EXISTENCE AND UNIQUENESS IN THE LINEARISED ONE AND TWO-DIMENSIONAL PROBLEM OF PARTIAL DIFFERENTIAL EQUATIONS WITH VARIATIONAL METHOD

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Abstract. The classical solution and the strong solution of a partial differential equation problem are continuously differentiable solutions. This solution has a derivative for a continuous infinity level. However, not all problems of partial differential equations can be easily obtained by strong solutions. Even the existence of a solution requires in-depth investigation. The variational formulation method can qualitatively analyze a unique solution to a partial differential equation problem. This study provides an alternative method in analyzing the problem model of partial differential equations analytically. In this study, the problem of partial differential equations will be solved analytically by using the variational method and the finite element method to confirm it numerically.

Kata Kunci: Weak solution, Hilbert space, finite element method

1. Introduction

Partial differential equations are widely used in modelling various physical, geometric and probabilistic phenomena. The partial differential equation modelling can be in the form of linear and nonlinear equations depending on the complexity of the model to be investigated. There is no known general theory of the solvency of all partial differential equations. Such a theory is implausible, given the large variety of possible models. This issue has led to the development of research on methods of solving partial differential equations problems.

Some cases of partial differential equations that are interesting and continue to be developed in various studies are the Laplace equation, Helmholtz equation, Liouville equation, wave equation, Korteweg-de Vries equation [49] [46]. As for the

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system of partial differential equations, especially in the case of fluid mechanics, the Navier-Stokes system of nonlinear equations and their linearized form, namely the Neumann-Kelvin system of equations, is still being developed to model the dynamics of fluid motion. Given the wide variety of cases that can be modelled with partial differential equations of varying degrees of difficulty, the question of what it means to "solve" a partial differential equation may be subtle. Classical and analytical solutions are perfect solutions to the problem of partial differential equations. From these solutions, it can be seen how the dynamics of the model under study are. However, not all problems of differential equations can be obtained with analytical solutions. The informal idea in building an excellent partial differential equation problem model is a model that captures many of the desired features and meets the following three criteria [12] (a) the problem has a solution; (b) the solution is single; (c) the solution depends continuously on the data given in the problem. The last condition is very important for problems arising from physical applications. Meanwhile, observing the existence and singularity of solutions is the main task for mathematicians and classifying and characterizing solutions.

Next is to define what is meant by 'solution'. The desired solution satisfies (a) (c). However, does the solution have to be genuinely analytic or at least infinitely differentiable? This solution is the perfect condition, but it will not be easy to achieve in some cases. Perhaps it would be wiser if the solutions of partial differential equations of order k could be differentiated as much as k times continuously. Then at least all the derivatives that appear in the problem of partial differential equations will be continuous, although it is possible that certain higher derivatives will not exist. The idea of solving a partial differential equation problem has always been an exciting topic to study. In this study, variational methods will be studied to analyze the existence and singularity of a solution qualitatively.

The principle of the variational approach to solving partial differential equations is to replace the equation with an equivalent formula called variational, which is obtained by integrating the equation multiplied by any function, called the test function. The solution generated by this method is a weak solution of partial differential equations. Through the Lax-Milgram Theorem, it can be shown that the solution exists and is singular. Then through the regulatory process, it will be proven that the weak solution is a strong solution to the problem of partial differential equations. This variational method is very useful in determining the existence and singularity of solutions to the boundary condition problems of partial differential equations. The applications of this method can be found in the article [49], [45]. In that article, they analyzed weak solutions by functional analysis to prove the existence of a single solution of the linear boundary value problem that describes the motion of the equilibrium state of a half-submerged cylinder in an ideal, incompressible heavy fluid. We can also refer to the following articles: [53], [62], [17], [56], [44], [57], [16] to better understand the concept of variational method.

This study provides an alternative method in analyzing the problem model of partial differential equations analytically. The partial differential equation modelling that will be analyzed in this research is built from fluid dynamics modelling.

2. Finite element method for one dimensional case

We seek to calculate an approximation of the solution [57] $u : [0,1] \to \mathbb{R}$ of the following problem:

$$-\epsilon u''(x) + \lambda u'(x) = f(x), \tag{2.1}$$

$$u(0) = u(1) = 0, \tag{2.2}$$

with $\epsilon > 0$, λ and f are given so that there is a unique solution to this problem and we want to approach this solution by a continuous function, polynomial by parts.

(1) **Write the variational formulation of the problem**

We construct the variational formulation by multiplying both side of (1) by a test function v satisfying the boundary conditions and then integrate over $[0,1]$. We thus obtain

$$-\int_0^1 -\epsilon u''(x)\, v(x)\, \mathrm{d}x + \int_0^1 \lambda u'(x)\, v(x)\mathrm{d}x = \int_0^1 f(x)\, v(x)\, \mathrm{d}x, \quad \forall v \in H^1(0,1).$$

Integrating by parts, and since v(0)=v(1)=0 then we have

$$\epsilon \int_0^1 u'(x)\, v'(x)\, \mathrm{d}x + \lambda \int_0^1 u'(x)\, v(x)\mathrm{d}x = \int_0^1 f(x)\, v(x)\, \mathrm{d}x, \quad \forall v \in H^1(0,1). \tag{2.3}$$

We obtain the bilinear and linear form:

$$B[u,v] = \epsilon \int_0^1 u'(x)\, v'(x)\, \mathrm{d}x + \lambda \int_0^1 u'(x)\, v(x)\mathrm{d}x,$$

$$L(v) = \int_0^1 f(x)\, v(x)\, \mathrm{d}x.$$

Therefore we can express the variational formulation as follows: find $u \in H_0^1([0,1])$ such that for all $v \in H_0^1([0,1])$, we have $B[u,v] = L(v)$.

(2) **Calculation of the exact solution**

We suppose the function f is constant not null.

(a) Find the exact solution of the problem.

$$-\epsilon u''(x) + \lambda u'(x) = f(x) \Leftrightarrow u''(x) + \frac{1}{-\epsilon}\lambda u'(x) = \frac{1}{-\epsilon}f(x).$$

- Subtitute :

$$r'(x) + \frac{\lambda}{-\epsilon}r = \frac{f}{-\epsilon}.$$

- Integrating factor:

$$\mu(x) = e^{\int \frac{\lambda}{-\epsilon}\, dx} = e^{-\frac{\lambda}{\epsilon}x + C_1} = C_2 e^{-\frac{\lambda}{\epsilon}x}, \quad C_2 = e^{C_1}.$$

- Solve for r:

$$r(x) = \frac{\int C_2 e^{\frac{-\lambda}{\epsilon}x} \frac{f}{-\epsilon} \, dx}{C_2 e^{\frac{-\lambda}{\epsilon}x}}$$

$$= \frac{f}{\lambda} + C_4 e^{\frac{\lambda}{\epsilon}x}, \quad C_4 = C_3/C_2.$$

- Solution :

$$u(x) = \int r(x) \, dx = \int \frac{f}{\lambda} + C_4 e^{\frac{\lambda}{\epsilon}x} \, dx = \frac{f}{\lambda}x + C_4 \frac{\epsilon}{\lambda} e^{\frac{\lambda}{\epsilon}x} + C_5.$$

- Subtitute the boundary conditions, we obtain the exact solution of (1) and (2):

$$u(x) = \frac{f}{\lambda}x + \frac{f}{\lambda(1 - e^{\frac{\lambda}{\epsilon}})} \left(e^{\frac{\lambda}{\epsilon}x} - 1 \right). \tag{2.4}$$

(b) Write a function to calculate the solution u on a grid of points given. Draw u for $f = 1, \lambda \in -1, 1$, and $\epsilon \in \{1, 0.5, 0.1, 0.01\}$.
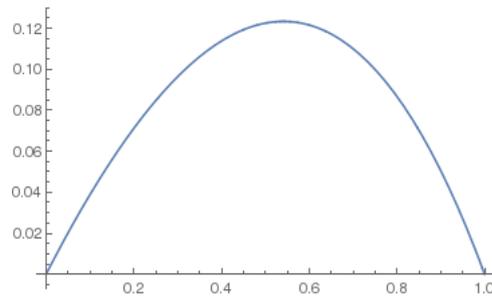
Let us see the graph of u :

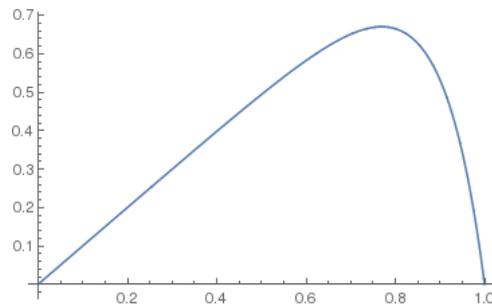Figure 1. $f = 1, \lambda = 1, \epsilon = 1$:

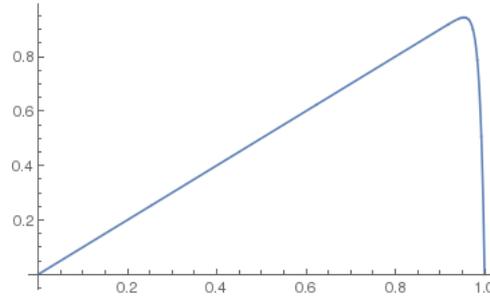Figure 2. $f = 1, \lambda = 1, \epsilon = 0.1$:

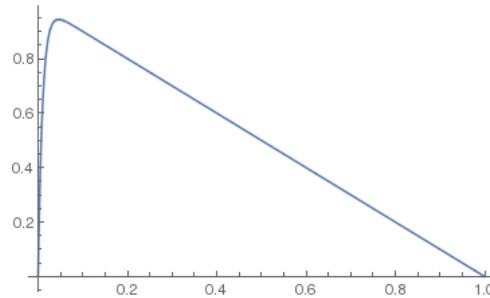Figure 3. $f = 1, \lambda = 1, \epsilon = 0.01$:

Figure 4. $f = 1, \lambda = -1, \epsilon = 0.1$:

(3) **We seek an approximate solution (finite element method P1)**

(a) Write the approximate variational formulation.

Let us generate a mesh, let a uniform Cartesian mesh $x_i = ih, i = 0, 1, 2, \cdots, n$ where $h = 1/n$, defining the intervals (x_{i-1}, x_i), $i = 1, 2, \cdots, n$. Then, we construct a set of basis functions bassed on the mesh, such as the piecewise linear functions $i = 1, 2, \cdots, n-1$

$$\phi_i(x) = \begin{cases} 0 & x < x_{i-1} \\ \frac{x - x_{i-1}}{h} & x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{h} & x_i \le x \le x_{i+1} \\ 0 & x > x_{i+1} \end{cases} . \tag{2.5}$$

Represent the approximate (FE) solution by a linear combination of the basis functions

$$u_h(x) = \sum_{j=1}^{n-1} c_j \phi_j(x), \tag{2.6}$$

where the coefficients c_j are the unknowns to be determined. Substituting the approximate solution $u_h(x)$ in the variational formulation:

$$\epsilon \int_0^1 u_h'(x)\, v'(x)\, \mathrm{d}x + \lambda \int_0^1 u_h'(x)\, v(x)\mathrm{d}x = \int_0^1 f(x)\, v(x)\, \mathrm{d}x \tag{2.7}$$

Let us define a finite dimensional space on the mesh. Let the solution be in the space V, which is $H_0^1(0,1)$ in the model problem. Based on the mesh,
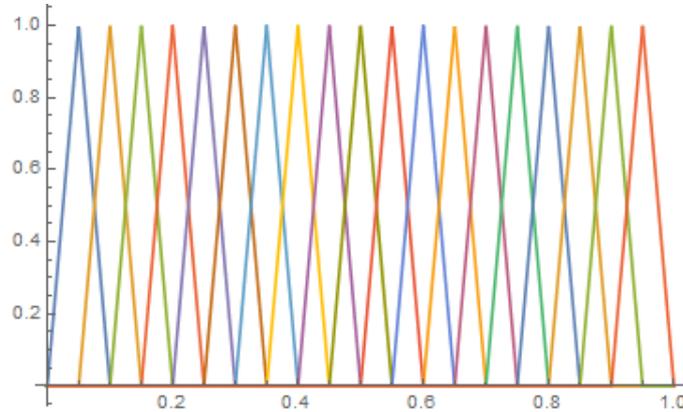
Figure 5. basis function for $n = 20$

we wish to construct a subspace V_h (a finite dimensional space) $\subset V$ (the solution space).

$$V_h = \{v_h(x)|v_h(x)\text{is continuous piecewise linear, } v_h(0) = v_h(1) = 0\}.$$

(b) Show that the approximate solution u_h can be found as the solution of a linear system

$$A_h u_h = b_h.$$

Deduce that $A_h = \epsilon B_h + \lambda C_h$, where B_h is a symmetric tridiagonal matrix and C_h is an antisymmetric tridiagonal matrix.

From (2.8), we have

$$\epsilon \int_0^1 \sum_{j=1}^{n-1} c_j \phi_j'(x)\, v'(x)\, \mathrm{d}x + \lambda \int_0^1 \sum_{j=1}^{n-1} c_j \phi_j'(x)\, v(x)\mathrm{d}x = \int_0^1 f(x)\, v(x)\, \mathrm{d}x.$$

(2.8)

$$\Rightarrow \epsilon \sum_{j=1}^{n-1} c_j \int_0^1 \phi_j'(x)\, v'(x)\, \mathrm{d}x + \lambda \sum_{j=1}^{n-1} c_j \int_0^1 \phi_j'(x)\, v(x)\mathrm{d}x = \int_0^1 f(x)\, v(x)\, \mathrm{d}x.$$

(2.9)

We choose the test function $v(x)$ as $\phi_1, \phi_2, \cdots, \phi_{n-1}$ successively, in the matrix vector form:

$$\left(\epsilon \begin{bmatrix} b(\phi_1, \phi_1) & b(\phi_1, \phi_2) & \dots & b(\phi_1, \phi_{n-1}) \\ b(\phi_2, \phi_1) & b(\phi_2, \phi_2) & \dots & b(\phi_2, \phi_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ b(\phi_{n-1}, \phi_1) & b(\phi_{n-1}, \phi_2) & \dots & b(\phi_{n-1}, \phi_{n-1}) \end{bmatrix} + \lambda \begin{bmatrix} c(\phi_1, \phi_1) & c(\phi_1, \phi_2) & \dots & c(\phi_1, \phi_{n-1}) \\ c(\phi_2, \phi_1) & c(\phi_2, \phi_2) & \dots & c(\phi_2, \phi_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ c(\phi_{n-1}, \phi_1) & c(\phi_{n-1}, \phi_2) & \dots & c(\phi_{n-1}, \phi_{n-1}) \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_3 \end{bmatrix} = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ \vdots \\ (f, \phi_{n-1}) \end{bmatrix}$$

(2.10)

where $b(\phi_i, \phi_j) = \int_0^1 \phi_i'(x)\, \phi_j'(x)\, \mathrm{d}x$, $c(\phi_i, \phi_j) = \int_0^1 \phi_i'(x)\, \phi_j(x)\, \mathrm{d}x$, $(f, \phi_i) = \int_0^1 f\, \phi_i(x)\, \mathrm{d}x$.

Then we can write the approximation solution u_h can be obtained by solve the following linear system:

$$A_h \, u_h = b_h,$$

where $A_h = \epsilon B_h + \lambda C_h$.

By standard calculation of integral, we will get B_h is a symmetric tridiagonal matrix and C_h is an antisymmetric tridiagonal matrix. Let us see the case: $\epsilon = 0.1$, $\lambda = 1$, $f = 1$, $n = 10$:

$$B_h = \begin{pmatrix}
20 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 20 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -10 & 20 & -10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -10 & 20 & -10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -10 & 20 & -10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -10 & 20 & -10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 20
\end{pmatrix}$$

,

$$C_h = \begin{pmatrix}
0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.5 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0
\end{pmatrix}.$$

Then we have

$$A_h = \begin{pmatrix}
2. & -0.5 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
-1.5 & 2. & -0.5 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -1.5 & 2. & -0.5 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -1.5 & 2. & -0.5 & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & -1.5 & 2. & -0.5 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & -1.5 & 2. & -0.5 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & -1.5 & 2. & -0.5 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & -1.5 & 2. & -0.5 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.5 & 2.
\end{pmatrix}.$$

We also have

$$(f, \phi_i) = \begin{pmatrix} \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \\ \frac{1}{10} \end{pmatrix}.$$

Hence by the calculation of $A_h^{-1}.(f, \phi_i)$, we obtain

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{pmatrix} = \begin{pmatrix} 0.0999661 \\ 0.199865 \\ 0.29956 \\ 0.398645 \\ 0.495902 \\ 0.587671 \\ 0.662979 \\ 0.688904 \\ 0.566678 \end{pmatrix}. \qquad (2.11)$$

Substituting (2.11) into (2.17) then we have the approximation solution. Let us see the table below:

$$\begin{pmatrix} x & u(x) & u_h(x) & u(x) - u_h(x) \\ 0 & 0. & 0. & 0. \\ 0.1 & 0.099922 & 0.0999661 & -0.0000441427 \\ 0.2 & 0.19971 & 0.199865 & -0.000154593 \\ 0.3 & 0.299133 & 0.29956 & -0.000426202 \\ 0.4 & 0.397567 & 0.398645 & -0.00107863 \\ 0.5 & 0.493307 & 0.495902 & -0.00259449 \\ 0.6 & 0.581729 & 0.587671 & -0.00594212 \\ 0.7 & 0.650256 & 0.662979 & -0.0127232 \\ 0.8 & 0.664704 & 0.688904 & -0.0242 \\ 0.9 & 0.532149 & 0.566678 & -0.0345287 \\ 1 & 0. & 0. & 0. \end{pmatrix}. \qquad (2.12)$$

For $n = 20$, we can obtain the solution with the similar calculation as follows:

$$\begin{pmatrix} u(x) & u_h(x) & u(x) - u_h(x) \\ 0. & 0. & 0. \\ 0.0499705 & 0.0499756 & -5.0779565 \times 10^{-6} \\ 0.099922 & 0.099935 & -0.0000130127 \\ 0.149842 & 0.149867 & -0.0000253658 \\ 0.19971 & 0.199754 & -0.0000445175 \\ 0.249492 & 0.249566 & -0.0000740682 \\ 0.299133 & 0.299253 & -0.000119414 \\ 0.348542 & 0.34873 & -0.000188551 \\ 0.397567 & 0.39786 & -0.000293164 \\ 0.445958 & 0.446408 & -0.000450015 \\ 0.493307 & 0.49399 & -0.000682575 \\ 0.538936 & 0.539958 & -0.0010226 \\ 0.581729 & 0.58324 & -0.00151086 \\ 0.619847 & 0.622042 & -0.00219529 \\ 0.650256 & 0.653379 & -0.00312278x \\ 0.667957 & 0.672274 & -0.00431704 \\ 0.664704 & 0.670432 & -0.00572785 \\ 0.626905 & 0.634029 & -0.00712355 \\ 0.532149 & 0.540023 & -0.00787414 \\ 0.343487 & 0.350015 & -0.00652742 \\ 0. & 0. & 0. \end{pmatrix} \quad (2.13)$$

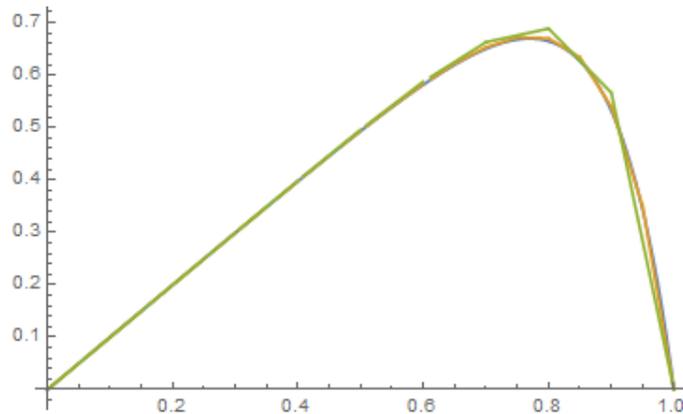Study of the error. Fix $\epsilon = 0.1, \lambda = 1, f = 1$. For n ranging from 10 to 100 (in

Figure 6. Comparison result between exact solution and approximation solution for $P1(n = 10, n = 20)$

steps of 10), plot the curves $\log n \to \log ||e_n||_\infty$ where e_n is the error vector defined by $e_n(k) = u_h(k) - u(x_k)$. Deduce a decay law of $||e_n||_\infty$ of the form $||e_n||_\infty \sim Constant/n^{s_1}$ where s_1 is to be determined.

4. We seek an approximate solution (finite element method P2)

The basis function for P2 is quadratic function. Let us generate a mesh, let a uniform Cartesian mesh $x_i = ih, i = 0, 1, 2, \cdots, n$ where $h = 1/n$, defining the intervals (x_{i-1}, x_i), $i = 1, 2, \cdots, n$. Then, we construct two sets of functions bassed on the mesh, such as the quadratic functions $i = 1, 2, \cdots, n - 1$. The first function is:

$$
p_i(x) = \begin{cases} 0 & x < x_{i-1} \\ \frac{(x-x_{i-0.5})(x-x_{i-1})}{0.5h^2} & x_{i-1} \le x \le x_i \\ \frac{(x-x_{i+0.5})(x-x_{i+1})}{0.5h^2} & x_i \le x \le x_{i+1} \\ 0 & x > x_{i+1} \end{cases} \tag{2.14}
$$

The second function is for fill the middle value of the first function

$$
q_j(x) = \begin{cases} -\frac{(x-x_i)(x-x_{i+1})}{0.25h^2} & x_i \le x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{2.15}
$$

for $j = 0, 1, 2, ..., n - 1$ Then we construct the basis function as the combination of that two functions such as:

$$
\psi_j = \{\psi_1 = p_1, \psi_2 = p_2, \cdots, \psi_{n-1} = p_{n-1}, \psi_n = q_0, \psi_{n+1} = q_1, \cdots, \psi_{2n-1} = q_{n-1}\}.
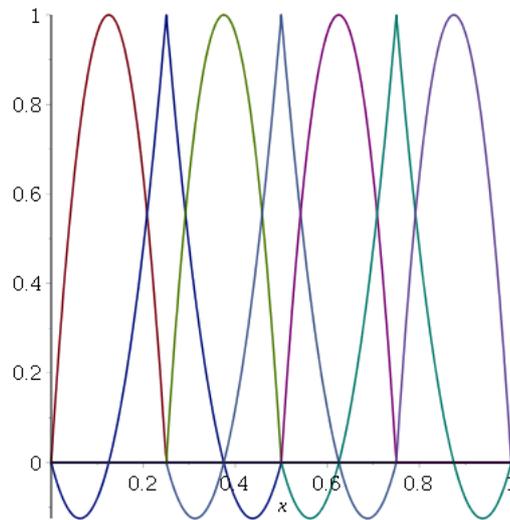$$

We can plot the basis function as follows (for n=10):

Figure 7. basis function for $P2$ case

Represent the approximate (FE) solution by a linear combination of the basis functions

$$
u_h(x) = \sum_{j=1}^{2n-1} c_j \psi_j(x), \tag{2.16}
$$

where the coefficients c_j are the unknowns to be determined. For the case, $n = 4, \epsilon = 0.1, \lambda = 1, f = 1$ we obtain these results:

$$
B_h = \begin{pmatrix}
18.667 & 1.333 & 0 & -10.667 & -10.667 & 0 & 0 \\
1.333 & 18.667 & 1.333 & 0 & -10.667 & -10.667 & 0 \\
0 & 1.333 & 18.667 & 0 & 0 & -10.667 & -10.667 \\
-10.667 & 0 & 0 & 21.333 & 0 & 0 & 0 \\
-10.667 & -10.667 & 0 & 0 & 21.333 & 0 & 0 \\
0 & -10.667 & -10.667 & 0 & 0 & 21.333 & 0 \\
0 & 0 & -10.667 & 0 & 0 & 0 & 21.333
\end{pmatrix}.
$$

$$
C_h = \begin{pmatrix}
0 & -0.167 & 0 & -0.667 & 0.667 & 0 & 0 \\
0.167 & 0 & -0.167 & 0 & -0.667 & 0.667 & 0 \\
0 & 0.167 & 0 & 0 & 0 & -0.667 & 0.667 \\
0.667 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.667 & 0.667 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.667 & 0.667 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.667 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

$$
A_h = 0.1 * B_h + C_h = \begin{pmatrix}
1.8667 & -0.0337 & 0. & -1.7337 & -0.3997 & 0. & 0. \\
0.3003 & 1.8667 & -0.0337 & 0. & -1.7337 & -0.3997 & 0. \\
0. & 0.3003 & 1.8667 & 0. & 0. & -1.7337 & -0.3997 \\
-0.3997 & 0. & 0. & 2.1333 & 0. & 0. & 0. \\
-1.7337 & -0.3997 & 0. & 0. & 2.1333 & 0. & 0. \\
0. & -1.7337 & -0.3997 & 0. & 0. & 2.1333 & 0. \\
0. & 0. & -1.7337 & 0. & 0. & 0. & 2.1333
\end{pmatrix}
$$

$$
A_h^{-1} \begin{pmatrix}
0.901412 & 0.0875424 & 0.00781085 & 0.732563 & 0.240035 & 0.0227499 & 0.00146346 \\
0.893776 & 0.981158 & 0.0875424 & 0.726358 & 0.964832 & 0.254976 & 0.0164022 \\
0.814177 & 0.893776 & 0.901412 & 0.661669 & 0.878904 & 0.900023 & 0.168891 \\
0.168891 & 0.0164022 & 0.00146346 & 0.606012 & 0.0449735 & 0.00426248 & 0.000274197 \\
0.900023 & 0.254976 & 0.0227499 & 0.731435 & 0.844603 & 0.0662615 & 0.00426248 \\
0.878904 & 0.964832 & 0.240035 & 0.714272 & 0.948777 & 0.844603 & 0.0449735 \\
0.661669 & 0.726358 & 0.732563 & 0.537729 & 0.714272 & 0.731435 & 0.606012
\end{pmatrix}.
$$

$$
F = \begin{pmatrix}
0.083 \\
0.083 \\
0.083 \\
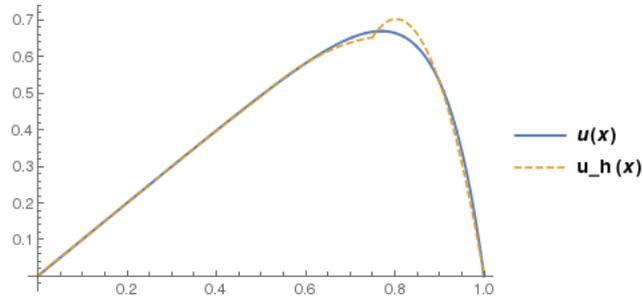0.167 \\
0.167 \\
0.167 \\
0.167
\end{pmatrix}.
$$

Figure 8. Comparison result between exact solution and approximation solution for $P2(n = 4)$

$$c = A_h^{-1}.F = \begin{pmatrix} 0.249199 \\ 0.490635 \\ 0.652362 \\ 0.124973 \\ 0.372729 \\ 0.599242 \\ 0.608447 \end{pmatrix}.$$

Then we have

$$u_h(x) = \sum_{j=1}^{7} c_j \psi_j(x). \tag{2.17}$$

$$\begin{pmatrix} x & u(x) & u_h(x) & u(x) - u_h(x) \\ 0 & 0. & 0 & 0 \\ 1/8 & 0.124887 & 0.124973 & -0.0000860471 \\ 2/8 & 0.249492 & 0.249199 & 0.000293245 \\ 3/8 & 0.373115 & 0.372729 & 0.000385755 \\ 4/8 & 0.493307 & 0.490635 & 0.00267265 \\ 5/8 & 0.601527 & 0.599242 & 0.00228501 \\ 6/8 & 0.667957 & 0.652362 & 0.0155949 \\ 7/8 & 0.588528 & 0.608447 & -0.0199193 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now let us check for $n = 10$:

$$
\begin{pmatrix}
x & u(x) & u_h(x) & u(x) - u_h(x) \\
0 & 0. & 0 & 0 \\
0.05 & 0.0499705 & 0.0499826 & -0.0000120677 \\
0.1 & 0.099922 & 0.0999511 & -0.000029145 \\
0.15 & 0.149842 & 0.149892 & -0.0000502498 \\
0.2 & 0.19971 & 0.199786 & -0.0000764664 \\
0.25 & 0.249492 & 0.249599 & -0.000106273 \\
0.3 & 0.299133 & 0.299273 & -0.000139483 \\
0.35 & 0.348542 & 0.348719 & -0.000177509 \\
0.4 & 0.397567 & 0.397779 & -0.000212591 \\
0.45 & 0.445958 & 0.446217 & -0.000258685 \\
0.5 & 0.493307 & 0.493591 & -0.000283661 \\
0.55 & 0.538936 & 0.539276 & -0.000340579 \\
0.6 & 0.581729 & 0.582057 & -0.00032816 \\
0.65 & 0.619847 & 0.620259 & -0.000411977 \\
0.7 & 0.650256 & 0.650558 & -0.000302309 \\
0.75 & 0.667957 & 0.668437 & -0.000480028 \\
0.8 & 0.664704 & 0.664853 & -0.000149252 \\
0.85 & 0.626905 & 0.627571 & -0.000666252 \\
0.9 & 0.532149 & 0.532032 & 0.000116878 \\
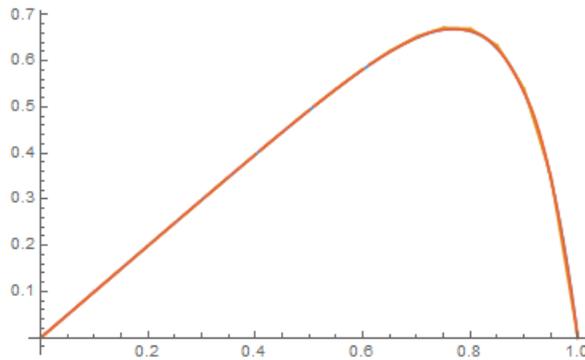0.95 & 0.343487 & 0.345065 & -0.00157802 \\
1 & 0. & 0 & 0
\end{pmatrix}
$$

Figure 9. Comparison result between exact solution and approximation solution for $P2(n = 10)$

3. Variational method for two dimensional case

Let Ω an open bounded related of $\mathbb{R}^N$, $N \geq 1$. For $v \in L^2(\Omega)$, we denote

$$
\bar{v} = \frac{1}{|\Omega|} \int_\Omega v(x) \mathrm{d}x \tag{3.1}
$$

as the average of v in Ω and $|\Omega|$ is the measure of Ω. We consider the Hilbert space V,

$$V = \{v \in H^1(\Omega), \bar{v} = 0\}, \tag{3.2}$$

endowed with norm:

$$\|v\|^2 = \int_\Omega |\nabla v|^2 \mathrm{d}x. \tag{3.3}$$

Theorem 3.1. *There exists a unique function $u \in V$ such that*

$$\int_\Omega \nabla u \nabla v \mathrm{d}x = \int_\Omega gv \mathrm{d}x \quad \forall v \in V \tag{3.4}$$

for $g \in L^2(\Omega)$.

Proof. We will apply the Lax-Milgram theorem [24] by checking the continuity of bilinear and linear form and also coercivity of the bilinear form. From (3.4) we have the bilinear form is

$$B[u, v] = \int_\Omega \nabla u \nabla v \mathrm{d}x \quad \forall u, v \in V, \tag{3.5}$$

and the linear form is

$$L(v) = \int_\Omega gv \mathrm{d}x \quad \forall u, v \in V \text{ and } g \in L^2(\Omega). \tag{3.6}$$

First we will check the continuity of B,

$$|B[u, v]| = \left| \int_\Omega \nabla u \nabla v \mathrm{d}x \right| \tag{3.7}$$

$$= \|\nabla u \nabla v\|_{L^1(\Omega)} \tag{3.8}$$

$$\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \tag{3.9}$$

$$= \|u\|_V \|v\|_V. \tag{3.10}$$

We obtain $|B[u, v]| \leq \|u\|_V \|v\|_V$, B continue in V.

Next we will check the coercivity of B

$$B[u, u] = \int_\Omega |\nabla u|^2 \mathrm{d}x = \|v\|_V^2. \tag{3.11}$$

The coercivity is satistied by the norm of V.

Finally we obtain the continuity of L by Cauchy-Schwarz inequality,

$$|L(v)| = \left| \int_\Omega gv \mathrm{d}x \right| \tag{3.12}$$

$$\leq \|g\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \tag{3.13}$$

$$\leq \|g\|_{L^2(\Omega)} \|v\|_V. \tag{3.14}$$

According to the Lax-Milgram theorem, we obtain that there exists a unique function $u \in V$ such that for all $v \in V$, we have $B[u, v] = L(v)$.　□

By using the FreeFem software, we can simulate the solution as follows:

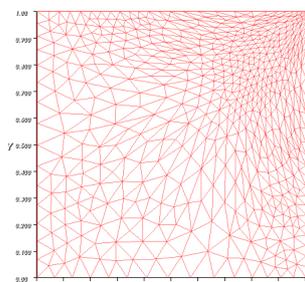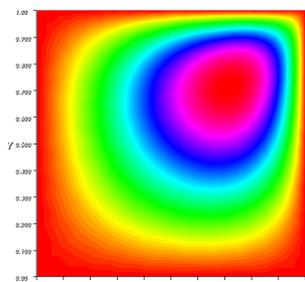Figure 10. Mesh of finite element method

Figure 11. Solution of variational formulation

4. Acknowledgements

References

[1] Asavanant, J., Vanden-Broeck, J.M., 1994, Free-surface flows past a surface-piercing object of finite length, *Journal of Fluid Mechanics* Vol. **273**: 109 – 124

[2] Audusse, E., 2005, A multilayer Saint-Venant model: Derivation and numerical validation, *Discrete and Continuous Dynamical Systems-B* Vol. **5**: 189

[3] Bai, Y., Cheung, K.F., 2015, Dispersion and kinematics of multi-layer non-hydrostatic models, *Ocean Modelling* Vol.**92**: 11 – 27

[4] Barber, B.C., 1993, On the dispersion relation for trapped internal waves, *Journal of Fluid Mechanics* Vol. **252**: 31 – 49

[5] Batchelor, G.K., 1967, *Introduction to Fluid Dynamics*, Collection Enseignement - INSTN CEA, Cambridge University Press, Cambridge, Mass, USA

[6] Bean, N.G., O'Reilly, M.M., 2008, Performance measures of a multi-layer Markovian fluid model, *Annals of Operations Research* Vol. **160**: 99 – 120

[7] Bhatia, H., Norgard, G., Pascucci, V., Bremer, P., 2012, The Helmholtz-Hodge Decomposition-A Survey, *IEEE Transactions on Visualization and Computer Graphics* Vol. **19**: 1386 – 1404

[8] Binder, B.J., Blyth, M.G., McCue, S.W., 2013, Free-surface flow past arbitrary topography and an inverse approach for wave-free solutions, *IMA Journal of Applied Mathematics* Vol. **78**(4): 685 – 696

[9] Binder, B.J., Dias, F., Vanden-Broeck, J.-M., 2007, Influence of rapid changes in a channel bottom on free-surface flows, *IMA Journal of Applied Mathematics* Vol. **73**: 254 – 273

[10] Binghong, Z., Liu, Q., Zemei, T., 2004, Rayleigh-Marangoni-Benard instability in two-layer fluid system, Acta Mechanica Sinica Vol. 20: 366 – 373

[11] Borovikov, V.A., Bulatov, V.V., Vladimirov, Y.V., 1995, Internal gravity waves excited by a body moving in a stratified fluid, *Fluid Dynamics Research* Vol. **15**(5): 325 – 336

[12] Brezis, H.,2010, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer New York

[13] Bulatov, V.V., Vladimirov, Y.V., 2011, The uniform asymptotic form of the internal gravity-wave field generated by a source moving above a smoothly varying bottom, *Journal of Engineering Mathematics* Vol. **69**: 243 – 259

[14] Buttle, N.R., Pethiyagoda, R., Moroney, T.J., McCue, S.W., 2018, Three-dimensional free-surface flow over arbitrary bottom topography, *Journal of Fluid Mechanics* Vol. **846**: 166 – 189

[15] Chapman, S.J., Vanden-Broeck, J.-M., 2006, Exponential asymptotics and gravity waves, *Journal of Fluid Mechanics* Vol. **567**: 299 – 326

[16] Chen, R.M., Walsh, S., Wheeler, M.H., 2016, Existence and qualitative theory for stratified solitary water waves, *Ann. Inst. H. Poincar Anal. Non Linaire* Vol. **35**(2): 517 – 576

[17] Constantin, A., Sattinger, D., Strauss, W., 2006, Variational formulations for steady water waves with vorticity, *Journal of Fluid Mechanics* Vol. **548**: 151 – 163

[18] Crapper, G.D., 1967, Ship waves in a stratified ocean, *Journal of Fluid Mechanics* Vol. **29**(4): 667 – 672

[19] Dautray, R., Lions, J.L., 1987, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Collection Enseignement - INSTN CEA, Masson

[20] Duchêne, V., 2010, Asymptotic Shallow Water Models for Internal Waves in a Two-Fluid System with a Free Surface, *SIAM Journal on Mathematical Analysis* Vol. **42**(5): 2229 – 2260

[21] Duchêne, V., 2011, Boussinesq/Boussinesq systems for internal waves with a free surface and the KdV approximation, *ESAIM: Mathematical Modelling and Numerical Analysis* Vol. **46**: 145 – 185

[22] Duchêne, V., 2015, The multilayer shallow water system in the limit of small density contrast, *Asymptotic Analysis* Vol **98**(3): 189 – 235

[23] Engqvist, A., 1996, Self-similar multi-layer exchange flow through a contraction, *Journal of Fluid Mechanics* Vol. **328**: 49 – 66

[24] Evans, L.C., 1997, *Partial Differential Equations*, Graduate Studies in Mathematics, American Mathematical Society

[25] Forbes, L.K., Schwartz, L.W., 1982, Free-surface flow over a semicircular obstruction, *Journal of Fluid Mechanics* Vol. **114**: 299 – 314

[26] Gang, W., Jia, C.L., Shi, Q.D., 2003, Surface effects of internal wave generated by a moving source in a two-layer fluid of finite depth, Applied Mathematics and Mechanics Vol.24(9): 1025 – 1040

[27] Groves, M.D., 2004, Steady Water Waves, *Journal of Nonlinear Mathematical Physics* Vol. **11**(4) : 435 – 460

[28] Hudimac, A.A., 1961, Ship waves in a stratified ocean, *Journal of Fluid Me-*

chanics Vol. **11**(2): 229 – 243

[29] Huttemann, H. Hutter, K., 2001, Baroclinic solitary water waves in a two-layer fluid system with diffusive interface, *Experiments in Fluids* Vol. **30**: 217 – 326

[30] Jamshidi, S., Trinh, P., 2019, Gravity-capillary waves in reduced models for wave-structure interactions, *Journal of Fluid Mechanics* Vol **890**: A18

[31] Kallen, E., 1987, Surface effects of vertically propagating gravity waves in a stratified fluid, *Journal of Fluid Mechanics* Vol. **182**: 111 – 125

[32] Keller, J.B., Levy, D.M, Ahluwalia, D.S., 1981, Internal and surface wave production in a stratified fluid, *Wave Motion* Vol. **3**(3): 215 – 229

[33] King, A.C., Bloor, M.I.G. 1987, Free-surface flow over a step, *Journal of Fluid Mechanics* Vol. **182**: 193 – 208

[34] King, A.C., Bloor, M.I.G., 1990, Free-surface flow of a stream obstructed by an arbitrary bed topography, *The Quarterly Journal of Mechanics and Applied Mathematics* Vol. **43**(1): 87 – 106

[35] Kress, R., 1989, *Linear Integral Equations*, Applied Mathematical Sciences, Springer New York

[36] Kuznetsov, N., Maz'ya, V., Vainberg, B., 2004, Linear Water Waves, Cambridge University Press, Cambrige

[37] Li, C., Campbell, B., Liu, Y., Yue, D., 2019, A fast multi-layer boundary element method for direct numerical simulation of sound propagation in shallow water environments, *Journal of Computational Physics* Vol. **392**: 694 – 712

[38] Lynett, P.J., Liu, P.L.-F., 2004, Linear analysis of the multi-layer model, *Coastal Engineering* Vol. **51**(5): 439 – 454

[39] Mercier, M.J., Vasseur, R., Dauxois, T., 2011, Resurrecting dead-water phenomenon, *Nonlinear Processes in Geophysics* Vol. **18**: 193 – 208

[40] Mielke, A., 1986, Steady flows of inviscid fluids under localized perturbations, *Journal of Differential Equations* Vol. **65**(1) : 89 – 116

[41] Monjarret, R., 2014, Local well-posedness of the multi-layer shallow-water model with free surface, *arXiv*:1411.2342 [math.AP]

[42] Motygin, O.V., Kuznetsov, N.G., 1997, The wave resistance of a two-dimensional body moving forward in a two-layer fluid, *Journal of Engineering Mathematics* Vol. **32**(1): 53 – 72

[43] Nguyen, T.C., Yeung, R.W., 2011, Unsteady three-dimensional sources for a two-layer fluid of finite depth and their applications, *Journal of Engineering Mathematics* Vol. **70**(1): 67 – 91

[44] Pagani, C.D., Pierotti, D., 1995, On the Neumann-Kelvin Problem in Bounded Domains, *Journal of Mathematical Analysis and Applications* Vol. **192**(1) : 41 – 62

[45] Pagani, C. D., Pierotti, D., 1999, The NeumannKelvin Problem for a Beam, *Journal of Mathematical Analysis and Applications* Vol. **240**(1) : 60 – 79

[46] Pagani, C.D., Pierotti, D., 1999, Exact Solution of the WaveResistance Problem for a Submerged Cylinder. I. Linearized Theory, *Archive for Rational Mechanics and Analysis* Vol. **149**(4): 271 – 288

[47] Pagani, C.D., Pierotti, D., 2001, The forward motion of an unsymmetric surface-piercing cylinder: the solvability of a nonlinear problem in the super-critical case, *Quarterly Journal of Mechanics and Applied Mathematics* Vol. **54**: 85 – 106

[48] Pagani, C.D., Pierotti, D., 2004, The Subcritical Motion of a Semisubmerged

Body: Solvability of the Free Boundary Problem, *SIAM J. Math. Analysis* Vol. **36**: 69 – 93

[49] Pagani, C. D., Pierotti, D., 2006, The NeumannKelvin problem revisited, *Applicable Analysis* Vol. **85**(1-3): 277 – 292

[50] Pagani, C. D., Pierotti, D., 2006, Variational linear problems in wave-obstacle interaction, *Proceedings of the Steklov Institute of Mathematics* Vol. **255**(1): 203 – 214

[51] Petcu, M., Temam, R., 2013, An interface problem: The two-layer shallow water equations, *Discrete and Continuous Dynamical Systems-A* Vol **33**(12): 5327 – 5345

[52] Pierotti, D., 2002, The subcritical motion of a surface-piercing cylinder: existence and regularity of waveless solutions of the linearized problem, *Advances in Differential Equations* Vol. **7**(4): 385 – 418

[53] Pierotti, D., 2003, On Unique Solvability and Regularity in the Linearized Two-Dimensional Wave Resistance Problem, *Quarterly of Applied Mathematics* Vol **61**(4) : 639 – 655

[54] Pierotti, D., 2006, Uniqueness and trapped modes in the linear problem of the steady flow over a submerged hollow, *Wave Motion* Vol. **43**(3): 222 – 231

[55] Pierotti, D., 2008, On the plane problem of the flow around a submerged beam, *Journal of Differential Equations* Vol. **244**(9): 2350 – 2371

[56] Pierotti, D., Simioni, P., 2008, The steady two-dimensional flow over a rectangular obstacle lying on the bottom, Journal of Mathematical Analysis and Applications Vol. 342: 1467 – 1480

[57] Rakotoson, J.-M., Rakotoson, J.-E., 1999, Analyse fonctionnelle applique aux quations aux drives partielles, Presses Universitaires de France - PUF

[58] Sato, K., 2015, *Complex Analysis for Practical Engineering*, Springer International Publishing, Switzerland

[59] Ter-Krikorov, A.M., 2002, Perturbations from a source in a three-layer atmosphere, *Journal of Applied Mathematics and Mechanics* Vol. **66**(1): 59 – 64

[60] Trinh, P.H., 2016, A topological study of gravity free-surface waves generated by bluff bodies using the method of steepest descents, *Proc. Math. Phys. Eng. Sci.* 472(2191):20150833

[61] Tuck, E., Stokes, Y., 2012, On thin or slender bodies, *ANZIAM Journal* Vol. **53**: 190 – 212

[62] Vainberg, B., Mazya, V.G., 1973, On the plane problem of the motion of a body immersed in a fluid, *Trans. Moscow Math. Soc.* Vol. **28**: 33 – 55

[63] Vasholz, D.P., 2011, Stratified wakes, the high Froude number approximation, and potential flow, *Theoretical and Computational Fluid Dynamics* Vol. **25**(6): 357 – 379

[64] Wei, G., Lu, D., Dai, S., 2005, Waves induced by a submerged moving dipole in a two-layer fluid of finite depth, *Acta Mechanica Sinica* Vol. **21**(1): 24 – 31

[65] Yeung, R.W., Nguyen, T.C., 1999, Waves Generated by a Moving Source in a Two-Layer Ocean of Finite Depth, *Journal of Engineering Mathematics* Vol. **35**(1): 85 – 107