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THE FIRST U-EXTENSION MODULE AS CLASSES OF SHORT U-EXACT SEQUENCES

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Abstract. Inspired by the notions of the U-exact sequence introduced by Davvaz and Parnian-Garamaleky in 1999, and of the chain U-complex introduced by Davvaz and Shabani-Solt in 2002, Mahatma and Muchtadi-Alamsyah in 2017 developed the concept of the U-projective resolution and the U-extension module, which are the generalizations of the concept of the projective resolution and the concept of extension module, respectively. It is already known that every element of a first extension module can be identified as a short exact sequence. To the simple, there is a relation between the first extension module and the short exact sequence. It is proper to expect the relation to be provided in the U-version. In this paper, we aim to construct a one-one correspondence between the first U-extension module and the set consisting of equivalence classes of short U-exact sequence.
Keywords: Chain U-complex, U-projective resolution, U-extension module

1. Motivation

In [1] Davvaz and Shabani-Solt introduced the notion of the chain U-complex which generalizes the concept of the chain complex. The main idea was by replacing the kernel of every homomorphism in the sequence with the inverse image of a possibly nonzero submodule. For more details, a sequence of modules and module homomorphisms

$$\cdots \xrightarrow{d_{p+2}} C_{p+1} \xrightarrow{d_{p+1}} C_p \xrightarrow{d_p} C_{p-1} \xrightarrow{d_{p-1}} \cdots$$

is called a *chain U-complex* if, for every $k \in \mathbb{Z}$, $U_k \subseteq \mathrm{Im}(d_{k+1}) \subseteq d_k^{-1}(U_{k-1})$ where U_k is submodule of C_k for every $k \in \mathbb{Z}$. By this definition, the ordinary chain complex now can be regarded as a chain U-complex with $U_k = 0$ for all $k \in \mathbb{Z}$. As an example of chain U-complex, consider the sequence

$$\begin{array}{ccccccccccc} \cdots \xrightarrow{4} & \frac{1}{3}\mathbb{Z} & \xrightarrow{3} & \frac{1}{2}\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{2} & 2\mathbb{Z} & \xrightarrow{3} & 3\mathbb{Z} & \xrightarrow{4} \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ & 4\mathbb{Z} & & 3\mathbb{Z} & & 2\mathbb{Z} & & 2\mathbb{Z} & & 6\mathbb{Z} & \end{array}$$

where the arrow "$m\mathbb{Z} \xrightarrow{k} n\mathbb{Z}$" denotes the map $x \mapsto kx$ for every $x \in m\mathbb{Z}$. The objects written in the bottom row are the submodule U_ks.

As the chain U-complex was defined, we can consider a modified concept of exactness of a sequence by replacing the subset relation $\mathrm{Im}(d_{k+1}) \subseteq d_k^{-1}(U_{k-1})$ with equality for all $k \in \mathbb{Z}$. In fact, Davvaz and Parnian-Garamaleky [2] has introduced in advance the notion of U-exact sequences before the chain U-complex was. Nevertheless, the definition does not yet contain the conditions necessary for a U-exact sequences to be seen as a special case of chain U-complex, for it does not require the submodule U_k to be contained in $\mathrm{Im}(d_{k+1})$ for every $k \in \mathbb{Z}$. However, experience shows that there are more advantages when a U-exact sequences is also a chain U-complex.

Projective resolution is a kind of exact sequence that is used widely in representation theory. As the concept of exact sequences was generalized, Mahatma and Muchtadi-Alamsyah [3] proposed a method to construct the U-projective resolution as the generalization of the projective resolution. Furthermore, they continued in the same article with a method to induce the k-th U-extension module form a U-projective resolution for all $k \in \mathbb{N}$, as the projective resolution does to the k-th extension module for all $k \in \mathbb{N}$.

We assume throughout this paper that R is commutative algebra. It is known that for any R-modules M and N there exists one-one correspondence between the first extension R-module $\mathrm{Ext}^1(M,N)$ and the set $e(M,N)$ consists of all equivalence classes of short exact sequence of the form $0 \to N \to E \to M \to 0$ (see Chapter 7 of [4]). By this result, we can define the R-module structure for $e(M,N)$. The goal of this paper is to investigate the analogous result in the U version where U is nonzero submodule of M.

2. The U-Extension

Given R-modules M and N, the short exact sequence $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ is also known as *the extension of N by M.* We start this paper with the notion generalizing the concept of the extension by replacing the property that $\mathrm{Im}(f) = \ker(g)$ with $\mathrm{Im}(f) = g^{-1}(U)$ where U is nonzero submodule of M. This concept would require that the module N should be large enough so that it can be mapped onto U.

Let M and N be R-modules and U be a submodule of M. The sequence $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ such that f is one-one, g is onto, and $f(N) = g^{-1}(U)$ is called the *U-extension of N by M.* We shall also call such sequence as a *short U-exact sequence.*

We restrict the discussion in this paper only for the module N, which is direct sum of U, and only for the short U-exact sequence $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ with property that if $N = U \oplus V$ then $f(V) = \ker(g)$ and $gf(u) = u$ for every $u \in U$.

Therefore, every short U-exact sequence throughout this paper will be assumed to be of that form. Notice that if we allow the submodule U to be 0, then the case $U = 0$ gives us exactly the ordinary extension of N by M.

Let $\mathcal{E}(M[U], N)$ denotes the set of all U-extension of N by M. Let $\mathbf{E} : 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ be an element of $\mathcal{E}(M[U], N)$. A short U-exact sequence $\mathbf{F} : 0 \to N \xrightarrow{f'} F \xrightarrow{g'} M \to 0$ in $\mathcal{E}(M[U], N)$ is said to be equivalent to $\mathbf{E}$, denoted by $\mathbf{E} \approx \mathbf{F}$, if there exists a morphism $\delta : E \to F$ such that $g'\delta = g$ and $f' = \delta f$, that is if the diagram

$$\begin{array}{ccccccccc} 0 \to & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \to 0 \\ & \downarrow 1 & & \downarrow \delta & & \downarrow 1 & \\ 0 \to & N & \xrightarrow{f'} & F & \xrightarrow{g'} & M & \to 0 \end{array} \tag{2.1}$$

commutes. It is easy to verify that δ is an isomorphism and hence "$\approx$" is an equivalence relation in $\mathcal{E}(M[U], N)$. For every $\mathbf{E} \in \mathcal{E}(M[U], N)$, the class of all short U-exact sequence equivalent to $\mathbf{E}$ will be denoted by $[\mathbf{E}]$. Thus, the set $\mathcal{E}(M[U], N)$ partitioned by "$\approx$" will consist of all classes $[\mathbf{E}]$ where $\mathbf{E} \in \mathcal{E}(M[U], N)$. We denote those set by $e(m[U], N)$. Thus,

$$e(M[U], N) = \mathcal{E}(M[U], N)/ \approx = \{[\mathbf{E}] | \mathbf{E} \in \mathcal{E}(M[U], N)\}.$$

3. The U-Projective Resolution and the U-Extension Module

Let M be R-module, and U be a nonzero submodule of M. Consider the sequence $P_0 \xrightarrow{d_0} M \to 0$ where P_0 is projective. Let P_1 be a projective module such that the sequence $P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} M \to 0$ is U-exact at P_0, that is $\mathrm{Im}(d_1) = d_0^{-1}(U)$. Set $U_0 := d_0^{-1}(U)$ and let P_2 be a projective module such that the sequence $P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0$ is U_0-exact at P_1, or $\mathrm{Im}(d_2) = d_1^{-1}(U_0)$. Set $U_1 := d_1^{-1}(\ker(d_0))$ and let P_3 be a projective module such that the sequence $P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1$ is U_1-exact at P_2, or $\mathrm{Im}(d_3) = d_2^{-1}(U_1)$. Continue the process by setting the submodule $U_k := d_k^{-1}(\ker(d_{k-1}))$ and choose the projective module P_{k+2} such that the sequence $P_{k+2} \xrightarrow{d_{k+2}} P_{k+1} \xrightarrow{d_k+1} P_k$ is U_k-exact at P_{k+1}, or $\mathrm{Im}(d_{k+2}) = d_{k+1}^{-1}(U_k)$.

The infinite sequence $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ obtained from the process above is called the *U-projective resoultion of M*, denoted by $\mathbf{P}$: $P_\bullet(U_\bullet) \xrightarrow{d_\bullet} M(U)$. From the construction above, it seems that the sequence obtained depends on the choice of the module P_ks. Nevertheless, in [3] Mahatma and Muchtadi-Alamsyah showed that the U-projective resolution is unique op to the so-called (U, U)-homotopy, that is if $\mathbf{P}$: $P_\bullet(U_\bullet) \xrightarrow{d_\bullet} M(U)$ and $\mathbf{Q}$: $P'_\bullet(U'_\bullet) \xrightarrow{d'_\bullet} M(U)$ both are U-projective resolution of M then there exist chain (U, U')-map $\mathbf{f} : \mathbf{P} \to \mathbf{Q}$ and chain (U', U)-map $\mathbf{g}$: $\mathbf{Q} \to \mathbf{P}$ such that $\mathbf{gf} \simeq 1_P$ and $\mathbf{fg} \simeq 1_Q$ (see also [1] for detail of the map between two U-complexes). Now notice that, in a U-projective resolution of M, since $\mathrm{Im}(d_1) = d_0^{-1}(U) = U_0$ then we may choose $P_2 := P_1$ and set $d_2 := 1_{P_1}$. Hence every U-projective resolution of M is of the form $\cdots \xrightarrow{d_3} P_1 \xrightarrow{1_{P_1}} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$.

For the example of the U-projective resolution, let us consider the case when the algebra R is hereditary. Here we consider two cases: when module M is projective and when it is not. According to the method given in the beginning of this section, if M is projective, then the U-projective resolution of M will be of the form

$$\begin{array}{ccccccccc} 0 \to U & \xrightarrow{1} & U & \xrightarrow{1} & M & \xrightarrow{1} & M & \to 0 \\ & & & & \uparrow & & \uparrow & \\ & & & & U & & U & \end{array}$$

while if M is not projective, then the U-projective resolution of M will be of the form

$$\begin{array}{ccccccccc} 0 \to \ker(d) & \xrightarrow{1} & d^{-1}(U) & \xrightarrow{1} & d^{-1}(U) & \xrightarrow{1} & P & \xrightarrow{d} & M \to 0. \\ & & & & \uparrow & & \uparrow & & \uparrow \\ & & & & \ker(d) & & d^{-1}(U) & & U \end{array}$$

Recall that the objects written in the bottom row denote the submodule $U_k s$. Here, when the U_k is not written, we mean that $U_k = 0$. The detail of these constructions can be found in [5].

Let $\mathbf{P} : P_\bullet \xrightarrow{d_\bullet} M(U)$ be the U-projective resolution of M. If $P_n \neq 0$ and $P_i = 0$ for all $i > n$ then we say that the length of $\mathbf{P}$ is n. Hence, if R is hereditary, we have that the U-projective resolution length is either 2 or 3. Moreover, in [5] Baur, Mahatma, and Muchtadi-Alamsyah showed that an algebra R is hereditary if and only if, for $U \neq 0$, every U-projective resolution of an R-module has length of either 2 or 3.

Given an R-module M, a nonzero submodule U of M, and U-projective resolution of M $\mathbf{P}$:$P_\bullet(U_\bullet) \xrightarrow{d_\bullet} M(U)$, let $(\mathbf{P}_M)$ be the sequence $\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$ obtained by removing M from $\mathbf{P}$. Given an R-module N, apply the functor $\mathrm{Hom}(\_, N)$ to $(\mathbf{P}_M)$ to obtain the sequence $0 \to \mathrm{Hom}(P_0, N) \xrightarrow{\bar{d}_1^N} \mathrm{Hom}(P_1, N) \xrightarrow{\bar{d}_2^N} \cdots$, where $\bar{d}_k^N$ denotes the map $\mathrm{Hom}(d_k, N)$ for every $k \in \mathbb{N}$. Now, for every $k \in \mathbb{N}$, define the submodules $A_k^N := \{\alpha d_{k-1} d_k | \alpha : \mathrm{Im}(d_{k-1}) \to N$ and $Z_k^N := \left(\bar{d}_{k+1}^N\right)^{-1} (A_{k+1}^M)$ of $\mathrm{Hom}(P_k, N)$. Note first that, for every $k \in \mathbb{N}$, a morphism $z \in \mathrm{Hom}(P_k, N)$ is in Z_k^N if and only if there exists a morphism $\alpha : \mathrm{Im}(d_k) \to N$ such that $\bar{d}_{k+1}^N(z) = z d_{k+1} = \alpha d_k d_{k+1}$. Next, for every $k \in \mathbb{N}$ define the submodule $B_k^N := \{\mu d_k + \lambda d_{(k-1,k)} | \mu : P_{k-1} \to N, \lambda : U_{k-2} \to N\}$ where $d_{(k-1,k)}$ is the morphism $d_{k-1} d_k$ regarded as single morphism. Finally, for every $k \in \mathbb{N}$ we define the *k-th U-extension module of N by M* by $\mathrm{Ext}^k(M[U], N) := Z_k^N / B_k^N$.

From the construction above, the module obtained depends on the choice of the U-projective resolution used as the basic material. Nevertheless, in [3] Mahatma and Muchtadi-Alamsyah showed that the k-th U-extension module is unique up to isomorphism for every $k \in \mathbb{N}$.

4. The First *U*- Extension Module

Let M be R-module, and U be a nonzero submodule of M. Given any R-module X, we have seen that, for every $k \in N$, the construction of $Ext^k(M[U], X)$ involves many steps, that make the structure of the module obtained seems so complicated. We can describe the module $Ext^1(M[U], X)$ very simply.

To do so, recall first that, in the U-projective resolution of M, $P_2 = P_1$ and $d_2 = 1_{P_1}$. Hence the module Z_1^X and B_1^N can be simplified to $Z_1^X = \{\alpha d_1 | \alpha : \mathrm{Im}(d_1) \to X\}$ and $B_1^X = \{\mu d_1 | \mu : P_0 \to X\}$, respectively. Thus the module $\mathrm{Ext}^1(M[U, X]) = Z_1^X / B_1^X$ consists of all classes $[z]$ of morphisms in $\mathrm{Hom}(P_1, X)$ whose form αd_1 where $\alpha : \mathrm{Im}(d_1) \to X$, where two classes $[z_1]$ and $[z_2]$ in $\mathrm{Ext}^1(M[U], X)$ are considered to be the same if and only if $z_1 - z_2$ whose form μd_1 where $\mu : P_0 \to X$. As a consequence, $\mathrm{Ext}^1(M[U], X) = 0$ if and only if every morphism $\alpha : \mathrm{Im}(d_1) \to X$ can be extended into $\alpha' : P_0 \to X$. This could happen when $\mathrm{Im}(d_1) = d_0^{-1}(U)$ is a direct summand of P_0. Especially, we have that $\mathrm{Ext}^1(M[M], X) = 0$ for any R-module, since if $U = M$ then $\mathrm{Im}(d_1) = d_0^{-1}(U) = d_0^{-1}(M) = P_0$.

5. Construction of the Correspondence

Suppose given an R-module M, nonzero submodule U of M, and an R-module $N = U \oplus V$. For every $\mathbf{E} \in \mathcal{E}(M[U], N)$, we will identify the class $[\mathbf{E}] \in e(M[U], N)$ by an element $[z_{\mathbf{E}}] \in \mathrm{Ext}^1(M[U], V)$ and vice versa.

Consider the sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$, where P_is are projective and $\mathrm{Im}(d_1) = d_0^{-1}(U)$. Let $\mathbf{E}$: $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ be an element in $\mathcal{E}(M[U], N)$. Consider the diagram

$$\begin{array}{ccccccc} & & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M \to 0 \\ & & & & & & \downarrow 1 \\ 0 \to & N & \xrightarrow{f} & E & \xrightarrow{g} & & M \to 0 \end{array}$$

Note that since g is surjective and P_0 is projective, then there exists a morphism $h : P_0 \to E$ satisfying $gh = d_0$. Let $d(0,1)$ be the composition $d_0 d_1$ regarded as a single morphism from P_1 to U. Consider that $hd_1 - fd_{(0,1)}$ is a morphism from P_1 to E satisfying $g\left(hd_1 - fd_{(0,1)}\right) = d_0 d_1 - gfd_{(0,1)} = 0$ since $gf|_U = 1_U$. Hence, $\mathrm{Im}(hd_1 - fd_{(0,1)}) \subseteq \ker(g) = f(V)$, and since P_1 is projective, then there exists a morphism $z : P_1 \to V$ satisfying $fz = hd_1 - fd_{(0,1)}$.

Define the morphism $\alpha : d_1(P_1) \to V$ by $\alpha d_1(p) := z(p)$ for every $p \in P_1$. Notice that if $d_1(p) = 0$ then $z(p) = 0$ and hence α is well-defined. Since $z = \alpha d_1$ then $z \in Z_1^V$.

Now suppose that $h' : P_0 \to E$ is another morphism satisfying $gh' = d_0$. Let $z' \in Z_1^V$ satisfy $fz' = h'd_1 - fd_{(0,1)}$. Since $g(h - h') = 0$ then $\mathrm{Im}(h - h') \subseteq \ker(g) = f(V)$. Since P_0 is projective, then there exists a morphism $\mu : P_0 \to V$ satisfying $f\mu = h - h'$. Thus we have $f(z - z') = (h - h')d_1 = f\mu d_1$, which implies $z - z' = \mu d_1 \in B_1^N$. Therefore $[z] = [z']$ in $\mathrm{Ext}^1(M[U], V)$.

The paragraph above shows how to construct a map from $\mathcal{E}(M[U], N)$ to $\mathrm{Ext}^1(M[U], V)$. For this map let us denote the image of $\mathbf{E}$ by $z_{\mathbf{E}}$. Now suppose that $\mathbf{F}\in[\mathbf{E}]$. Let $\delta : E \to F$ be the isomorphism that makes the Diagram 2.1 commutes. Now, to obtain the morphism $z_{\mathbf{F}} \in Z_1^V$ which represents the image of $\mathbf{F}$, we may set $h_{\mathbf{F}} := \delta h$ and choose $z_{\mathbf{F}}$ as the morphism satisfying $f'z_{\mathbf{F}} = h_{\mathbf{F}}d_1 - f'd_{(0,1)}$ or $\delta f z_{\mathbf{F}} = \delta h_{\mathbf{E}} d_1 - \delta f d_{(0,1)}$. Since δ is an isomorphism, then we get $fz_{\mathbf{F}} = h_{\mathbf{E}}d_1 - fd_{(0,1)} = fz_{\mathbf{E}}$ which implies $z_{\mathbf{F}} = z_{\mathbf{E}}$.

We have just constructed a map $\varphi : e(M[U], N) \to \mathrm{Ext}^1(M[U], V)$ where $\varphi([\mathbf{E}]) = [z_{\mathbf{E}}]$ for every $[\mathbf{E}] \in e(m[U], N)$. Our goal is to show that φ is a one-one correspondence.

Theorem 5.1. *The map φ is onto.*

Proof. Suppose that $[z] \in \mathrm{Ext}^1(M[U], V)$. Define the submodule $I := \left\{\left(d_{(0,1)}(x) \oplus z(x)\right) \oplus d_1(-x) | x \in P_1\right\}$ of $N \oplus P_0$ and the module $E := (N \oplus P_0)/I$. Remember that every $n \in N$ can be written uniquely as $n_U \oplus n_V$ where $n_U \in U$ and $n_v \in V$. Create a sequence $N \xrightarrow{f} E \xrightarrow{g} M$ where $f(n) := (n \oplus 0) + I$ for every $n \in N$ and $g((a \oplus b) + I) := a_U + d_0(b)$ for every $(a \oplus b) + I \in E$. To show that the morphism g is well-defined, notice that if $a \oplus b \in I$ then there exists an $x \in P_1$ such that $\alpha = d_{(0,1)}(x) \oplus z(x)$ and $b = d_1(-x)$. Since $\mathrm{Im}(z) \subseteq V$ then we have $a_U = d_{(0,1)}(x)$ and hence $g((a \oplus b) + I) = d_{(0,1)}(x) + d_0 d_1(-x) = 0$. So g is well-defined. Furthermore, since d_0 is onto then g is onto.

Now, if $f(n) = 0$ then $n \oplus 0 \in I$. Hence, there exists an $x \in P_1$ such that $n = d_{(0,1)}(x) \oplus z(x)$ and $0 = d_1(-x)$. Since $z \in Z_1^V$ then $z = \alpha d_1$ for a morphism $\alpha : d_1(P_1) \to V$. Therefore, $n = 0 \oplus 0$ and hence f is one-one.

Next, notice that for every $n \in N$, $gf(n) = g\left((n \oplus 0) + I\right) = n_U \in U$. Hence $\mathrm{Im}(gf) \subseteq U$ and so $\mathrm{Im}(f) \subseteq g^{-1}(U)$. Now, if $(a \oplus b) + I \in g^{-1}(I)$ then $g\left((a \oplus b) + I\right) = a_U + d_0(b) \in U$. Since $a_u \in U$ then we have $d_0(b) \in U$ and so $b \in d_0^{-1}(U) = \mathrm{Im}(d_1)$. Let $b = d_1(p)$ where $p \in P_1$. We see that

$$\begin{aligned}
(a \oplus b) + I &= (a \oplus d_1(p)) + I \\
&= \left(a + (d_{(0,1)}(p) \oplus z(p)) \oplus 0\right) + I \\
&= f\left(a + (d_{(0,1)}(p) \oplus z(p))\right) \\
&\in f(N).
\end{aligned}$$

Then $g^{-1}(U) \subseteq \mathrm{Im}(f)$. Hence we have $\mathrm{Im}(f) = g^{-1}(U)$.

Next, if $v \in V$ then $v_U = 0$ and hence $gf(v) = g\left((v \oplus 0) + I\right) = 0 + d_0(0) = 0$. Therefore, $f(V) \subseteq \ker(g)$. Now suppose that $(a \oplus b) + I \in \ker(g)$. Since $g\left((a \oplus b) + I\right) = a_U + d_0(b) = 0$ then $d_0(b) = -a_U \in U$. Hence $b \in d_0^{-1} = \mathrm{Im}(d_1)$. Let $b = d_1(x)$ where $x \in P_1$. Then

$$\begin{aligned}
(a \oplus b) + I &= \left((-d_{(0,1)}(x) \oplus a_V) \oplus d_1(x)\right) + I \\
&= ((0 \oplus (a_V + z(x))) \oplus 0) + I \\
&= f\left(0 \oplus (a_V + z(x))\right) \\
&\in f(V).
\end{aligned}$$

Hence $\ker(g) \subseteq f(V)$ and we have $f(V) = \ker(g)$.

Finally, if $u \in U$ then $u_V = u$ and hence $gf(u) = g\left((u \oplus 0) + I\right) = u + d_0(0) = u$. We have just shown that the sequence $\mathbf{E}$: $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ satisfies all criterions in $\mathcal{E}(M[U], N)$. Thus, $\mathbf{E} \in \mathcal{E}(M[U], N)$. We will show that $\varphi([\mathbf{E}]) = [z]$.

Define the morphism $h : P_0 \to E$ by $h(q) := (0 \oplus q) + I$ for every $q \in P_0$. We see that for every $q \in P_0$, $gh(q) = g((0 \oplus q) + I) = d_0(q)$. Hence $gh = d_0$. Next, notice that for every $p \in P_1$,

$$\begin{aligned} hd_1(p) &= (0 \oplus d_1(p)) + I \\ &= \big((d_{(0,1)}(p) \oplus z(p)) \oplus 0\big) + I \\ &= f\left(d_{(0,1)}(p) \oplus z(p)\right) \\ &= fd_{(0,1)}(p) + fz(p). \end{aligned}$$

The equation above gives us $fz = hd_1 - fd_{(0,1)}$. Hence $[z]$ is the map of the class $[\mathbf{E}] \in e(M[U], N)$ by φ. thus, $[z]$ has a pre-image in $e(M[U], N)$ by φ. Since $[z]$ is arbitrary then φ is onto. $\square$

To show that φ is one-one we will show that every element in $\mathrm{Ext}^1(M[U], V)$ has unique pre-image in $e(m[U], N)$ by φ.

Theorem 5.2. *For every $[z] \in Ext^1(M[U], V)$, the pre-image of $[z]$ by φ is unique*

Proof. Given $z \in \mathrm{Ext}^1(M[U], V)$, suppose that $\mathbf{E} \in \mathcal{E}(M[U], N)$ is the U-exact sequence constructed using the method given in the proof of Theorem 5.1. Thus, $[z] = [z_{\mathbf{E}}]$. Let $\mathbf{F}$: $0 \to N \xrightarrow{f'} F \xrightarrow{g'} M \to 0$ be an element in $\mathcal{E}(M[U], N)$ with $[z_{\mathbf{F}}] = [z_{\mathbf{E}}]$. Our goal is to show that $[\mathbf{E}] = [\mathbf{F}]$, that is there exists a morphism $\delta : E \to F$ which makes the Diagram 2.1 commutes, that is $g'\delta = g$ and $f' = \delta f$.

Consider the diagram

$$\begin{array}{ccccccccc} & & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \to & 0 \\ & & \downarrow z' & & \downarrow h' & & \downarrow 1 & & \\ 0 & \to & N & \xrightarrow{f'} & F & \xrightarrow{g'} & M & \to & 0 \end{array}$$

with $g'h' = d_0$ and $f'z' = h'd_1 - f'd_{(0,1)}$. Since $[z_{\mathbf{F}}] = [z_{\mathbf{E}}]$ then $z' - z \in B_1^V$ and hence there exists a morphism $\mu : P_0 \to V$ such that $z' = z + \mu d_1$. Let us define the morphism $\delta : E \to F$ by $\delta\left((a \oplus b) + I\right) := f'(a) + (h' - f'\mu)(b)$ for every $(a \oplus b) + I \in E$. Notice that if $(a \oplus b) + I = I$ then $\alpha = d_{(0,1)}(x) \oplus z(x)$ and $b = d_1(-x)$ for an $x \in P_1$. Consequently,

$$\begin{aligned} \delta\left((a \oplus b) + I\right) &= f'\left(d_{(0,1)}(x) \oplus z(x)\right) + (h' - f'\mu)d_1(-x) \\ &= f'd_{(0,1)}(x) + f'z(x) + f'(z' - z)(x) \\ &= f'z(x) - f'z'(x) + f'(z' - z)(x) \\ &= 0. \end{aligned}$$

Hence, δ is well-defined.

Next, note that for every $a \in N$, $g'f'(a) = g'f'(a_U \oplus a_V) = g'f'(a_U \oplus 0) + g'f'(0 \oplus a_V) = a_U + 0 = a_U$. Hence, for every $(a \oplus b) + I \in E$ we have

$$\begin{aligned} g'\delta((a\oplus b)+I) &= g'\left(f'(a)+(h'-f'\mu)(b)\right)\\ &= g'f'(a)+g'h'(b)-g'f'\mu(b)\\ &= a_U+d_0(b)\\ &= g((a\oplus b)+I) \end{aligned}$$

where the third row holds since $\mathrm{Im}(\mu)\subseteq V$ and $f'(V)\subseteq \ker(g')$. Hence $g'\delta = g$. Next, for every $n\in N$, we have $\delta f(n)=\delta((n\oplus 0)+I)=f'(n)+(h'-f'\mu)(0)=f'(n)$. Hence, $\delta f = f'$. We have shown that $[\mathbf{E}]=[\mathbf{F}]$. □

6. Result and Discussion

We have shown that given any R-module M and nonzero submodule U of M, if $N=U\oplus V$ then there exists one-one correspondence between the set $e(M[U],N)$ of all equivalence classes in $\mathcal{E}(M[U],N)$ with the module $\mathrm{Ext}^1(M[U],V)$. We have known in Section 4 that $\mathrm{Ext}^1(M[M],V)=0$. Clearly, the only M-extension of $M\oplus V$ by M is given by the sequence of the form $0\to M\oplus V\xrightarrow{1+\varphi} M\oplus W\xrightarrow{1\oplus 0} M\to 0$ where $\varphi : V\to W$ is an isomorphism.

As we know that there exists one-one correspondence between the module $\mathrm{Ext}^k(M,N)$ with the set of equivalence classes of exact sequences off the form $0\to N\to E_k\to\cdots\to E_2\to E_1\to M\to 0$, it would be interesting to investigate whether the result in this paper could be extended for another value of k. But we must leave a note here that the construction of U-extension module give results that $\mathrm{Ext}^2(M[U],X)=0$ for any module X. Nevertheless, we may expect that there will be relation between the exact sequence $0\to N\to E_2\to E_1\to M\to 0$ and some nonzero modules $\mathrm{Ext}^k(M[U],N)$ with $k>2$.

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