Jurnal Matematika UNAND
Vol. **11** No. **1** Hal. 74 – 81
Edisi Januari 2022
ISSN : 2303–291X
e-ISSN : 2721–9410

ON CHARACTERISTIC POLYNOMIAL OF ANTIADJACENCY MATRIX OF A LINE DIGRAPH

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Received March 22, 2022 Revised April 1, 2022 Published April 7, 2022

Abstract. In this paper we find the characteristic polynomial of the antiadjacency matrix of a line digraph. There are recent studies on the relation between the characteristic polynomial of the adjacency matrix and its line digraph, we are also interested in finding the connection between the antiadjacency matrix of a digraph and its line digraph. In this paper, we show the connection of characteristic polynomial of the antiadjacency matrix between an acyclic digraph and its line digraph.
Keywords: Acyclic digraph, antiadjacency matrix, characteristic polynomial, digraph, line digraph

1. Intoduction

1.1. *Motivation*

The antiadjacency of a digraph was introduced by Bapat [4] in 2010, but there are still few results related to this topic that have been discovered. Some results of the antiadjacency matrix of several classes of digraph can be found in [7], [16], and [1]. Therefore, in this paper, we find more properties of the antiadjacency matrix of a digraph.

Recent studies, such as in [12], [6], and [11] have discussed the characteristic polynomial of a digraph. From [12], we knew some properties of the transformation of the characteristic polynomial of the adjacency matrix of a digraph. In [6], it discussed the characteristic polynomial of the adjacency of a digraph of complementary spectrum. Meanwhile, in [11], the characteristic polynomial of lifting a digraph is discussed.

We can use line digraphs of regular digraphs, such as de Bruijn digraphs and Reddy-Pradhan-Kuhl digraph, to design point-to-point interconnection network (see, e.g., [3] and [13]). From [2], [18], and [17], we knew the properties of the adjacency matrix of line digraph, such as it can be decomposed into a multiplication of two matrices and the coefficients of its characteristic polynomial. Therefore,

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we are interested in finding the properties of the other representation matrix of line digraph, that is the antiadjacency matrix. Hence, this paper discusses the coefficient of the characteristic polynomial of the antiadjacency matrix of a line digraph.

From [2], we know that the adjacency matrix of a digraph D and its line digraph $L(D)$ can be expressed as $A(D) = H^T T$ and $A(L(D)) = TH^T$ (where H and T are the incidence matrix of head and tail of D), respectively. Moreover, in [15], we find the relation between the characteristic polynomial of the matrix AB and BA (that is $P_{BA}(\lambda) = \lambda^{n-m} P_{AB}$). Therefore, we are also interested to find the relation between the characteristic polynomial of the antiadjacency matrix of a digraph and its line digraph.

In this paper we show that we cannot implement the results of [2] directly to the characteristic polynomial of the antiadjacency matrix of a digraph. This implies that the antiadjacency matrix of a digraph and its line digraph can not be expressed as AB and BA, respectively. Thus, we have to use another method to find the relation between the characteristic polynomial of the antiadjacency matrix of a digraph and its digraph.

1.2. *Preliminaries*

A directed graph (digraph) D is an ordered pair $(V(D), A(D))$ that consists of a set of vertices $VV(D)$ and a set of arcs (directed edges) $A(D)$, that is disjoint from $V(D)$, together with an incidence function ψ_D associated each arc of D as an ordered pair of vertices in D [5].

Let u, v be the distinct vertices of a digraph D. If D has either an arc (u, v) or an arc (v, u), then D is called an oriented digraph [8].

Let the set of vertices of the digraph D and H, respectively, be $V(D)$ and $V(H)$ where $V(H) \subseteq V(D)$, and let the set of arcs of the digraph D and H, respectively, be $A(D)$ and $A(H)$ where $A(H) \subseteq A(D)$. Then, the digraph H is a subdigraph of a digraph D. A subdigraph H of a digraph D is called an induced subdigraph of D if u and v are the vertices of H and (u, v) is the arc of D, then (u, v) is also the arc of H as well. [9].

Let $W = (u = u_1, u_2, \cdots, u_k = v)$ be a sequence of vertices of D such that the vertex u_i is adjacent to u_{i+1}, where $i \in \{0, 1, \cdots, k-1\}$. Then W is called an $u - v$ directed walk in D. The directed walk W length is the number of visited arcs on D. If we have $u = v$ in the $u - v$ directed walk then the directed walk is called a closed walk. On the other hand, if we have $u \neq v$ in an $u - v$ directed walk, then the directed walk is called an open walk. If W is not passing through the vertex more than once, then W is a directed path. A directed cycle is a closed directed walk length of at least two, where no vertex is repeated except for the initial and terminal vertices. If a digraph D does not have a directed cycle subgraph, then D is called an acyclic digraph. Meanwhile, if a digraph D is had a directed cycle, then D is called a cyclic digraph [10]. A Hamiltonian directed path in a digraph D is a directed path that includes all vertices in D [4].

Let D be a digraph with a set of vertices $V := V(D)$, a set of arcs $A := A(D)$, $u, v, w, z \in V(D)$, and $uv, wz \in A(D)$. A line digraph $L(D)$ of a digraph D is a digraph with every vertex of $L(D)$ representing an arc of D; that is $V(L(D)) = \{uv|(u,v) \in A(D)\}$, and the vertex uv is adjacent to the vertex wz if $v = w$. Furthermore, an iterated line digraph is $L^n(D)$, with every vertex x of $L^n(D)$ representing a directed walk $v_0v_1 \cdots v_n$ which has length n in D, and the vertex $x = v_0v_1 \cdots v_n$ is adjacent to the vertex $y = v_1v_2 \cdots v_nv_{n+1}$ where $(v_n, v_{n+1}) \in A(D)$ [14].

A digraph can be represented by the representation matrices, which are adjacency and antiadjacency matrices. An adjacency matrix of a digraph D (with a set of vertices $\{v_1, v_2, \cdots, v_n\}$) is an $n \times n$ matrix defined by $A(D) = [a_{ij}]$, where a_{ij} is equal to 1 if there exists an arc from v_i to v_j, and equal to 0 elsewhere. On the other hand, the antiadjacency matrix of a digraph D (with a set of vertices $\{v_1, v_2, \cdots, v_n\}$) is an $n \times n$ matrix defined by $B(D) = J - A(D)$, where J is an $n \times n$ matrix in which every entry is equal to 1 and $A(D)$ is an adjacency matrix of D [4].

2. Related Results

Theorem 2.1 show the relation between the characteristic polynomial of AB and BA, where $A \in M_{mn}$ and $B \in M_{nm}$ with $m \leq n$.

Theorem 2.1. [15] *If $A \in M_{m,n}$ and $B \in M_{n,m}$ where $m \leq n$, then the n eigenvalues of BA are the m eigenvalues of AB together with $n - m$ zeroes; that is, $P_{BA}(\lambda) = \lambda^{(n-m)} P_{AB}(\lambda)$. If $m = n$ and at least one of A or B is nonsingular, then AB and BA are similar.*

Theorem 2.2 shows the determinant of the antiadjacency matrix of a simple digraph with a hamiltonian path.

Theorem 2.2. [4] *Let D be an acyclic digraph with the set of vertices $V(D) = v_1, v_2, \ldots, v_n$ and $B(D)$ be the antiadjacency matrix of D. Then $det(B(D)) = 1$ if D has a Hamiltonian path, and* $\det(B(D)) = 0$ *otherwise.*

3. Results and Discussions

3.1. *The Coefficients of Characteristic Polynomial of Antiadjacency Matrix of Line Digraph*

We present our results concerning the coefficient of the antiadjacency matrix of an iterated line digraph, where the digraph is acyclic and oriented, in Proposition 3.1.

Proposition 3.1. *Let D be an acyclic-oriented digraph (with order n and size m) and $L^k(D)$ be its iterated line digraph. where $k \in \mathbb{Z}^+$. If $P(\lambda; B(L^k(D)) = \lambda^l + \sum_{i=1}^{l} a_i \lambda^{l-i}$ is the characteristic polynomial of the antiadjacency matrix of $L^k(D)$, then $|a_i|$ is equal to the number of directed path of D which involving $(i+k)$-vertices, where $i \in \{1, 2, 3, \cdots, n\}$ and $k \in \mathbb{Z}^+$.*

Proof. Let D be an acyclic-oriented digraph (with order n and size m) and $L^k(D)$ be its iterated line digraph. where $k \in \mathbb{Z}^+$. Moreover, let $P(\lambda; B(L^k(D)) = \lambda^l + \sum_{i=1}^{l} a_i \lambda^{l-i}$ is the characteristic polynomial of the antiadjacency matrix of $L^k(D)$, where $i \in \{1, 2, 3, \cdots, n\}$ and $k \in \mathbb{Z}^+$.

Using the elementary methods of finding the principal minor, we have $|a_i|= \sum$ (all $i \times i$ principal minors of $B(L^k(D))$ where $i \in \{1, 2, \cdots, l\}$.

Note that all the $i \times i$ principal minors of $B(L^k(D))$ are the determinant of the antiadjacency matrix of the subdigraphs of $L^k(D)$. The subdigraphs of $L(D)$ respectively are the directed walk of $L^k(D)$. If the subdigraphs $L^k(D)$ are the directed path of $L^k(D)$, their subdigraphs contain a directed Hamiltonian path because their subdigraphs meet all of their vertices.

All of the directed paths of $L^k(D)$ involve (l) vertices of D, for $i \in \{1, 2, \cdots, n\}$. Therefore, from Theorem 2.2, the determinant of the antiadjacency matrices of the subdigraphs $L^k(D)$ which are a directed path are equal to 1 and equal to 0 elsewhere. Furthermore, we have $|a_i|$ is equal to the number of the directed path in $L^k(D)$ involving i-vertices. Meanwhile, $|b_j|$ is equal to the number of directed path in $L(D)$ involving j-vertices, where $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$.

Since D and $L^k(D)$ are an acyclic-oriented digraph where $L(D)$ of a digraph D is a digraph with every vertex of $L^k(D)$ representing a directed walk of D involving $(k+1)$-vertices of D, then the directed paths of $L^k(D)$ involve $i+k$ vertices, where $i \in \{1, 2, \cdots, n\}$. □

Since a line digraph of an acyclic-oriented digraph is an iterated line digraph $L^1(D) = L(D)$, then from Proposition 3.1, we obtain the coefficients of the characteristic polynomial of the antiadjacency matrix of a line digraph, that is presented in Corollary 3.2.

Corollary 3.2. *Let D be an acyclic-oriented digraph (with n-vertices and size m) and $L(D)$ be its line digraph. If $P(\lambda; B(L(D)) = \lambda^m + \sum_{i=1}^{m} b_i \lambda^{m-i}$ is the characteristic polynomial of the antiadjacency matrix of $L(D)$, then $|b_i|$ is equal to the number of directed path of D which involving $(i+1)$-vertices.*

Proof. Let D be an acyclic-oriented digraph (with n-vertices and size m) and $L^k(D)$ be its iterated line digraph. Moreover, let $P(\lambda; B(L(D)) = \lambda^m + \sum_{i=1}^{m} b_i \lambda^{m-i}$ be the characteristic polynomial of the antiadjacency matrix of $L(D)$

Since a line digraph is an iterated line digraph $L^1(D) = L(D)$, then from Proposition 3.1, we have its coefficients are equal to the number of the directed path of D, which involves $(i+1)$-vertices. □

3.2. *The Relation Between The Antiadjacency matrix a digraph and its line digraph*

We present our results concerning the relation between the antiadjacency matrix of an acyclic digraph and its line digraph in Proposition 3.3.

Proposition 3.3. *Let D be an acyclic-oriented digraph (with n-vertices and size m) and $L(D)$ be its line digraph. If $P(\lambda; B(D)) = \lambda^n + \sum_{i=1}^{n} a_i \lambda^{n-i}$ is the characteristic polynomial of the antiadjacency matrix of D and $P(\lambda; B(L(D)) = \lambda^m + \sum_{i=1}^{m} b_i \lambda^{m-i}$ is the characteristic polynomial of the antiadjacency matrix of $L(D)$, then $|b_i|= |a_{i+1}|$ and $|b_j|= 0$ where $i \in \{1, 2, \cdots, n-1\}$ and $j \in \{n, n+1, \cdots, m\}$.*

Proof. Let D be an acyclic-oriented digraph with order n and size m, and $L(D)$ be its line digraph. Let $P(\lambda; B(D)) = \lambda^n + \sum_{i=1}^{n} a_i \lambda^{n-i}$ be the characteristic polynomial of the antiadjacency matrix of D and $P(\lambda; B(L(D)) = \lambda^m + \sum_{i=1}^{m} b_i \lambda^{m-i}$ be the characteristic polynomial of the antiadjacency matrix of $L(D)$. Using elementary methods of finding the principal minor, we have $|a_i| = \sum$ (all $i \times i$ as principal minors of $B(D)$). Note that all the $i \times i$ principal minors of $B(D)$ are the determinant of the antiadjacency matrix of the subdigraphs of D. The subdigraphs of D are the directed walk of D. If the subdigraph of D is the directed path of D, it contains a directed Hamiltonian path because their subdigraphs meet all of their vertices. All of the directed path of D involve i-vertices, for $i \in \{1, 2, \cdots, n\}$. Therefore, from Theorem 2.2, the determinant of the antiadjacency matrices of the subdigraphs D, a directed path, is equal to 1 and equal to 0 elsewhere. Furthermore, we have $|a_i|$ is equal to the number of the directed path in D involving i-vertices. According to Corollary 3.2, we have $|b_i|$ equal to the number of directed path of D, which involves $(i+1)$-vertices. In conclusion, we have $|b_i| = |a_{i+1}|$ and $|b_j| = 0$ where $i \in \{1, 2, \cdots, n-1\}$ and $j \in \{n, n+1, \cdots, m\}$. □

Let D be an acyclic-oriented digraph (with n-vertices and size m) and $L(D)$ be its line digraph, where $P(\lambda; B(D))$ is the characteristic polynomial of the antiadjacency matrix of D and $P(\lambda; B(L(D))$ is the characteristic polynomial of the antiadjacency matrix of $L(D)$. Since from Proposition 3.1 we have $P(\lambda; B(D)) \neq \lambda^{n-m} P(\lambda; B(L(D))$, then from Theorem 2.1 we have the antiadjacency matrix of a digraph and its line digraph respectively cannot be decomposed into AB and BA. We present our results concerning the relationship between the antiadjacency matrix of an acyclic digraph and its line digraph in Example 3.4.

Example 3.4. Let D_1 be a directed cyclic digraph with a set of vertices $V(D_1) = \{v_1, v_2, v_3, v_4\}$, set of arcs $A(D_1) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, and incidence function $\psi_{D_1}(a_1) = (v_1, v_2)$, $\psi_{D_1}(a_2) = (v_2, v_3)$, $\psi_{D_1}(a_3) = (v_3, v_1)$, $\psi_{D_1}(a_4) = (v_1, v_4)$, and $\psi_{D_1}(a_5) = (v_4, v_3)$. Then, its line digraph is $L(D_1)$ which has a set of vertices $V(L(D_1)) = \{v_1, v_2, v_3, v_4\}$, set of arcs $A(L(D_1)) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, and incidence function $\psi_{L(D_1)}(a_1) = (v_1v_2, v_2v_3)$, $\psi_{L(D_1)}(a_2) = (v_2v_3, v_3v_1)$, $\psi_{L(D_1)}(a_3) = (v_3v_1, v_1v_2)$, $\psi_{L(D_1)}(a_4) = (v_3v_1, v_1v_4)$, $\psi_{L(D_1)}(a_5) = (v_1v_4, v_4v_3)$, and $\psi_{L(D_1)}(a_6) = (v_4v_3, v_3v_1)$.

Therefore, the antiadjacency matrix $B(D_1)$ and $B(L(D_1))$ are the following.

$$B(D_1) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

$$B(L(D_1)) = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Figure 1 shows the directed graph D_1 and its line digraph $L(D_1)$.

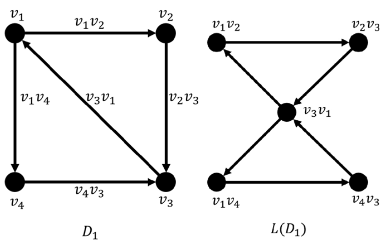

Figure. 1. Digraph D_1 and its line digraph $L(D_1)$

Hence, the characteristic polynomial of $B(D_1)$ and $B(L(D_1))$ are the following.

$$P(\lambda; B(D_1)) = \lambda^4 - 4\lambda^3 + 5\lambda^2 - 4\lambda,$$
$$P(\lambda; B(L(D_1))) = \lambda^5 - 5\lambda^4 - 6\lambda^3 + 6\lambda^2.$$

We present our results concerning the relationship between the antiadjacency matrix of an acyclic digraph and its iterated line digraph in Example 3.5.

Example 3.5. Let D_2 a directed acyclic digraph with a set of vertices $V(D_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, a set of arcs $A(D_2) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, and incidence functions $\psi_{D_2}(a_1) = (v_1, v_5)$, $\psi_{D_2}(a_2) = (v_5, v_2)$, $\psi_{D_2}(a_3) = (v_2, v_3)$, $\psi_{D_2}(a_4) = (v_3, v_6)$, $\psi_{D_2}(a_5) = (v_1, v_2)$, $\psi_{D_2}(a_6) = (v_2, v_4)$, $\psi_{D_2}(a_7) = (v_1, v_3)$, and $\psi_{D_2}(a_8) = (v_1, v_4)$. Then, its iterated line digraph $L^2(D_2)$ has a set of vertices $V(L^2(D_2)) = \{v_1v_5v_2, v_2v_3v_6, v_1v_2v_3, v_3v_6v_4, v_5v_2v_3, v_1v_3v_6, v_5v_2v_4, v_1v_2v_4\}$, a set of arcs $A(L^2(D_2)) = \{a_1, a_2, a_3\}$, and incidence functions $\psi_{L^2(D_2)}(a_1) = (v_1v_5v_2, v_2v_3v_6)$, $\psi_{L^2(D_2)}(a_2) = (v_1v_2v_3, v_3v_6v_4)$, and $\psi_{L^2(D_2)}(a_3) = (v_5v_2v_3, v_3v_6v_4)$.

Therefore, the antiadjacency matrix $B(D_1)$ and $B(L(D_1))$ are the following.

$$B(D_2) = \begin{pmatrix} 1&0&0&0&0&1 \\ 1&1&0&0&1&1 \\ 1&1&1&1&1&0 \\ 1&1&1&1&1&1 \\ 1&0&1&1&1&1 \\ 1&1&1&0&1&1 \end{pmatrix},$$

$$B(L^2(D_2)) = \begin{pmatrix} 1&0&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \end{pmatrix}.7$$

Figure 2 shows the directed graph D_2 and its iterated line digraph $L^2(D_2)$.

Hence, the characteristic polynomial of $B(D_1)$ and $B(L(D_1))$ are the following.

$$P(\lambda; B(D_2)) = \lambda^6 - 6\lambda^5 + 9\lambda^4 - 8\lambda^3 + 6\lambda^2 - 3\lambda,$$
$$P(\lambda; B(L^2(D_2))) = \lambda^8 - 8\lambda^7 + 3\lambda^6.$$

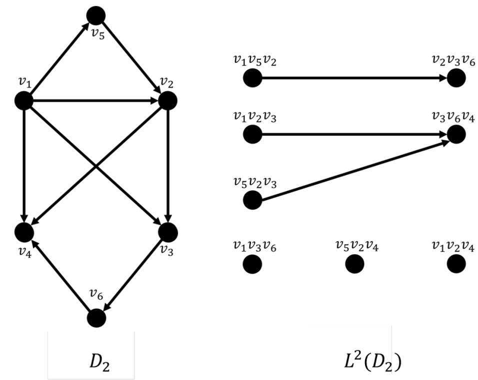

Figure. 2. Digraph D_2 and its iterated line digraph $L^2(D_2)$

In Example 3.5, we show a counterexample if we assume the relation between the antiadjacency matrix of a cyclic-oriented digraph and its iterated line digraph are similar as in Proposition 3.1 or similar as in [2]. This implies that the antiadjacency matrix of a cylic-oriented digraph and its line digraph respectively cannot be decomposed into a matrix AB and BA.

4. Conclusion

In this paper we found the characteristic polynomial coefficient of antiadjacency matrix of iterated line digraph. Since a line digraph is also an iterated line digraph, we also found its characteristic polynomial. After finding the coefficient of the characteristic polynomial of antiadjacency matrix of iterated line digraph, we can find the relation between the characteristic polynomial of an acyclic digraph and its line digraph. On the other hand, we found the counterexample that made the relationship between the characteristic polynomial of a cyclic digraph and its line digraph similar to an acyclic digraph. We showed that the properties of characteristic polynomial of antiadjacency matrix of a digraph is not always the same with the adjacency one.

5. Acknowledgement

We are gratefully thank the reviewer whose suggestions make this paper better.

Daftar Pustaka

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