THE FIRST $U$-EXTENSION MODULE AS CLASSES OF SHORT $U$-EXACT SEQUENCES

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Abstract. Inspired by the notions of the $U$-exact sequence introduced by Davvaz and Parnian-Garamaleky in 1999, and of the chain $U$-complex introduced by Davvaz and Shabani-Solt in 2002, Mahatma and Muchtadi-Alamsyah in 2017 developed the concept of the $U$-projective resolution and the $U$-extension module, which are the generalizations of the concept of the projective resolution and the concept of extension module, respectively. It is already known that every element of a first extension module can be identified as a short exact sequence. To the simple, there is a relation between the first extension module and the short exact sequence. It is proper to expect the relation to be provided in the $U$-version. In this paper, we aim to construct a one-one correspondence between the first $U$-extension module and the set consisting of equivalence classes of short $U$-exact sequence.

Keywords: Chain $U$-complex, $U$-projective resolution, $U$-extension module

1. Motivation

In [1] Davvaz and Shabani-Solt introduced the notion of the chain $U$-complex which generalizes the concept of the chain complex. The main idea was by replacing the kernel of every homomorphism in the sequence with the inverse image of a possibly nonzero submodule. For more details, a sequence of modules and module homomorphisms

$$\cdots \xrightarrow{d_{p+2}} C_{p+1} \xrightarrow{d_{p+1}} C_p \xrightarrow{d_p} C_{p-1} \xrightarrow{d_{p-1}} \cdots$$

is called a chain $U$-complex if, for every $k \in \mathbb{Z}$, $U_k \subseteq \text{Im}(d_{k+1}) \subseteq d_k^{-1}(U_{k-1})$ where $U_k$ is submodule of $C_k$ for every $k \in \mathbb{Z}$. By this definition, the ordinary chain complex now can be regarded as a chain $U$-complex with $U_k = 0$ for all $k \in \mathbb{Z}$. As an example of chain $U$-complex, consider the sequence
\[
\begin{array}{cccccccc}
\ldots & 4 & Z & 3 & \uparrow & 2 & Z & 2 & \uparrow & 3 & Z & 3 & \uparrow & \ldots
\end{array}
\]

where the arrow "\( m \mathbb{Z} \to n \mathbb{Z} \)" denotes the map \( x \mapsto kx \) for every \( x \in m \mathbb{Z} \). The objects written in the bottom row are the submodule \( U_k \).s.

As the chain \( U \)-complex was defined, we can consider a modified concept of exactness of a sequence by replacing the subset relation \( \text{Im}(d_{k+1}) \subseteq d_k(U_k) \) with equality for all \( k \in \mathbb{Z} \). In fact, Davvaz and Parnian-Garamaleky [2] has introduced in advance the notion of \( U \)-exact sequences before the chain \( U \)-complex was. Nevertheless, the definition does not yet contain the conditions necessary for a \( U \)-exact sequences to be seen as a special case of chain \( U \)-complex, for it does not require the submodule \( U_k \) to be contained in \( \text{Im}(d_k(U_{k+1})) \) for every \( k \in \mathbb{Z} \). However, experience shows that there are more advantages when a \( U \)-exact sequences is also a chain \( U \)-complex.

Projective resolution is a kind of exact sequence that is used widely in representation theory. As the concept of exact sequences was generalized, Mahatma and Muchtadi-Alansyah [3] proposed a method to construct the \( U \)-projective resolution as the generalization of the projective resolution. Furthermore, they continued in the same article with a method to induce the \( k \)-th \( U \)-extension module form a \( U \)-projective resolution for all \( k \in \mathbb{N} \), as the projective resolution does to the \( k \)-th extension module for all \( k \in \mathbb{N} \).

We assume throughout this paper that \( R \) is commutative algebra. It is known that for any \( R \)-modules \( M \) and \( N \) there exists one-one correspondence between the first extension \( R \)-module \( \text{Ext}^1(M, N) \) and the set \( e(M, N) \) consists of all equivalence classes of short exact sequence of the form \( 0 \to N \to E \to M \to 0 \) (see Chapter 7 of [4]). By this result, we can define the \( R \)-module structure for \( e(M, N) \). The goal of this paper is to investigate the analogous result in the \( U \) version where \( U \) is nonzero submodule of \( M \).

2. The \( U \)-Extension

Given \( R \)-modules \( M \) and \( N \), the short exact sequence \( 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0 \) is also known as the extension of \( N \) by \( M \). We start this paper with the notion generalizing the concept of the extension by replacing the property that \( \text{Im}(f) = \ker(g) \) with \( \text{Im}(f) = g^{-1}(U) \) where \( U \) is nonzero submodule of \( M \). This concept would require that the module \( N \) should be large enough so that it can be mapped onto \( U \).

Let \( M \) and \( N \) be \( R \)-modules and \( U \) be a submodule of \( M \). The sequence \( 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0 \) such that \( f \) is one-one, \( g \) is onto, and \( f(N) = g^{-1}(U) \) is called the \( U \)-extension of \( N \) by \( M \). We shall also call such sequence as a short \( U \)-exact sequence.

We restrict the discussion in this paper only for the module \( N \), which is direct sum of \( U \), and only for the short \( U \)-exact sequence \( 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0 \) with property that if \( N = U \oplus V \) then \( f(V) = \ker(g) \) and \( gf(u) = u \) for every \( u \in U \).
Therefore, every short $U$-exact sequence throughout this paper will be assumed to be of that form. Notice that if we allow the submodule $U$ to be 0, then the case $U = 0$ gives us exactly the ordinary extension of $N$ by $M$.

Let $\mathcal{E}(M[U], N)$ denotes the set of all $U$-extension of $N$ by $M$. Let $E : 0 \to N \xrightarrow{f'} E \xrightarrow{\delta} M \to 0$ be an element of $\mathcal{E}(M[U], N)$. A short $U$-exact sequence $F : 0 \to N \xrightarrow{f'} F \xrightarrow{\delta'} M \to 0$ in $\mathcal{E}(M[U], N)$ is said to be equivalent to $E$, denoted by $E \approx F$, if there exists a morphism $\delta : E \to F$ such that $g\delta = g$ and $f' = \delta f$, that is if the diagram

\[
\begin{array}{ccc}
0 & \to & N \\
\downarrow & & \downarrow \\
0 & \to & N
\end{array}
\]

commutes. It is easy to verify that $\delta$ is an isomorphism and hence "$\approx$" is an equivalence relation in $\mathcal{E}(M[U], N)$. For every $E \in \mathcal{E}(M[U], N)$, the class of all short $U$-exact sequence equivalent to $E$ will be denoted by $[E]$. Thus, the set $\mathcal{E}(M[U], N)$ partitioned by "$\approx$" will consist of all classes $[E]$ where $E \in \mathcal{E}(M[U], N)$. We denote those set by $e(M[U], N)$. Thus,

\[
e(M[U], N) = \mathcal{E}(M[U], N)/\approx = \{[E] | E \in \mathcal{E}(M[U], N)\}.
\]

3. The $U$-Projective Resolution and the $U$-Extension Module

Let $M$ be $R$-module, and $U$ be a nonzero submodule of $M$. Consider the sequence $P_0 \xrightarrow{d_0} M \to 0$ where $P_0$ is projective. Let $P_1$ be a projective module such that the sequence $P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} M \to 0$ is $U$-exact at $P_0$, that is $\text{Im}(d_1) = d_0^{-1}(U)$. Set $U_0 := d_0^{-1}(U)$ and let $P_2$ be a projective module such that the sequence $P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0$ is $U_0$-exact at $P_1$, or $\text{Im}(d_2) = d_1^{-1}(U_0)$. Set $U_1 := d_1^{-1}(\text{ker}(d_0))$ and let $P_3$ be a projective module such that the sequence $P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1$ is $U_1$-exact at $P_2$, or $\text{Im}(d_3) = d_2^{-1}(U_1)$. Continue the process by setting the submodule $U_k := d_k^{-1}(\text{ker}(d_{k-1}))$ and choose the projective module $P_{k+2}$ such that the sequence $P_{k+2} \xrightarrow{d_{k+2}} P_{k+1} \xrightarrow{d_{k+1}} P_k$ is $U_k$-exact at $P_{k+1}$, or $\text{Im}(d_{k+2}) = d_{k+1}^{-1}(U_k)$.

The infinite sequence $\cdots d_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{d_0} M \to 0$ obtained from the process above is called the $U$-projective resolution of $M$, denoted by $P_\bullet(U_\bullet) \xrightarrow{d_\bullet} M(U)$. From the construction above, it seems that the sequence obtained depends on the choice of the module $P_k$s. Nevertheless, in [3] Mahatma and Muchtadi-Alamsyah showed that the $U$-projective resolution is unique up to the so-called $(U, U')$-homotopy, that is if $P : P_\bullet(U_\bullet) \xrightarrow{d_\bullet} M(U)$ and $Q : Q_\bullet(U'_\bullet) \xrightarrow{d'_\bullet} M(U)$ both are $U$-projective resolution of $M$ then there exist chain $(U, U')$-map $f : P \to Q$ and chain $(U', U)$-map $g : Q \to P$ such that $gf \simeq 1_P$ and $fg \simeq 1_Q$ (see also [1] for detail of the map between two $U$-complexes). Now notice that, in a $U$-projective resolution of $M$, since $\text{Im}(d_1) = d_0^{-1}(U) = U_0$ then we may choose $P_2 := P_1$ and set $d_2 := 1_{P_1}$. Hence every $U$-projective resolution of $M$ is of the form

\[
\cdots d_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{d_0} M \to 0.
\]
For the example of the $U$-projective resolution, let us consider the case when the algebra $R$ is hereditary. Here we consider two cases: when module $M$ is projective and when it is not. According to the method given in the beginning of this section, if $M$ is projective, then the $U$-projective resolution of $M$ will be of the form

$$
0 \rightarrow U \xrightarrow{1} U \xrightarrow{1} M \xrightarrow{1} M \rightarrow 0
$$

while if $M$ is not projective, then the $U$-projective resolution of $M$ will be of the form

$$
0 \rightarrow \ker(d) \xrightarrow{1} d^{-1}(U) \xrightarrow{1} d^{-1}(U) \xrightarrow{1} P \xrightarrow{d} M \rightarrow 0.
$$

Recall that the objects written in the bottom row denote the submodule $U_k$s. Here, when the $U_k$ is not written, we mean that $U_k = 0$. The detail of these constructions can be found in [5].

Let $P : P_0 \xrightarrow{d_0} M(U)$ be the $U$-projective resolution of $M$. If $P_n \neq 0$ and $P_1 = 0$ for all $i > n$ then we say that the length of $P$ is $n$. Hence, if $R$ is hereditary, we have that the $U$-projective resolution length is either 2 or 3. Moreover, in [5] Baur, Mahatma, and Muchtadi-Alamsyah showed that an algebra $R$ is hereditary if and only if, for $U \neq 0$, every $U$-projective resolution of an $R$-module has length of either 2 or 3.

Given an $R$-module $M$, a nonzero submodule $U$ of $M$, and $U$-projective resolution of $M$ $P : P_0 \xrightarrow{d_0} M(U)$, let $(P_M)$ be the sequence $\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow 0$ obtained by removing $M$ from $P$. Given an $R$-module $N$, apply the functor $\Hom(\_ , N)$ to $(P_M)$ to obtain the sequence $0 \rightarrow \Hom(P_0, N) \xrightarrow{d_0^N} \Hom(P_1, N) \xrightarrow{d_1^N} \cdots$, where $d_k^N$ denotes the map $\Hom(d_k, N)$ for every $k \in \mathbb{N}$. Now, for every $k \in \mathbb{N}$, define the submodules $A_k^N := \{ \alpha d_{k-1} - d_k \alpha : \text{Im}(d_{k-1}) \rightarrow N \}$ and $Z_k^N := (d_{k+1}^N)^{-1}(A_{k+1}^N)$ of $\Hom(P_k, N)$. Note first that, for every $k \in \mathbb{N}$, a morphism $z \in \Hom(P_k, N)$ is in $Z_k^N$ if and only if there exists a morphism $\alpha : \text{Im}(d_k) \rightarrow N$ such that $d_{k+1}^N(z) = zd_{k+1} = \alpha d_k$. Next, for every $k \in \mathbb{N}$ define the submodule $B_k^N := \{ \mu d_k + \lambda d_{k-1} \mu : P_{k-1} \rightarrow N, \lambda : U_{k-2} \rightarrow N \}$ where $d_{k-1} \mu$ is the morphism $d_{k-1} d_k$ regarded as single morphism. Finally, for every $k \in \mathbb{N}$ we define the $k$-th $U$-extension module of $N$ by $M$ by $\Ext^k(M[U], N) := Z_k^N / B_k^N$.

From the construction above, the module obtained depends on the choice of the $U$-projective resolution used as the basic material. Nevertheless, in [3] Mahatma and Muchtadi-Alamsyah showed that the $k$-th $U$-extension module is unique up to isomorphism for every $k \in \mathbb{N}$.
4. The First $U$-Extension Module

Let $M$ be $R$-module, and $U$ be a nonzero submodule of $M$. Given any $R$-module $X$, we have seen that, for every $k \in \mathbb{N}$, the construction of $\text{Ext}^k(M[U], X)$ involves many steps, that make the structure of the module obtained seems so complicated. We can describe the module $\text{Ext}^1(M[U], X)$ very simply.

To do so, recall first that, in the $U$-projective resolution of $M$, $P_2 = P_1$ and $d_2 = 1_{P_1}$. Hence the module $Z_1^X$ and $B_1^N$ can be simplified to $Z_1^X = \{\alpha d_1 | \alpha : \text{Im}(d_1) \to X\}$ and $B_1^X = \{\mu d_1 | \mu : P_0 \to X\}$, respectively. Thus the module $\text{Ext}^1(M[U], X) = Z_1^X/B_1^X$ consists of all classes $[z]$ of morphisms in $\text{Hom}(P_1, X)$ whose form $\alpha d_1$ where $\alpha : \text{Im}(d_1) \to X$, where two classes $[z_1]$ and $[z_2]$ in $\text{Ext}^1(M[U], X)$ are considered to be the same if and only if $z_1 - z_2$ whose form $\mu d_1$ where $\mu : P_0 \to X$. As a consequence, $\text{Ext}^1(M[U], X) = 0$ if and only if every morphism $\alpha : \text{Im}(d_1) \to X$ can be extended into $\alpha' : P_0 \to X$. This could happen when $\text{Im}(d_1) = d_0^{-1}(U)$ is a direct summand of $P_0$. Especially, we have that $\text{Ext}^1(M[M], X) = 0$ for any $R$-module, since if $U = M$ then $\text{Im}(d_1) = d_0^{-1}(U) = d_0^{-1}(M) = P_0$.

5. Construction of the Correspondence

Suppose given an $R$-module $M$, nonzero submodule $U$ of $M$, and an $R$-module $N = U \oplus V$. For every $E \in \mathcal{E}(M[U], N)$, we will identify the class $[E] \in e(M[U], N)$ by an element $[z_E] \in \text{Ext}^1(M[U], V)$ and vice versa.

Consider the sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$, where $P_i$s are projective and $\text{Im}(d_1) = d_0^{-1}(U)$. Let $E : 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ be an element in $\mathcal{E}(M[U], N)$. Consider the diagram

$$
P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0
\xrightarrow{1}
0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0
$$

Note that since $g$ is surjective and $P_0$ is projective, then there exists a morphism $h : P_0 \to E$ satisfying $gh = d_0$. Let $d(0, 1)$ be the composition $d_0 d_1$ regarded as a single morphism from $P_1$ to $U$. Consider that $hd_1 - fd(0,1)$ is a morphism from $P_1$ to $E$ satisfying $g(hd_1 - fd(0,1)) = d_0 d_1 - gfd(0,1) = 0$ since $gf|_U = 1_U$. Hence, $\text{Im}(hd_1 - fd(0,1)) \subseteq \ker(g) \subseteq \text{ker}(f) \subseteq \text{ker}(f|_V)$, and since $P_1$ is projective, then there exists a morphism $z : P_1 \to V$ satisfying $f z = hd_1 - fd(0,1)$.

Define the morphism $\alpha : d_1(P_1) \to V$ by $d_1(P_0) := z(p)$ for every $p \in P_1$. Notice that if $d_1(p) = 0$ then $z(p) = 0$ and hence $\alpha$ is well-defined. Since $z = \alpha d_1$ then $z \in Z_1^Y$.

Now suppose that $h' : P_0 \to E$ is another morphism satisfying $gh' = d_0$. Let $z' \in Z_1^Y$ satisfy $f z' = h'd_1 - fd(0,1)$. Since $g(h - h') = 0$ then $\text{Im}(h - h') \subseteq \ker(g) \subseteq \text{ker}(f) \subseteq \text{ker}(f|_V)$. Since $P_0$ is projective, then there exists a morphism $\mu : P_0 \to V$ satisfying $f \mu = h - h'$. Thus we have $f(z - z') = (h - h')d_1 = f \mu d_1$, which implies $z - z' = \mu d_1 \in B_1^N$. Therefore $[z] = [z']$ in $\text{Ext}^1(M[U], V)$.

The paragraph above shows how to construct a map from $\mathcal{E}(M[U], N)$ to $\text{Ext}^1(M[U], V)$. For this map let us denote the image of $E$ by $z_E$. Now sup-
pose that $F \in [E]$. Let $\delta : E \to F$ be the isomorphism that makes the Diagram 2.1 commutes. Now, to obtain the morphism $2F \in Z_1^V$ which represents the image of $F$, we may set $h_F := \delta h$ and choose $z_F$ as the morphism satisfying $f'z_F = h_F d_1 - f'd_{(0,1)}$ or $\delta f z_F = \delta h d_1 - \delta f d_{(0,1)}$. Since $\delta$ is an isomorphism, then we get $f z_F = h_E d_1 - f d_{(0,1)} = f z_E$ which implies $z_F = z_E$.

We have just constructed a map $\varphi : e(M[U], N) \to \text{Ext}^1(M[U], V)$ where $\varphi([E]) = [z_E]$ for every $[E] \in e(m[U], N)$. Our goal is to show that $\varphi$ is a one-one correspondence.

**Theorem 5.1.** The map $\varphi$ is onto.

**Proof.** Suppose that $[z] \in \text{Ext}^1(M[U], V)$. Define the submodule $I := \{(d_{(0,1)}(x) \oplus z(x)) \oplus d_1(-x) | x \in P_1\}$ of $N \oplus P_0$ and the module $E := (N \oplus P_0)/I$. Remember that every $n \in N$ can be written uniquely as $n_U \oplus n_V$ where $n_U \in U$ and $n_v \in V$. Create a sequence $N \xrightarrow{f} E \xrightarrow{g} M$ where $f(n) := (n \oplus 0) + I$ for every $n \in N$ and $g((a \oplus b) + I) := a_U + d_0(b)$ for every $(a \oplus b) + I \in E$. To show that the morphism $g$ is well-defined, notice that if $a \oplus b \in I$ then there exists an $x \in P_1$ such that $\alpha = d_{(0,1)}(x) \oplus z(x)$ and $b = d_1(-x)$. Since $\text{Im}(z) \subseteq V$ then we have $a_U = d_{(0,1)}(x)$ and hence $g((a \oplus b) + I) = d_{0,1}(x) + d_0d_1(-x) = 0$. So $g$ is well-defined. Furthermore, since $d_0$ is onto then $g$ is onto.

Now, if $f(n) = 0$ then $n \oplus 0 \in I$. Hence, there exists an $x \in P_1$ such that $n = d_{(0,1)}(x) \oplus z(x)$ and $0 = d_1(-x)$. Since $z \in Z_1^N$ then $z = \alpha d_1$ for a morphism $\alpha : d_1(P_1) \to V$. Therefore, $n = 0 \oplus 0$ and hence $f$ is one-one.

Next, notice that for every $n \in N$, $g(f(n)) = g((n \oplus 0) + I) = n_U \in U$. Hence $\text{Im}(g f) \subseteq U$ and so $\text{Im}(f) \subseteq g^{-1}(U)$. Now, if $(a \oplus b) + I \in g^{-1}(I)$ then $g((a \oplus b) + I) = a_U + d_0(b) \in U$. Since $a_U \in U$ then we have $d_0(b) \in U$ and so $b \in d_0^{-1}(U) = \text{Im}(d_1)$. Let $b = d_1(p)$ where $p \in P_1$. We see that

$$(a \oplus b) + I = (a \oplus d_1(p)) + I$$

$$= (a + (d_{(0,1)}(p) \oplus z(p)) \oplus 0) + I$$

$$= f((a + (d_{(0,1)}(p) \oplus z(p))))$$

$$\in f(N).$$

Then $g^{-1}(U) \subseteq \text{Im}(f)$. Hence we have $\text{Im}(f) = g^{-1}(U)$.

Next, if $v \in V$ then $v_U = 0$ and hence $g f(v) = g((v \oplus 0) + I) = 0 + d_0(0) = 0$. Therefore, $f(V) \subseteq \text{ker}(g)$. Now suppose that $(a \oplus b) + I \in \text{ker}(g)$. Since $g((a \oplus b) + I) = a_U + d_0(b) = 0$ then $d_0(b) = -a_U \in U$. Hence $b \in d_0^{-1} = \text{Im}(d_1)$. Let $b = d_1(x)$ where $x \in P_1$. Then

$$(a \oplus b) + I = ((-d_{(0,1)}(x) \oplus a_U) \oplus d_1(x)) + I$$

$$= ((0 \oplus (a_U + z(x))) \oplus 0) + I$$

$$= f(0 \oplus (a_U + z(x)))$$

$$\in f(V).$$
Hence \( \ker(g) \subseteq f(V) \) and we have \( f(V) = \ker(g) \).

Finally, if \( u \in U \) then \( uV = u \) and hence \( gf(u) = g((u \oplus 0) + I) = u + d_0(0) = u \).

We have just shown that the sequence \( E : 0 \to N \xrightarrow{J} E \xrightarrow{g} M \to 0 \) satisfies all criterions in \( E(M[U], N) \). Thus, \( E \in \mathcal{E}(M[U], N) \). We will show that \( \varphi([E]) = [z] \).

Define the morphism \( h : P_0 \to E \) by \( h(q) := (0 \oplus q) + I \) for every \( q \in P_0 \). We see that for every \( q \in P_0 \), \( gh(q) = g((0 \oplus q) + I) = d_0(q) \). Hence \( gh = d_0 \). Next, notice that for every \( p \in P_1 \),

\[
hd_1(p) = (0 \oplus d_1(p)) + I = (\langle d_{(0,1)}(p) \oplus z(p) \rangle 0) + I = f(d_{(0,1)}(p) \oplus z(p)) = f d_{(0,1)}(p) + f z(p).
\]

The equation above gives us \( f z = h d_1 - f d_{(0,1)} \). Hence \( [z] \) is the map of the class \([E] \in e(M[U], N) \) by \( \varphi \). thus, \([z] \) has a pre-image in \( e(M[U], N) \) by \( \varphi \). Since \([z] \) is arbitrary then \( \varphi \) is onto.

\( \square \)

To show that \( \varphi \) is one-one we will show that every element in \( \text{Ext}^1(M[U], V) \) has unique pre-image in \( e(m[U], N) \) by \( \varphi \).

**Theorem 5.2.** For every \([z] \in \text{Ext}^1(M[U], V) \), the pre-image of \([z] \) by \( \varphi \) is unique

**Proof.** Given \( z \in \text{Ext}^1(M[U], V) \), suppose that \( E \in \mathcal{E}(M[U], N) \) is the \( U \)-exact sequence constructed using the method given in the proof of Theorem 5.1. Thus, \([z] = [z_E] \). Let \( F : 0 \to N \xrightarrow{f'} M \to 0 \) be an element in \( \mathcal{E}(M[U], N) \) with \([z_F] = [z_E] \). Our goal is to show that \([E] = [F] \), that is there exists a morphism \( \delta : E \to F \) which makes the diagram 2.1 commutes, that is \( g' \delta = g \) and \( f' = \delta f \).

Consider the diagram

\[
\begin{array}{ccc}
P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \to & 0 \\
\downarrow z' & & \downarrow h' & & \downarrow 1 \\
0 & \to & N & \xrightarrow{f'} & M & \to & 0
\end{array}
\]

with \( g'h' = d_0 \) and \( f'z' = h'd_1 - f'd_{(0,1)} \). Since \([z_F] = [z_E] \) then \( z' - z \in B_Y \) and hence there exists a morphism \( \mu : P_0 \to V \) such that \( z' = z + \mu d_1 \). Let us define the morphism \( \delta : E \to F \) by \( \delta ((a \oplus b) + I) := f'(a) + (h' - f' \mu)(b) \) for every \((a \oplus b) + I \in E \). Notice that if \((a \oplus b) + I = I \) then \( \alpha = d_{(0,1)}(x) \oplus z(x) \) and \( b = d_1(-x) \) for an \( x \in P_1 \). Consequently,

\[
\delta ((a \oplus b) + I) = f'(d_{(0,1)}(x) \oplus z(x)) + (h' - f' \mu)d_1(-x) = f'd_{(0,1)}(x) + f'z(x) + f'(z' - z)(x) = f'z(x) - f'z'(x) + f'(z' - z)(x) = 0.
\]

Hence, \( \delta \) is well-defined.

Next, note that for every \( a \in N \), \( g'f'(a) = g'f'(a_U \oplus a_V) = g'f'(a_U \oplus 0) + g'f'(0 \oplus a_V) = a_U + 0 = a_U \). Hence, for every \((a \oplus b) + I \in E \) we have
\[ g'(a \oplus b) + I = g' (f'(a) + (h' - f' \mu)(b)) \\
= g'f'(a) + g'h'(b) - g'f' \mu(b) \\
= a_U + d_0(b) \\
= g((a \oplus b) + I) \]

where the third row holds since \( \text{Im}(\mu) \subseteq V \) and \( f'(V) \subseteq \ker(g') \). Hence \( g' \delta = g \).

Next, for every \( n \in N \), we have \( \delta f(n) = \delta((n \oplus 0) + I) = f'(n) + (h' - f' \mu)(0) = f'(n) \).

Hence, \( \delta f = f' \). We have shown that \([E] = [F]\). \( \square \)

6. Result and Discussion

We have shown that given any \( R \)-module \( M \) and nonzero submodule \( U \) of \( M \), if \( N = U \oplus V \) then there exists one-one correspondence between the set \( e(M[U], N) \) of all equivalence classes in \( \mathcal{E}(M[U], N) \) with the module \( \text{Ext}^1(M[U], V) \). We have known in Section 4 that \( \text{Ext}^1(M[M], V) = 0 \). Clearly, the only \( M \)-extension of \( M \oplus V \) by \( M \) is given by the sequence of the form \( 0 \rightarrow M \oplus V \xrightarrow{1+\varphi} M \oplus W \xrightarrow{1+0} M \rightarrow 0 \) where \( \varphi : V \rightarrow W \) is an isomorphism.

As we know that there exists one-one correspondence between the module \( \text{Ext}^k(M, N) \) with the set of equivalence classes of exact sequences of the form \( 0 \rightarrow N \rightarrow E_2 \rightarrow E_1 \rightarrow M \rightarrow 0 \), it would be interesting to investigate whether the result in this paper could be extended for another value of \( k \). But we must leave a note here that the construction of \( U \)-extension module give results that \( \text{Ext}^2(M[U], X) = 0 \) for any module \( X \). Nevertheless, we may expect that there will be relation between the exact sequence \( 0 \rightarrow N \rightarrow E_2 \rightarrow E_1 \rightarrow M \rightarrow 0 \) and some nonzero modules \( \text{Ext}^k(M[U], N) \) with \( k > 2 \).

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Daftar Pustaka


[5] Baur, K., Mahatma, Y., Muchtadi-Alamsyah, I., 2019, The \( U \)-projective resolution of modules over path algebras of type \( A_n \) and \( \overline{A_n} \), \textit{Communications of the Korean Mathematical Society}, 34 : 701 – 718