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AN AREA OF RIGHT TRIANGLE FOR TRIGONOMETRY

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Abstract. *In this paper, we shall investigate the tangent function's formula by determining the area of the right triangle and rectangle. The consequence of this formula is used to obtain the other two trigonometric functions, i.e., sine and cosine functions. Moreover, we also reprove the Pythagorean theorem.*
Keywords: Pythagorean Theorem, Right Triangle, Trigonometry

1. Introduction

One of the popular branches of mathematics is trigonometry. In many reference books, it is mentioned that the origin of the word "trigonometry" comes from two Greek words. In [1], the first trigon means "triangle," and the second, metria/ metron, means "measure." Specifically, in literature, trigonometry is the study of measuring triangles. Since middle school, we have been introduced to understanding sine, cosine, and tangent functions. The history of trigonometry spans a period of more than two millennia and crosses several cultural spheres, including the Hellenistic world, India, and Islam (see [2]). The study of trigonometry still exists. In the recent era, research on trigonometry can be seen in [3,4]. There are many ways to know about trigonometry. Usually, teachers directly define sine, cosine, and tangent functions when introducing them to their students. On the other hand, some researchers used algebra to understand trigonometry. In the trigonometry study, we can associate an angle (formed by two sides taken from three sides of a right triangle). Most trigonometry is defined by taking a ratio of two sides of the right triangle. Considering the similarity of right triangles, we knew that these ratios do not change whatever their side changed as long as the triangle is similar. The question: how to prove it? In this article, we will find the answer. After that, we will define a function representing this ratio that depends on the angle measure. Then

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we come to three well-known trigonometric functions: sine, cosine, and tangent. Furthermore, the results will be used to prove the Pythagorean theorem.

2. Concept Area of Rectangle and Right Triangle

It is well known that a rectangle is a quadrilateral in $\mathbb{R}^2$, with four sides and four vertices. Check that All sides are right sides (equal to 90^o), and the opposite sides are equal and resemblant. How about the area of the rectangle? The area of a rectangle depends on its length (L) and width (W). The formula is

$$\text{Area}_{\text{rectangle}} = L \times W.$$

A rectangle can be divided into two equal parts by cutting through one of its diagonals. The result is two congruent right triangles. The rectangle's length and width become the right triangle's legs. How about the area of the right triangle? The formula is

$$\text{Area}_{\text{right triangle}} = \frac{L \times W}{2}.$$

3. Main Results

The discussion of angles begins by investigating the ratio of the two sides of a right triangle. We will use the formula for calculating the area of a rectangle and the area of a right triangle. This investigation was carried out to convince us that the ratio of two sides of the right triangle does not depend on the size of the sides of the right triangle.

3.1. *Three Trigonometry Functions*

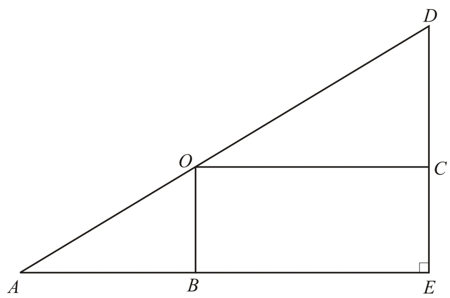

Figure. 1. A right triangle AED

In this subsection, we want to obtain definitions of three trigonometry functions. The definitions are developed through the area of the right triangle, more simple than before. With this definition, trigonometry can introduce earlier, even in elementary school. We use $\overline{AB}$ to denote a segment from A to B and AB denote length of $\overline{AB}$.

Proposition 3.1. *Let $\triangle AED$ be a right triangle with $\angle AED = 90^0$ (see Figure 1). If B, C and O located on $\overline{AE}, \overline{ED}$ and $\overline{AD}$, respectively, such that $\overline{OC}$ is parallel to $\overline{AE}$ and $\overline{OB}$ is perpendicular to $\overline{AE}$, then we have*

$$\frac{OB}{AB} = \frac{CD}{OC} = \frac{ED}{AE}.$$

Proof. Let $AB = a, BE = b, EC = c$ and $CD = d$. The area of right triangle $\triangle AED$ is:

$$\text{Area}_{AED} = \frac{(a+b)(c+d)}{2},$$

or

$$\text{Area}_{AED} = \frac{ac}{2} + \frac{bd}{2} + bc.$$

Therefore we have

$$\frac{(a+b)(c+d)}{2} = \frac{ac}{2} + \frac{bd}{2} + bc,$$
$$ad = bc,$$
$$\frac{c}{a} = \frac{d}{b}.$$

From the equation above, we also obtain

$$ad = bc,$$
$$ad + bd = bc + bd.$$

Consequently, $\frac{c+d}{a+b} = \frac{d}{b}$. Hence $\frac{c}{a} = \frac{d}{b} = \frac{c+d}{a+b}$ or

$$\frac{OB}{AB} = \frac{CD}{OC} = \frac{ED}{AE}.$$

The proof is complete. □

It follows from Proposition 3.1 that we can get that ratio whenever we construct a segment that parallels with a leg of the right triangle. In other words, we have a constant $K \in \mathbb{R}^+$ such that

$$K = \frac{c}{a} = \frac{d}{b} = \frac{c+d}{a+b} \tag{3.1}$$

holds.

The numerator of those ratios in (3.1) is the length of the opposite side of the angle of three similar right triangles, and the denominator is the length of the adjacent side of the angle. These ratios do not depend on the triangle formed but on the measure of angle $\angle EAD$.

Now, we can define K as *tangent function* of angle θ. Formally,

$$\tan\theta := \frac{y}{x},$$

where x is adjacent side of θ and y is opposite side of θ. So far, in trigonometry books, the authors immediately define the tangent function, but without investigating the truth of the ratio.

Moreover, we can find another ratio that involves hypotenuse from the right triangle. In the following proposition, we again determine it through an area of the right triangle.

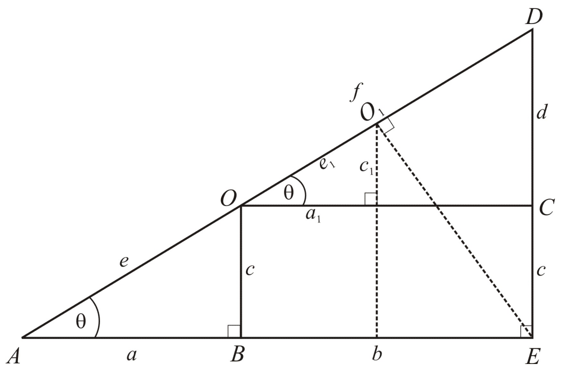

Figure. 2. A right triangle AED with a description of the angles and side lengths

Proposition 3.2. *Let $\triangle AED$ be a right triangle with $\angle AED = 90^0$. If B, C and O located on $\overline{AE}, \overline{ED}$ and $\overline{AD}$, respectively, such that $\overline{OC}$ is parallel to $\overline{AE}$ and $\overline{OB}$ is perpendicular to $\overline{AE}$, then we have:*

$$\frac{AB}{AO} = \frac{OC}{OD} = \frac{AE}{AD}$$

and

$$\frac{BO}{AO} = \frac{CD}{OD} = \frac{ED}{AD}.$$

Proof. Let $AO = e, OD = f, OC_1 = a_1, C_1O_1 = c_1$ and $OO_1 = e_1$ (see Figure 2). If $O = O_1$, then $a_1 = c_1 = e_1 = 0$. The area of right triangle $\triangle AED$ is

$$\frac{(e+f)v}{2} = \frac{(a+b)(c+d)}{2}.$$

We also determine area of right triangle $\triangle AOE$

$$\text{Area}_{AOE} = \frac{ve}{2} = \frac{(a+b)c}{2}.$$

Next, we have $\dfrac{c}{e} = \dfrac{v}{a+b}$. Meanwhile $\dfrac{v}{a+b} = \dfrac{c+d}{e+f}$, so $\dfrac{c}{e} = \dfrac{c+d}{e+f}$. Moreover we obtain

$$\frac{c}{e} = \frac{d}{f} = \frac{c+d}{e+f}, \tag{3.2}$$

or

$$\frac{BO}{AO} = \frac{CD}{OD} = \frac{ED}{AD}.$$

On the other side, the area of the right triangle $\triangle DOE$ is

$$\text{Area}_{DOE} = \frac{vf}{2} = \frac{(c+d)b}{2}.$$

So, $\dfrac{b}{f} = \dfrac{v}{c+d}$. We also have $\dfrac{v}{c+d} = \dfrac{a+b}{e+f}$. Consequently

$$\frac{a}{e} = \frac{b}{f} = \frac{a+b}{e+f} \tag{3.3}$$

or

$$\frac{AB}{AO} = \frac{OC}{OD} = \frac{AE}{AD}.$$

If $O \neq O_1$, then $a_1, c_1, e_1 > 0$. By (3.2), and (3.3), we have

$$\frac{c+c_1}{e+e_1} = \frac{d-c_1}{f-e_1} = \frac{c+d}{e+f},$$

and

$$\frac{a+a_1}{e+e_1} = \frac{b-a_1}{f-e_1} = \frac{a+b}{e+f}.$$

Since $\dfrac{c+c_1}{e+e_1} = \dfrac{d-c_1}{f-e_1}$ and $\dfrac{a+a_1}{e+e_1} = \dfrac{b-a_1}{f-e_1}$, then

$$f(d-c_1) + e(d-c_1) = f(c+d) - e_1(c+d)$$
$$d(e+e_1) = f(c+c_1) + (ec_1 - ce_1),$$

and

$$f(b-a_1) + e(b-a_1) = f(a+b) - e_1(a+b), b(e+e_1) = f(a+a_1) + (ea_1 - ae_1).$$

Recall Proposition 3.1, we have:

$$\frac{c+c_1}{a+a_1} = \frac{d}{b} = \frac{f(c+c_1) + (ec_1 - ce_1)}{f(a+a_1) + (ea_1 - ae_1)}.$$

Here, $ec_1 - ce_1$ and $ea_1 - ae_1$ must be equal 0. Thus

$$\frac{c_1}{e_1} = \frac{c}{e} = \frac{c+c_1}{e+e_1} = \frac{d-c_1}{f-e_1} = \frac{c+d}{e+f} = \frac{d}{f},$$

and

$$\frac{a_1}{e_1} = \frac{a}{e} = \frac{a+a_1}{e+e_1} = \frac{b-a_1}{f-e_1} = \frac{a+b}{e+f} = \frac{b}{f}.$$

Our conclusions are $\dfrac{BO}{AO} = \dfrac{CD}{OD} = \dfrac{ED}{AD}$ and $\dfrac{AB}{AO} = \dfrac{OC}{OD} = \dfrac{AE}{AD}$. □

Based on Proposition 3.2, there are two constants $L, M \in \mathbb{R}^+$ such that:

$$L = \frac{a}{e} = \frac{b}{f} = \frac{a+b}{e+f}, \tag{3.4}$$

and

$$M = \frac{c}{e} = \frac{d}{f} = \frac{c+d}{e+f} \tag{3.5}$$

hold. Then, we can define cosine and sine function of angle θ based on (3.4) and (3.5) as

$$\cos\theta := \frac{x}{z} \text{ and } \sin\theta := \frac{y}{z},$$

where x is adjacent of right triangle, y is opposite of right triangle, and y is hypotenuse of right triangle. Consequently, we obtain other formula of tangent function as follow

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

3.2. *Pythagorean Theorem*

Consider a right triangle. We have known a famous statement: the sum of the area of two square on the legs equals the area of the square on the hypotenuse. It is called Pythagorean Theorem. Until now, there have been many ways to prove it (there are hundreds). Here, we try to prove it using the results above.

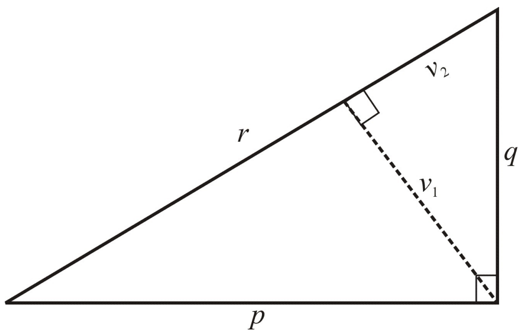

Figure. 3. A right triangle

Theorem 3.3. *Consider a right triangle with legs p and q and hypotenuse r. Then we have*

$$p^2 + q^2 = r^2.$$

Proof. See Figure 3 and we know that the area of a right triangle (with legs p and q) is

$$\text{Area}_{\text{ right triangle}} = \frac{pq}{2} = \frac{rv_1}{2}. \tag{3.6}$$

Proposition 3.1 says that

$$\frac{v_1}{v_2} = \frac{r - v_2}{v_1} = \frac{p}{q},$$

so we get $v_1 = \frac{p}{q}v_2$ and $v_1^2 = v_2 r - v_2^2$. Consequently, we obtain

$$\frac{p^2}{q^2}v_2 = r - v_2,$$

or

$$r = v_2(1 + \frac{p^2}{q^2}).$$

Meanwhile by (3.6),

$$q^2 = rv_2$$

holds. We conclude that

$$r = \frac{q^2}{r}(1 + \frac{p^2}{q^2})$$

or $p^2 + q^2 = r^2$. □

4. Applications

In [10], Sparks said trigonometry rests on five pillars constructed by the Pythagorean principle. We now apply Theorem 3.3 to construct two well-known pillars of trigonometry: identity, sine, and cosine law.

Corollary 4.1. *Let a right triangle (see Figure 2). Then we have trigonometry identity*

$$\cos^2\theta + \sin^2\theta = 1.$$

Proof. From Figure 3 and Theorem 3.3, we have

$$\frac{p^2}{r^2} + \frac{q^2}{r^2} = \frac{r^2}{r^2} = 1.$$

Since $\cos\theta = \frac{p}{r}$ and $\sin\theta = \frac{q}{r}$, then $\cos^2\theta + \sin^2\theta = 1$. □

Next, let $\triangle ABC$ be an acute triangle (Figure 4). By definition of sine function,

$$\begin{aligned} CC_1 &= AC\sin\beta = BC\sin\alpha \\ BB_1 &= BC\sin\gamma = AB\sin\beta. \end{aligned}$$

We conclude that $\frac{AC}{\sin\alpha} = \frac{BC}{\sin\beta} = \frac{AB}{\sin\gamma}$. It is called *sine law.* We also show *cosine law* using Theorem 3.3 (Pythagorean formula) and Figure 4.

$$\begin{aligned} (CC_1)^2 + (AC_1)^2 &= (AC)^2, \\ (CC_1)^2 + (AB)^2 + (C_1B)^2 - 2(AB)(C_1B) &= (AC)^2. \end{aligned}$$

and

$$(CC_1)^2 + (C_1B)^2 = (BC)^2.$$

Thus, $(AB)^2 - 2(AB)(C_1B) = (AC)^2 - (BC)^2$ holds. Recall definition of cosine to get

$$(C_1B) = (BC)\cos\beta.$$

We obtain the cosine law:

$$(AC)^2 = (BC)^2 + (AB)^2 - 2(AB)(BC)\cos\beta.$$

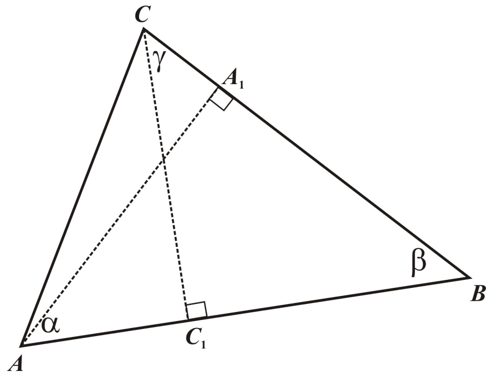

Figure. 4. An acute triangle

5. Concluding Remarks

We have constructed only a tangent function instead three essential trigonometric functions. Consequently, construction of five pillars of trigonometry as told in [10] can reduce to just one tangent function through area of right triangle. Hence, other pillars of trigonometry can be seen as the result of propositions and theorems. Furthermore, until the recent era, angle calculations can also be performed on two subspaces (see [5,6]). We know that the cosine function has a significant role. Additionally, several types of angles can be calculated in a normed space. The result can be checked in [7,8,9].

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