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SOLUSI SISTEM PERSAMAAN DIFERENSIAL LINIER ORDE FRACTIONAL DENGAN TURUNAN TIPE CAPUTO

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Abstrak. Dalam makalah ini diselesaikan sistem persamaan diferensial linier orde *fractional* dengan turunan tipe Caputo. Teorema utama yang menyajikan bentuk umum solusi didiskusikan. Beberapa contoh yang mengilustrasikan teorema utama dipaparkan.

Kata Kunci: Sistem Persamaan Diferensial *Fractional*, Turunan Tipe Caputo

1. Pendahuluan

Diberikan suatu sistem persamaan diferensial *fractional* linier sebagai berikut.

$$\frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} = A\mathbf{x}(t) + B\mathbf{u}(t),\ \mathbf{x}(0) = \mathbf{x}_0 \tag{1.1}$$

dengan:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{dt^m} f(\tau) d\tau, \tag{1.2}$$

dimana $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\mathbf{u}(t) \in \mathbb{R}^m$, untuk $m-1 < \alpha < m, m \in \mathbb{N}$, dan Γ adalah fungsi gamma.

Dalam makalah ini akan diselesaikan sistem persamaan diferensial (1.1), dimana $\frac{d^\alpha \mathbf{x}(t)}{dt^\alpha}$ adalah turunan *fractional* tipe Caputo.

2. Landasan Teori

2.1. *Transformasi Laplace*

Definisi 2.1. [6] *Misalkan $f(t)$ adalah suatu fungsi yang didefinisikan untuk $t > 0$ dan $s \in \mathbb{R}$. Maka Transformasi Laplace dari $f(t)$, didefinisikan sebagai berikut*

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt. \tag{2.1}$$

Transformasi Laplace dari *F(s)* dikatakan ada apabila integral pada Definisi 2.1 konvergen untuk beberapa nilai s, bila tidak demikian maka transformasi Laplace dari *F(s)* dikatakan tidak ada [6].

Teorema 2.2. [4] *Transformasi Laplace dari turunan orde ke-m dari suatu fungsi $f(t)$ diberikan sebagai berikut*

$$\mathcal{L}[f^{(m)}(t)] = s^m F(s) - s^{(m-1)} f(0) - s^{(m-2)} f^{'}(0) - \cdots - f^{(m-1)}(0). \qquad (2.2)$$

Teorema 2.3. [3] *Jika transformasi Laplace dari fungsi $f(t)$ dan $g(t)$ adalah $F(s)$ dan $G(s)$ maka transformasi Laplace untuk konvolusi kedua fungsi tersebut adalah*

$$\mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right] = F(s) * G(s).$$

2.2. Fungsi Gamma dan Fungsi Beta

Definisi 2.4. [4] *Fungsi Gamma didefinisikan sebagai berikut.*

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt, \ n > 0.$$

Definisi 2.5. [4] *Fungsi Beta didefinisikan sebagai berikut.*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \ x \in \mathbb{R}, \ p,q \in \mathbb{C},$$

dimana $p > 0$ dan $q > 0$.

2.3. Turunan *Fractional* Tipe Caputo

Definisi 2.6. [3] *Turunan fractional Caputo orde $\alpha \in \mathbb{R}$ dari fungsi $f(t)$* dengan *$m-1 < \alpha < m, m \in \mathbb{N}$, dinotasikan dengan $\dfrac{d^\alpha f(t)}{dt^\alpha}$, didefinisikan sebagai berikut.*

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}},$$

*dimana $f^{(m)}(\tau)$ adalah turunan ke-*m *dari fungsi $f(\tau)$ dan $\Gamma(m-\alpha)$ menyatakan fungsi gamma dari $(m-\alpha)$.*

Teorema 2.7. [3] *Turunan fractional Caputo memenuhi sifat kelinieran, yaitu*

$$\frac{d^\alpha}{dt^\alpha}[\lambda f(t) + \mu g(t)] = \lambda \frac{d^\alpha}{dt^\alpha} f(t) + \mu \frac{d^\alpha}{dt^\alpha} g(t), \ \lambda, \mu \in \mathbb{R}.$$

Teorema 2.8. [3] *Transformasi Laplace dari turunan fractional Caputo adalah*

$$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha} f(t)\right] = s^\alpha F(s) - \sum_{k=1}^{m} s^{(\alpha-k)} f^{(k-1)}(0). \qquad (2.3)$$

Bukti.

$$\begin{aligned}
\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha}f(t)\right] &= \mathcal{L}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}\right], \\
&= \frac{1}{\Gamma(m-\alpha)}\mathcal{L}\left[\int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}\right], \\
&= \frac{1}{\Gamma(m-\alpha)}\mathcal{L}\left[\int_0^t (t-\tau)^{m-1-\alpha}f^{(m)}(\tau)d\tau\right], \\
&= \frac{1}{\Gamma(m-\alpha)}\mathcal{L}\left[t^{m-\alpha-1}\right]\mathcal{L}\left[f^{(m)}(t)\right], \quad \text{(dari Teorema 2.3)}, \\
&= \frac{1}{\Gamma(m-\alpha)}\frac{\Gamma(m-\alpha)}{s^{m-\alpha}}\left[s^m F(s) - \sum_{k=1}^{m} s^{(m-k)}f^{(k-1)}(0)\right], \\
&= \frac{1}{s^{m-\alpha}}\left[s^m F(s) - \sum_{k=1}^{m} s^{(m-k)}f^{(k-1)}(0)\right], \\
&= s^\alpha F(s) - \sum_{k=1}^{m} s^{(\alpha-k)}f^{(k-1)}(0).
\end{aligned}$$

□

3. Pembahasan

Perhatikan kembali sistem persamaan (1.1), yaitu

$$\frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} = A\mathbf{x}(t) + B\mathbf{u}(t),\ \mathbf{x}(0) = \mathbf{x}_0,$$

dimana $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ dan $\dfrac{d^\alpha \mathbf{x}(t)}{dt^\alpha}$ menyatakan turunan *fractional* orde α dari $\mathbf{x}(t)$.

Teorema 3.1. [3] *Solusi dari sistem persamaan* (1.1) *untuk* $m-1 < \alpha < m$, $m \in \mathbb{N}$ *adalah*

$$\boldsymbol{x}(t) = \sum_{k=1}^{m} \Phi_k(t)\boldsymbol{x}^{(k-1)}(0) + \int_0^t \Phi(t-\tau)B\boldsymbol{u}(\tau)d\tau, \tag{3.1}$$

dimana

$$\Phi_k(t) = \sum_{n=0}^{\infty} A^n \mathcal{L}^{-1}\left[s^{-(\alpha n+k)}\right] = \sum_{n=0}^{\infty} \frac{A^n t^{(n\alpha+k)-1}}{\Gamma(n\alpha+k)}, \tag{3.2}$$

$$\begin{aligned}
\Phi(t) &= \sum_{n=0}^{\infty} A^n \mathcal{L}^{-1}[s^{-(n+1)\alpha}], \\
&= \sum_{n=0}^{\infty} \frac{A^n t^{(n+1)\alpha-1}}{\Gamma(n+1)\alpha}.
\end{aligned} \tag{3.3}$$

Bukti. Dengan menerapkan transformasi Laplace pada sistem (1.1), diperoleh

$$\begin{aligned}\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha}\mathbf{x}(t)\right] &= \mathcal{L}\left[A\mathbf{x}(t)+B\mathbf{u}(t)\right],\\ &= \mathcal{L}\left[A\mathbf{x}(t)\right]+\mathcal{L}\left[B\mathbf{u}(t)\right].\end{aligned}$$

Berdasarkan persamaan (2.3) pada Teorema 2.8, diperoleh

$$\begin{aligned}s^\alpha\mathbf{X}(s)-\sum_{k=1}^{m}s^{(\alpha-k)}\mathbf{x}^{(k-1)}(0) &= A\mathbf{X}(s)+B\mathbf{U}(s),\\ s^\alpha\mathbf{X}(s)-A\mathbf{X}(s) &= B\mathbf{U}(s)+\sum_{k=1}^{m}s^{(\alpha-k)}\mathbf{x}^{(k-1)}(0),\\ (I_n s^\alpha - A)\mathbf{X}(s) &= B\mathbf{U}(s)+\sum_{k=1}^{m}s^{(\alpha-k)}\mathbf{x}^{(k-1)}(0).\end{aligned}$$

Sehingga,

$$\mathbf{X}(s) = [I_n s^\alpha - A]^{-1}\left[B\mathbf{U}(s)+\sum_{k=1}^{m}s^{(\alpha-k)}\mathbf{x}^{(k-1)}(0)\right]. \tag{3.4}$$

Karena

$$\begin{aligned}[I_n s^\alpha - A]^{-1} &= \left[\frac{I}{s^\alpha}+\frac{A}{s^{2\alpha}}+\frac{A^2}{s^{3\alpha}}+\cdots.+\frac{A^n}{s^{(n+1)\alpha}}+\cdots.\right],\\ &= \sum_{n=0}^{\infty}A^n s^{-(n+1)\alpha}, \qquad (3.5)\end{aligned}$$

maka persamaan (3.4) menjadi

$$\begin{aligned}\mathbf{X}(s) &= \sum_{n=0}^{\infty}A^n s^{-(n+1)\alpha}\left[B\mathbf{U}(s)+\sum_{k=1}^{m}s^{(\alpha-k)}\mathbf{x}^{(k-1)}(0)\right],\\ &= \sum_{n=0}^{\infty}A^n s^{-(n+1)\alpha}\sum_{k=1}^{m}s^{(\alpha-k)}\mathbf{x}^{(k-1)}(0)+\sum_{n=0}^{\infty}A^n s^{-(n+1)\alpha}BU(s),\\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{m}A^n s^{-(\alpha n+k)}\mathbf{x}^{(k-1)}(0)+\sum_{n=0}^{\infty}A^n s^{-(n+1)\alpha}BU(s).\end{aligned}$$

Dengan mentransformasi Laplace, inverskan $\mathbf{X}(s)$ diperoleh

$$\begin{aligned}\mathbf{x}(t) &= \mathcal{L}^{-1}\left[\mathbf{X}(s)\right],\\ &= \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\sum_{k=1}^{m}A^n s^{-(\alpha n+k)}\mathbf{x}^{(k-1)}(0)+\sum_{n=0}^{\infty}A^n s^{-(n+1)\alpha}BU(s)\right],\\ &= \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\sum_{k=1}^{m}A^n s^{-(\alpha n+k)}\mathbf{x}^{(k-1)}(0)\right]+\mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}A^n s^{-(n+1)\alpha}BU(s)\right],\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{k=1}^{m} A^n \mathcal{L}^{-1}\left[s^{-(\alpha n+k)}\right]\mathbf{x}^{(k-1)}(0) + \sum_{n=0}^{\infty} A^n \mathcal{L}^{-1}\left[s^{-(n+1)\alpha}BU(s)\right],$$

$$= \sum_{n=0}^{\infty}\sum_{k=1}^{m} \frac{A^n t^{(n\alpha+k)-1}}{\Gamma(n\alpha+k)}\mathbf{x}^{(k-1)}(0) + \sum_{n=0}^{\infty} A^n \mathcal{L}^{-1}\left[s^{-(n+1)\alpha}B\mathbf{U}(s)\right],$$

$$= \sum_{n=0}^{\infty}\sum_{k=1}^{m} \frac{A^n t^{(n\alpha+k)-1}}{\Gamma(n\alpha+k)}\mathbf{x}^{(k-1)}(0) + \int_0^t \sum_{n=0}^{\infty} A^n (t-\tau)^{(n+1)\alpha-1}B\mathbf{u}(\tau)d\tau.$$

Dengan demikian,

$$\mathbf{x}(t) = \sum_{k=1}^{m} \Phi_k(t)\mathbf{x}^{(k-1)}(0) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau,$$

dimana

$$\Phi_k(t) = \sum_{n=0}^{\infty} A^n \mathcal{L}^{-1}\left[s^{-(\alpha n+k)}\right] = \sum_{n=0}^{\infty} \frac{A^n t^{(n\alpha+k)-1}}{\Gamma(n\alpha+k)},$$

$$\Phi(t) = \sum_{n=0}^{\infty} A^n \mathcal{L}^{-1}[s^{-(n+1)\alpha}],$$

$$= \sum_{n=0}^{\infty} \frac{A^n t^{(n+1)\alpha-1}}{\Gamma(n+1)\alpha}.$$

□

Contoh 3.2. Tentukanlah solusi dari sistem (1.1) dengan

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{x}^{'}(0) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix},\ \mathbf{x}^{''}(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

untuk $\alpha = 3/2$, dan $u(t) = 1$.

Karena

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\quad \text{maka } A^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad \text{untuk } n = 3, 4, 5, \cdots.$$

Dengan menggunakan persamaan (3.2) dan (3.3), diperoleh

$$\Phi_k(t) = \sum_{n=0}^{\infty} \frac{A^n t^{(n\alpha+k)-1}}{\Gamma(n\alpha+k)},$$

$$= \frac{A^0 t^{(k)-1}}{\Gamma(k)} + \frac{At^{(\alpha+k)-1}}{\Gamma(\alpha+k)} + \frac{A^2 t^{(2\alpha+k)-1}}{\Gamma(2\alpha+k)}, \tag{3.6}$$

$$\Phi(t) = \sum_{n=0}^{\infty} \frac{A^n t^{(n+1)\alpha-1}}{\Gamma(n+1)\alpha},$$

$$= I_3\frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + A^2\frac{t^{3\alpha-1}}{\Gamma(3\alpha)}. \tag{3.7}$$

Untuk $\alpha = \frac{3}{2}$, diperoleh $m = 2$ karena $1 < \frac{3}{2} < 2$.

Subtitusikan persamaan (3.6) dan (3.7) dengan $u(t) = 1$ ke dalam persamaan (3.1), maka diperoleh

$$
\begin{aligned}
\mathbf{x}(t) &= \sum_{k=1}^{m} \Phi_k(t)\mathbf{x}^{(k-1)}(0) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau,\\
&= \sum_{k=1}^{2} \Phi_k(t)\mathbf{x}^{(k-1)}(0) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau,\\
&= \Phi_1(t)\mathbf{x}(0) + \Phi_2(t)\mathbf{x}^{'}(0) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau,\\
&= \left[I_3 + \frac{At^{\alpha}}{\Gamma(\alpha+1)} + \frac{A^2t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\mathbf{x}(0) + \left[\frac{A^0t^1}{\Gamma(2)} + \frac{At^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{A^2t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\mathbf{x}^{'}(0),\\
&\quad + \int_0^t \left[\frac{B}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{AB}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1} + \frac{A^2B}{\Gamma(3\alpha)}(t-\tau)^{3\alpha-1}\right]d\tau,\\
&= \mathbf{x}(0) + \frac{A\mathbf{x}(0)t^{\alpha}}{\Gamma(\alpha+1)} + \frac{A^2\mathbf{x}(0)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{A^0\mathbf{x}^{'}(0)t^1}{\Gamma(2)} + \frac{A\mathbf{x}^{'}(0)t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{A^2\mathbf{x}^{'}(0)t^{2\alpha+1}}{\Gamma(2\alpha+2)},\\
&\quad + \frac{Bt^{\alpha}}{\Gamma(\alpha+1)} + \frac{ABt^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{A^2Bt^{3\alpha}}{\Gamma(3\alpha+1)},\\
&= \begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \begin{bmatrix}-1\\1\\-1\end{bmatrix}\frac{t}{\Gamma(2)}\\
&\quad + \begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}-1\\1\\-1\end{bmatrix}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}-1\\1\\-1\end{bmatrix}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \begin{bmatrix}1\\-2\\2\end{bmatrix}\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
&\quad + \begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}1\\-2\\2\end{bmatrix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1\\-2\\2\end{bmatrix}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&= \begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \begin{bmatrix}0\\0\\1\end{bmatrix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \begin{bmatrix}-1\\1\\-1\end{bmatrix}\frac{t}{\Gamma(2)} + \begin{bmatrix}1\\0\\-1\end{bmatrix}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},\\
&\quad + \begin{bmatrix}0\\0\\1\end{bmatrix}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \begin{bmatrix}1\\-2\\2\end{bmatrix}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \begin{bmatrix}-2\\0\\1\end{bmatrix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \begin{bmatrix}0\\0\\-2\end{bmatrix}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&= \begin{bmatrix}
1 - \frac{t}{\Gamma(2)} + \frac{2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}\\
1 + \frac{t}{\Gamma(2)} - \frac{2t^{\alpha}}{\Gamma(\alpha+1)}\\
1 - \frac{t}{\Gamma(2)} + \frac{3t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{2t^{3\alpha}}{\Gamma(3\alpha+1)}
\end{bmatrix},
\end{aligned}
$$

$$= \begin{bmatrix} 1 - t + \frac{8}{3}t\sqrt{\frac{t}{\Pi}} + \frac{4}{3}t^2\sqrt{\frac{t}{\Pi}} - \frac{1}{3}t^3 \\ 1 + t - \frac{8}{3}t\sqrt{\frac{t}{\Pi}} \\ 1 - t + 4t\sqrt{\frac{t}{\Pi}} - \frac{15}{8}t^2\sqrt{\frac{t}{\Pi}} + \frac{1}{3}t^3 + \frac{1}{24}t^4 - \frac{64}{945}t^4\sqrt{\frac{t}{\Pi}} \end{bmatrix}$$

4. Kesimpulan

Solusi sistem persamaan diferensial linier *fractional* pada persamaan (1.1) adalah:

$$\mathbf{x(t)} = \sum_{k=1}^{m} \Phi_k(t)\mathbf{x}^{(k-1)}(0) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau, \text{ dimana}$$

$$\Phi_k(t) = \sum_{n=0}^{\infty} A^n \mathcal{L}^{-1}\left[s^{-(\alpha n+k)}\right] = \sum_{n=0}^{\infty} \frac{A^n t^{(n\alpha+k)-1}}{\Gamma(n\alpha+k)},$$

$$\Phi(t) = \sum_{n=0}^{\infty} A^n \mathcal{L}^{-1}[s^{-(n+1)\alpha}] = \sum_{n=0}^{\infty} \frac{A^n t^{(n+1)\alpha-1}}{\Gamma(n+1)\alpha}.$$

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