have

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{-2} & \omega^{-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{-4} & \omega^{-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^4 & \omega^2 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^2 & \omega \end{pmatrix}\begin{pmatrix} A \\ Ar \\ Ar^2 \\ \vdots \\ Ar^{n-2} \\ Ar^{n-1}\end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-2} \\ \lambda_{n-1}\end{pmatrix}. \tag{4.2}$$

From Proposition 2.1, Equation (2.4), it is easy to see $\lambda_0 = \dfrac{A\left(1-r^{n-1}\right)}{1-r}$, and for the case of n is even, based on Equation (2.5), we also have $\lambda_{m+1} = \lambda_{\frac{n}{2}} = \sum_{t=0}^{n-1} Ar^t\omega^{\frac{n}{2}t} = A\sum_{t=0}^{n-1}(-r)^t = \dfrac{A\left(1-r^n\right)}{1+r}$. Next, for $k = 1, 2, \cdots, m$, consider that:

$$\begin{aligned}\lambda_k + \lambda_{n-k} &= \sum_{t=0}^{n-1} Ar^t\left(\omega^{tk} + \omega^{t(n-k)}\right) = 2A + A\sum_{t=1}^{n-1} r^t\left(\omega^{tk}+\omega^{-tk}\right) \\ &= 2A\left(1+\sum_{t=1}^{m}\left(r^t + r^{n-t}\right)\cos\left(tk\theta\right)\right),\end{aligned} \tag{4.3}$$

but for the case of n is even,

$$\begin{aligned}\lambda_k + \lambda_{n-k} &= 2A + 2(-1)^k Ar^{m+1} + A\sum_{t=1}^{m}\left(r^t + r^{n-t}\right)\left(\omega^{tk}+\omega^{-tk}\right), \\ &= 2A\left(1+(-1)^k r^{m+1} + \sum_{t=1}^{m}\left(r^t + r^{n-t}\right)\cos\left(tk\theta\right)\right).\end{aligned} \tag{4.4}$$

Analogously, consider that

$$\begin{aligned}\lambda_k - \lambda_{n-k} &= A\sum_{t=0}^{n-1} r^t\left(\omega^{tk} - \omega^{t(n-k)}\right) = A\sum_{t=1}^{n-1} r^t\left(\omega^{tk}-\omega^{-tk}\right), \\ &= 2Ai\sum_{t=1}^{m}\left(r^t - r^{n-t}\right)\sin\left(tk\theta\right).\end{aligned} \tag{4.5}$$

Finally, by adding and subtracting Equations: (4.4) with (4.5), and when n is even, Equation (4.4) with (4.5), we have $\lambda_k = R_k + iC_k$ and $\lambda_{n-k} = R_k - iC_k$, where:

$$R_k = A\left(1+\sum_{t=1}^{m}\left(r^t + r^{n-t}\right)\cos\left(tk\theta\right)\right) \text{ and } C_k = A\sum_{t=1}^{m}\left(r^t - r^{n-t}\right)\sin\left(tk\theta\right),$$

and for the case of n is even, R_k becomes:

$$R_k = A\left(1 + (-1)^k r^{m+1} + \sum_{t=1}^{m} \left(r^t + r^{n-t}\right)\cos(tk\theta)\right). \qquad \square$$

Now, Theorem 4.1 is combined with the result by Bueno [15] to get the following theorem as the main result.

Theorem 4.2. *Given the definition and notations are asserted in Theorem 4.1. If we set* $\alpha = A(1 - r^n)$, *then we have* $\lambda_0 = \frac{\alpha}{1-r}$, *and for* $k = 1, 2, \cdots, m$, *we obtain* $\lambda_k = R_k + C_k i$ *and* $\lambda_{n-k} = \overline{\lambda_k} = R_k - C_k i$ *where* $R_k = \mu(1 - x_k)$ *and* $C_k = \mu y_k$ *with*

$$x_k = r\cos\frac{2\pi k}{n},\ y_k = r\sin\frac{2\pi k}{n},\ \text{and}\ \mu = \frac{\alpha}{r^2 - 2x_k + 1}.$$

For the case of n *is even, we add* $\lambda_{m+1} = \dfrac{\alpha}{1+r}$.

Proof. From Equations (4.2), for $k = 1, 2, \cdots, m$, we have:

$$\lambda_k = \sum_{j=0}^{n-1} Ar^j\omega^{kj} = \sum_{j=0}^{n-1} A\left(r\omega^k\right)^j,$$

and then, we apply Equation (2.4) to get:

$$\lambda_k = \frac{A\left(1 - \left(r\omega^k\right)^n\right)}{1 - r\omega^k} = \frac{A\left(1 - r^n\omega^{kn}\right)}{1 - r\omega^k}.$$

Since $\omega^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$ (see Section 2), then:

$$\begin{aligned}
\lambda_k &= \frac{A\left(1 - r^n\left(\cos 2\pi + i\sin 2\pi\right)\right)}{1 - r\left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right)} = \frac{A(1 - r^n)}{1 - r\left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right)} \\
&= \frac{A(1 - r^n)}{\left(1 - r\cos\frac{2\pi k}{n}\right) - \left(r\sin\frac{2\pi k}{n}\right)i} \Leftrightarrow \\
\lambda_k &= \frac{A(1 - r^n)\left[\left(1 - r\cos\frac{2\pi k}{n}\right) + \left(r\sin\frac{2\pi k}{n}\right)i\right]}{\left(1 - r\cos\frac{2\pi k}{n}\right)^2 + \left(r\sin\frac{2\pi k}{n}\right)^2} \\
&= \frac{A(1 - r^n)\left[\left(1 - r\cos\frac{2\pi k}{n}\right) + \left(r\sin\frac{2\pi k}{n}\right)i\right]}{r^2 - 2\left(r\cos\frac{2\pi k}{n}\right) + 1}.
\end{aligned}$$

Finally, combining the above formulation with the notation and result of Theorem 4.1, then we write:

$$\begin{aligned}
&\lambda_k = R_k + C_k i \text{ and } \lambda_{n-k} = \overline{\lambda_k} = R_k - C_k i,\ \text{where} \\
&R_k = \mu(1 - x_k)\ \text{and}\ C_k = \mu y_k \text{ with } \alpha = A(1 - r^n), \\
&x_k = r\cos\frac{2\pi k}{n},\ y_k = r\sin\frac{2\pi k}{n},\ \text{and}\ \mu = \frac{\alpha}{r^2 - 2x_k + 1}.
\end{aligned}$$

We make sure that:

$$\lambda_0 = \frac{A(1 - r^n)}{(1 - r\cos 0) - (r\sin 0)i} = \frac{\alpha}{1 - r},$$

and the case of n is even,

$$\lambda_{m+1} = \lambda_{\frac{n}{2}} = \frac{A\left(1 - r^n\right)}{\left(1 - r\cos\pi\right) - \left(r\sin\pi\right)i} = \frac{\alpha}{1+r}. \qquad \square$$

4.1. *Computation Perspective*

We present first the following two simple illustrations of how to apply the formulation to compute the eigenvalues based on Theorem 4.1 and 4.2. Then, by considering those illustrations, we will analyse them to construct the algorithms.

Example 4.3. For the matrix $G_{5,2,3}$, we have $n = 5$, $A = 3$, $r = 2$, $m = 2$, and $\theta = \frac{2\pi}{5}$. Based on Theorem 4.1, $\lambda_0 = \frac{3\left(1-2^5\right)}{1-2} = 93$, $\lambda_1 = R_1 + C_1 i$, $\lambda_4 = R_1 - C_1 i$, where

$$R_1 = 3\left(1 + (2+16)\cos\frac{2\pi}{5} + (4+8)\cos\frac{4\pi}{5}\right) \approx -9.44.$$
$$C_1 = 3\left((2-16)\sin\frac{2\pi}{5} + (4-8)\sin\frac{4\pi}{5}\right) \approx: -47.00.$$

$\lambda_2 = R_2 + C_2 i$, $\lambda_4 = R_2 - C_2 i$, where

$$R_2 = 3\left(1 + (2+16)\cos\frac{4\pi}{5} + (4+8)\cos\frac{8\pi}{5}\right) \approx -29.56.$$
$$C_2 = 3\left((2-16)\sin\frac{4\pi}{5} + (4-8)\sin\frac{8\pi}{5}\right) \approx -13.27.$$

Based on Theorem 4.2, $\alpha = 3\left(1 - 2^5\right) = -93$, for $k = 1$, $x_1 = 2\cos\frac{2\pi}{5} \approx 0.618$, $y_1 = 2\sin\frac{2\pi}{5} \approx 1.9021$, $\mu = \frac{-93}{2^2 - 2(0.618)+1} \approx -24.708$, then:

$$R_1 = -24.708 \times (1 - 0.618) \approx -9.44 \text{ and}$$
$$C_1 = -24.708 \times (1.9021) \approx -47.00.$$

For $k = 2$, $x_2 = 2\cos\frac{4\pi}{5} \approx -1.618$, $y_2 = 2\sin\frac{4\pi}{5} \approx 1.1756$, $\mu = \frac{-93}{2^2-2(-1.618)+1} \approx -11.292$, then

$$R_2 = -11.292 \times (1 + 1.618) \approx: -29.56 \text{ and}$$
$$C_2 = -11.292 \times 1.1756 \approx -13.27.$$

Below is an illustration for the case of n is even.

Example 4.4. For the matrix $G_{6,3,2}$, we have $n = 6$, $A = 2$, $r = 3$, $m = 2$, and $\theta = \frac{\pi}{3}$. Based on Theorem 4.1, $\lambda_0 = \frac{2\left(1-3^6\right)}{1-3} = 728$, $\lambda_0 = \frac{2\left(1-3^6\right)}{1+3} = -364$, $\lambda_1 = R_1 + C_1 i$, $\lambda_5 = R_1 - C_1 i$, where

$$R_1 = 2\left(1 - 27 + (3+243)\cos\frac{\pi}{3} + (9+81)\cos\frac{2\pi}{3}\right) = 104.$$
$$C_1 = 2\left((3-243)\sin\frac{\pi}{3} + (9-81)\sin\frac{2\pi}{3}\right) \approx -540.40.$$

$\lambda_2 = R_2 + C_2 i$, $\lambda_4 = R_2 - C_2 i$, where:

$$R_2 = 2\left(1 + 27 + (3+243)\cos\frac{2\pi}{3} + (9+81)\cos\frac{4\pi}{3}\right) = -280.$$

$$C_2 = 2\left((3-243)\sin\frac{2\pi}{3} + (9-81)\sin\frac{4\pi}{3}\right) \approx -290.98.$$

Based on Theorem 4.2, $\alpha = 2\left(1-3^6\right) = -1456$, for $k=1$, $x_1 = 3\cos\frac{\pi}{3} = \frac{3}{2}$, $y_1 = 3\sin\frac{\pi}{3} = \frac{3}{2}\sqrt{3}$, $\mu = \frac{-1456}{3^2-2\left(\frac{3}{2}\right)+1} = -208$, then $R_1 = -208\left(1-\frac{3}{2}\right) = 104$ and $C_1 = -208\left(\frac{3}{2}\sqrt{3}\right) \approx -540.40$. For $k=2$, $x_2 = 3\cos\frac{2\pi}{3} = -\frac{3}{2}$, $y_1 = 3\sin\frac{4\pi}{3} = -\frac{3}{2}\sqrt{3}$,, $\mu = \frac{-1456}{3^2-2\left(-\frac{3}{2}\right)+1} = -112$, then $R_2 = -112\left(1-\left(-\frac{3}{2}\right)\right) = -280$ and $C_1 = -112\,(2.60) \approx -540.80$.

From the above illustrations, based on Theorem 4.1, it is so easy to see that only $m = \left\lfloor\frac{n-1}{2}\right\rfloor$ eigenvalues are computed iteratively, each of those also needs only m iterations, and the most important thing is that all computations without any complex number arithmetic used. So, it must be much faster than applying the general formula as presented in Equation 2.1, in Section 2. Meanwhile, Theorem 4.2 improves Theorem 4.1 that there is no iteration involved in each eigenvalue calculation, so the computation will be much faster when n is getting larger. Below we present the algorithm based on Theorem 4.1

Algorithm 4.5. ***(Computing eigenvalues based on Theorem 4.1)***
INPUT: $G_{n,r,A}$ *is a circulant matrix with entry geometric sequence.*
OUTPUT: $\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1}$ *are the eigenvalues of* $G_{n,r,A}$.

(1) $m \leftarrow \left\lfloor\frac{n-1}{2}\right\rfloor$; $\theta \leftarrow \frac{2\pi}{n}$; $p \leftarrow r^n$; $q \leftarrow A(1-p)$; $\lambda_0 \leftarrow \frac{q}{1-r}$;
(2) ***if*** $(n \mod 2) = 0$ ***then*** $\lambda_{m+1} \leftarrow \frac{q}{1+r}$ ***endif;***
(3) ***for*** $k = 1$ ***to*** m ***do***
  $R \leftarrow 0$; $C \leftarrow 0$; $S \leftarrow 0$; $T \leftarrow k\theta$; $u \leftarrow 1$; $v \leftarrow p$;
  for $l = 1$ ***to*** m ***do***
   $u \leftarrow u.r$; $v \leftarrow \frac{v}{r}$; $S \leftarrow S + T$; $c \leftarrow \cos S$; $s \leftarrow \sin S$;
   $x \leftarrow (u+v).c$; $y \leftarrow (u-v).s$; $R \leftarrow R + x$; $C \leftarrow C + y$;
  end do;
  if $(n \mod 2) = 0$ ***then*** $R \leftarrow \left(R + (-1)^k .r.u\right)$ ***endif;***
  $R \leftarrow A.(R+1)$; $C \leftarrow A.C$;
  $\lambda_k \leftarrow R + C.\mathbf{i}$; $\lambda_{n-k} \leftarrow R - C.\mathbf{i}$;
 end do;
(4) ***return***$(\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1})$.

Then, the algorithm based on Theorem 4.2 is as follows.

Algorithm 4.6. ***(Computing eigenvalues based on Theorem 4.2)***
INPUT: $G_{n,r,A}$ *is a circulant matrix with entry geometric sequence.*
OUTPUT: $\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1}$ *are the eigenvalues of* $G_{n,r,A}$.

(1) $m \leftarrow \left\lfloor\frac{n-1}{2}\right\rfloor$; $\theta \leftarrow \frac{2\pi}{n}$; $\alpha \leftarrow A(1-p)$; $\lambda_0 \leftarrow \frac{\alpha}{1-r}$;
(2) ***if*** $(nmod2) = 0$ ***then*** $\lambda_{m+1} \leftarrow \frac{\alpha}{1+r}$ ***endif;***

(3) ***for*** $k = 1$ ***to*** m ***do***
$S \leftarrow k\theta$; $x \leftarrow r\cos S$; $y \leftarrow r\sin S$; $\mu \leftarrow \frac{\alpha}{r^2-2x+1}$;
$R \leftarrow \mu(1-x)$; $C \leftarrow \mu y$;
$\lambda_k \leftarrow R + C.\mathbf{i}$; $\lambda_{n-k} \leftarrow R - C.\mathbf{i}$;
end do;
(4) ***return***$(\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1})$.

5. Concluding Remark

The results of formulations for the determinant and inverse of the matrices involving geometric sequences are presented in a simple way in one theorem. These are similar to the results by Bueno [15], but we have a slightly different approach in the proofs. The method of deriving the determinant formulation is simply using elementary row operation to get an equivalent matrix of the form upper triangular matrix; the method of deriving the inverse formulation is simply using the combination of elementary row and column operation to get an equivalent matrix of the form diagonal matrix.

The main result of this paper is the eigenvalue formulation. The previous formulation from the case of general circulant matrices can be simplified by considering the speciality of the geometric sequence and using cyclic group properties of the unit circle in the complex plane, so the computation can be done efficiently without involving any complex number arithmetic, i.e. all complex number eigenvalues are constructed. Then, we combine this method with the result by Bueno [15] to remove the iteration in the computation process of each eigenvalue to get a better result.

The methods we propose in this article could be applicable for any variant of circulant matrices (such as skew or more general r-circulant) with any specific formation of numbers (such as arithmetic, Fibonacci, Lucas, Pell, etc.). These would become the nearly future works.

Bibliography

[1] Aliatiningtyas N., Guritman T., Wulandari T., 2022, On the Explicit Formula for Eigenvalues, Determinant, and Inverse of Circulant Matrices, *Jurnal Teori dan Aplikasi Matematika*, Vol. **6**(3), DOI: https://doi.org/10.31764/jtam.v6i3.8616

[2] Ma J., Qiu T., and He C., 2021, A New Method of Matrix Decomposition to Get the Determinants and Inverses of r-Circulant Matrices with Fibonacci and Lucas Numbers, *Journal of Mathematics*, Hindawi, Vol. **2021**: Article ID 4782594, https://doi.org/10.1155/2021/4782594

[3] Wei Y., Zheng Y., Jiang Z., Shon S., 2020, Determinants, inverses, norms, and spreads of skew circulant matrices involving the product of Fibonacci and Lucas numbers, *Journal Mathematics and Computer Sciences*, Vol. **20**: 64 – 78

[4] Bueno, A.C.F., 2020, On r-circulant matrices with Horadam numbers having arithmetic indices, *Notes on Number Theory and Discrete Mathematics*, Vol. **26**(2): 177 – 197

[5] Liu Z., Chen S., Xu W., Zhang Y., 2019, The eigen-structures of real (skew)

circulant matrices with some applications, *Journal Computational and Applied Mathematics*, Springer, Vol. **38**(178)

[6] Radicic B., 2019, On k-Circulant Matrices Involving the Jacobsthal Numbers, *Revista de la Union Matematica Argentina*, Vol. **60**(2): 431 – 442

[7] Bahs M., Solak S., 2018, On The g-Circulant Matrices, *Commun. Korean Math. Soc.,* Vol. **33**(3): 695 – 704

[8] Bozkurt D., Tam T.-Y., 2016, Determinants and inverses of r-circulant matrices associated with a number sequence, *Linear and Multilinear Algebra*, Taylor & Francis Online, Vol. **63**(10): 2079 – 2088

[9] Radicic B., 2016, On k-circulant matrices (with geometric sequence), *Quaestiones Mathematicae*, Taylor & Francis Online, Vol. **39**(1): 135 – 144

[10] Jiang X., Hong K., 2015, Explicit inverse matrices of Tribonacci skew circulant type matrices. *Applied Mathematics and Computation*, EIsevier, Vol. **268**: 93 – 102

[11] Jia J. and Li S., 2015, On the inverse and determinant of general bordered tridiagonal matrices, *Applied Mathematics and Computation*, EIsevier, Vol. **69**(6): 503 – 509

[12] Li J., Jiang Z., and Lu F., 2014, Determinants, Norms, and the Spread of Circulant Matrices with Tribonacci and Generalized Lucas Numbers, *Abstract and Applied Analysis*, Hindawi Publishing Corporation, Vol. **2014**: Article ID 381829, http://dx.doi.org/10.1155/2014/381829.

[13] Jiang Z. and Li D., 2014, The Invertibility, Explicit Determinants, and Inverses of Circulant and Left Circulant and $\mathcal{G}$-Circulant Matrices Involving Any Continuous Fibonacci and Lucas Numbers. In *Abstract and Applied Analysis*, Hindawi Publishing Corporation, Vol. **2014**: Article ID 931451, http://dx.doi.org/10.1155/2014/931451.

[14] Jiang Z., Gong Y., and Gao Y., 2014, Invertibility and Explicit Inverses of Circulant-Type Matrices with k-Fibonacci and k-Lucas Numbers. In *Abstract and Applied Analysis*, Hindawi Publishing Corporation, Vol. **2014**: Article ID 238953, http://dx.doi.org/10.1155/2014/238953.

[15] Bueno A. C. F., 2012, Right Circulant Matrices With Geometric Progression, *International Journal of Applied Mathematical Research*, Vol. **1**(4): 593 – 603

[16] Shen S.-Q., Cen J.-M., and Hao Y., 2011On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers. *Applied Mathematics and Computation*, EIsevier, Vol. **217**(23): 9790 – 9797

[17] Aldrovandi R., 2011, *Special Matrices of Mathematical Physics: Stochastic, Circulant and Bell Matrices*, World Scientific, Singapore.

[18] Grimaldi R. P., 1999, *Discrete and Combinatorial Mathematics*, 4th Edition, North-Holland Mathematical Library, Addison Wesley Longman Inc.

[19] Lancaster P. and Tismenetski M., 1985, *The Theory of Matrices*, 2nd ed., Academic Press Inc.

[20] Davis P. J., 1979, *Circulant Matrices*, Wiley, New York