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THE MATRIX TRANSFORMS THE VECH* TO VECD* OPERATORS

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Abstract. *This article introduces two new matrix operators inspired by the definition of vecd operators, and we call these the vecd*. This operator related to vech* operators. This operator is constructed the same way as the vec operator, i.e. a matrix of size $m \times n$ becomes a column vector, size $mn \times 1$. The difference is in the arrangement of entries, which become column vectors. In the vec operator, all entries in the matrix $m \times n$ will become entries in the column vector, but the vech and vecd only make some of the entries in the matrix $m \times n$ into column vectors with specific rules. We try to define the column vector Also we use the vech* operator with a different construct from vecd*. We explicitly construct a matrix that transforms vech*(A) to vecd*(A), where A is an $n \times n$ matrix for $n \in N$. We also derive some properties from this transform matrix.*
Keywords: *vec*, *vecd**, *vech**

1. Introduction

The *vec* operator or the *vec* matrix $(mn \times 1)$ is a column matrix obtained from a matrix, $m \times n$, i.e., change the matrix into a vector by stacking the column vertically [6]. This definition of the *vec* operator inspired several operators, such as *vech*, *vecd*, and *vecp* (see [7], [12], [13]). Based on these operator definitions, a transformation matrix connects operators with the original matrix and operators with operators. For the *vec* operator and transpose *vec* operator, there is a transformation matrix, known as the commutation matrix [3], and the duplicate matrix that transforms *vec* operator to *vech* operators [15].

For the square matrix, [7] defines *vech* operator in the same way that *vec* is defined, except that for each column of the square matrix only that part of which is on or below the diagonal of the square matrix is put into *vech* operator (*vech* =vector-half). For the square matrix, [12] defines the *vecd* operator, i.e., by stacking the entries diagonally and eliminating supra-diagonal entries of the square matrix. Here, d

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represents the diagonal. This article presents the new operators, based on definition *vecd* and it is called by *vecd**. Furthermore, it presents the matrix that transforms *vech** to *vecd** operators for the arbitrary matrix $n \times n$.

The method in this study is a literature study. The first step of this research is to define the *vecd** operator for arbitrary $n \times n$ matrix (see Section 3). Next, define the unique matrix associated with *vech** and *vecd**, and find its properties.

2. Basic Theory

Definition 2.1. [10] *Let $A = [a_{ij}]$ $m \times n$ matrix, and A_j is the j-th column of A. The $vec(A)$ is the $mn \times 1$ vector given by*

$$vec(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}. \tag{2.1}$$

Definition 2.2. [1] *Let $A = [a_{ij}]$ $n \times n$ matrix. $vech(A)$ is the $\dfrac{n(n+1)}{2} \times 1$ column vector that is obtained from $vec(A)$ by eliminating all supra-diagonal elements of A, it is given as*

$$vech(A) = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} & a_{22} & \cdots & a_{n2} & \cdots & a_{(n-1)(n-1)} & a_{n(n-1)} & a_{nn} \end{bmatrix}^T. \tag{2.2}$$

Definition 2.3. [8] *Let $A = [a_{ij}]$ $n \times n$ matrix. The $vech^*(A)$ is the $\dfrac{n(n+1)}{2} \times 1$ column vector that is obtained from $vec(A)$ by eliminating all below main diagonal elements of A, it is given as*

$$vech^*(A) = \begin{bmatrix} a_{11} & a_{12} & a_{22} & a_{13} & a_{23} & a_{33} & \cdots & a_{1n} & a_{2n} & \cdots & a_{(n-1)n} & a_{nn} \end{bmatrix}^T. \tag{2.3}$$

Definition 2.4. [12] *Let $A = [a_{ij}]$ $n \times n$ matrix. The $vecd(A)$ is the $\dfrac{n(n+1)}{2} \times 1$ column vector that is obtained by stacking main diagonal entry, then diagonal entry below the main diagonal up to entry a_{n1}, it is given as*

$$vecd(A) = \begin{bmatrix} a_{11} & a_{22} & \cdots & a_{nn} & a_{21} & a_{32} & \cdots & a_{n(n-1)} & \cdots & a_{(n-1)1} & a_{n2} & a_{n1} \end{bmatrix}^T. \tag{2.4}$$

Example 2.5. Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\begin{aligned} vec(A) &= \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{12} & a_{22} & a_{32} & a_{13} & a_{23} & a_{33} \end{bmatrix}^T, \\ vech(A) &= \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{22} & a_{32} & a_{33} \end{bmatrix}^T, \\ vech^*(A) &= \begin{bmatrix} a_{11} & a_{12} & a_{22} & a_{13} & a_{23} & a_{33} \end{bmatrix}^T, \\ vecd(A) &= \begin{bmatrix} a_{11} & a_{22} & a_{33} & a_{21} & a_{32} & a_{31} \end{bmatrix}^T. \end{aligned}$$

Let S_n be the set of all permutations of the n element set $[n] = \{1, 2, \ldots, n\}$. A permutation is a one-to-one function from $[n]$ onto $[n]$. If σ is a permutation, we have the identity matrix as follows.

Definition 2.6. [13] *Let σ be a permutation in S_n. Define the permutation matrix $P(\sigma) = [\delta_{i,\sigma(j)}]$, $\delta_{i,\sigma(j)} = ent_{ij}(P(\sigma))$ where*

$$\delta_{i,\sigma(j)} = \begin{cases} 1, & \text{if } i = \sigma(j), \\ 0, & \text{if } i \neq \sigma(j). \end{cases} \tag{2.5}$$

We present definitions of inversion on permutation and elementary product.

Definition 2.7. [2] *Inversion is the occurrence of a larger integer preceding a smaller integer. In comparison, the number of inversions is the total number of integers preceded by a smaller integer in each inversion according to the permutations.*

Definition 2.8. [2] *If the number of inversions of a permutation is an even number, then it is said to be an even permutation, and if it is an odd number, then it is said to be an odd permutation.*

Definition 2.9. [2] *Let A $n \times n$ matrix. The elementary product of A is the product of n elements from A without taking elements from the same row or column. In contrast, the signed elementary product of A is the elementary product which is marked $(+1)$ if the permutation is even and (-1) if the permutation is odd.*

Definition 2.10. [2] *An $m \times m$ matrix P whose columns form an orthonormal set of vectors is called an orthogonal matrix. It immediately follows that $P^T P = PP^T = I_m$.*

Theorem 2.11. [2] *Let P be $m \times m$ orthogonal matrix. Then $|P| = \pm 1$, so that P is nonsingular. Consequently, $P^{-1} = P^T$.*

Theorem 2.12. [1] *Every permutation matrix is an orthogonal matrix.*

Definition 2.13. [1] [15] *The Moore-Penrose inverse of the $m \times n$ matrix A is the $n \times m$ matrix, denoted by A^+ which satisfies the conditions*

$$AA^+A = A, \tag{2.6}$$

$$A^+AA^+ = A^+, \tag{2.7}$$

$$(AA^+)^T = AA^+, \tag{2.8}$$

$$(A^+A)^T = A^+A. \tag{2.9}$$

Theorem 2.14. *Corresponding to each $m \times n$ matrix A, one and only one $n \times m$ matrix A^+ exists, satisfying conditions (2.6) - (2.9).*

Theorem 2.15. [15] *Let A $m \times n$ matrix. Then*

$$A^+ = (A^TA)^{-1}A^T \text{ dan } A^+A = I_n \text{ jika } rank(A) = n.$$

Definition 2.16. [15] *Let A $n \times n$ symmetric matrix. The matrix that transforms $vech(A)$ into $vec(A)$ is called the duplication matrix.*

Definition 2.17. [1] *Let an $m \times n$ matrix $A = [a_{ij}]$ and a $p \times q$ matrix $B = [b_{ij}]$, is denoted by the symbol $A \otimes B$ and is defined to be the $mp \times nq$ matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \tag{2.10}$$

obtained by replacing each element a_{ij} of A with the $p \times q$ matrix B.

3. Results and Discussion

This paper aims to introduce a new operator like *vech* and *vecd*, and we call the operator with $vecd^*$, and then to present the relationship between $vech^*$ and $vecd^*$.

Definition 3.1. *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $vecd^*(A)$ is the $\frac{n(n+1)}{2} \times 1$ column vector that is obtained by stacking the main diagonal entry, then the diagonal entry above the main diagonal up to entry a_{1n}, it is given as*

$$vecd^*(A) = \begin{bmatrix} a_{11} & a_{22} & \cdots & a_{nn} & a_{12} & a_{23} & \cdots & a_{(n-1)n} & \cdots & a_{1(n-1)} & a_{2n} & a_{1n} \end{bmatrix}^T. \tag{3.1}$$

Based on Example 2.5, we have

$$vecd^*(A) = \begin{bmatrix} a_{11} & a_{22} & a_{33} & a_{12} & a_{23} & a_{13} \end{bmatrix}^T.$$

Example 3.2. Let A be an matrix of size 4 as follows.

$$A = \begin{bmatrix} 4 & 1 & -2 & 3 \\ -1 & 5 & -1 & 2 \\ -2 & -1 & 6 & -4 \\ 3 & 2 & -4 & 7 \end{bmatrix}.$$

Then,

$$vech^*(A) = \begin{bmatrix} 4 & 1 & 5 & -2 & -1 & 6 & 3 & 2 & -4 & 7 \end{bmatrix}^T$$

and

$$vecd^*(A) = \begin{bmatrix} 4 & 5 & 6 & 7 & 1 & -1 & -4 & -2 & 2 & 3 \end{bmatrix}^T$$

Let A be an $n \times n$ matrix. In [12], it is stated that there is an $n \times n$ matrix B_n^* that transforms $vech(A)$ to $vecd(A)$, i.e: $B_n^* vech(A) = vecd(A)$. This article will construct a matrix similar to B_n^*, symbolized by $B_n^{*(d)}$, which transforms $vech^*(A)$ to $vecd^*(A)$, where it explicitly assigns each $vech^*(A)$ entry exactly one to the $vecd^*(A)$ entry, i.e.,

$$B_n^{*(d)} vech^*(A) = vecd^*(A). \tag{3.2}$$

Given $B_n^{*(d)}$ for $n = 2, 3, 4, 5$ as follows.

$$B_2^{*(d)} = \begin{bmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{bmatrix},\ B_3^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,1\,0\,0 \end{bmatrix},\ B_4^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \end{bmatrix},$$

$$B_5^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \end{bmatrix}.$$

We denote $\boldsymbol{e}_{m,n}$ as an $n \times 1$ unit vector for entry 1 in the m-row, and $0'$s elsewhere, and $O_{m\times n}$ as a zero matrix consisting of m-row and n-column. Thus, $B_n^{*(d)}$ for $n = 2, 3, 4, 5$ can be written as follows.

$$B_2^{*(d)} = \left[\begin{array}{cc} e_{1,2}^T & O_{1\times 1} \\ O_{1\times 2} & e_{1,1}^T \\ \hline O_{1\times 1} & e_{1,2}^T \end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \end{bmatrix},\ B_3^{*(d)} = \left[\begin{array}{ccc} e_{1,3}^T & O_{1\times 2} & O_{1\times 1} \\ O_{1\times 2} & e_{1,2}^T & O_{1\times 2} \\ O_{1\times 3} & O_{1\times 2} & e_{1,1}^T \\ \hline O_{1\times 1} & e_{1,3}^T & O_{1\times 2} \\ O_{1\times 2} & O_{1\times 2} & e_{1,2}^T \\ \hline O_{1\times 1} & O_{1\times 2} & e_{1,3}^T \end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \\ E_3^{*(d)} \end{bmatrix},$$

$$B_4^{*(d)} = \left[\begin{array}{cccc} e_{1,4}^T & O_{1\times 3} & O_{1\times 2} & O_{1\times 1} \\ O_{1\times 2} & e_{1,3}^T & O_{1\times 3} & O_{1\times 2} \\ O_{1\times 3} & O_{1\times 2} & e_{1,2}^T & O_{1\times 3} \\ O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,1}^T \\ \hline O_{1\times 1} & e_{1,4}^T & O_{1\times 3} & O_{1\times 2} \\ O_{1\times 2} & O_{1\times 2} & e_{1,3}^T & O_{1\times 3} \\ O_{1\times 3} & O_{1\times 3} & O_{1\times 2} & e_{1,2}^T \\ \hline O_{1\times 1} & O_{1\times 2} & e_{1,4}^T & O_{1\times 3} \\ O_{1\times 2} & O_{1\times 3} & O_{1\times 2} & e_{1,3}^T \\ \hline O_{1\times 1} & O_{1\times 2} & O_{1\times 3} & e_{1,4}^T \end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \\ E_3^{*(d)} \\ E_4^{*(d)} \end{bmatrix},$$

$$B_5^{*(d)} = \left[\begin{array}{ccccc}
e_{1,5}^T & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & O_{1\times 1} \\
O_{1\times 2} & e_{1,4}^T & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} \\
O_{1\times 3} & O_{1\times 2} & e_{1,3}^T & O_{1\times 4} & O_{1\times 3} \\
O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,2}^T & O_{1\times 4} \\
O_{1\times 5} & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,1}^T \\
\hline
O_{1\times 1} & e_{1,5}^T & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} \\
O_{1\times 2} & O_{1\times 2} & e_{1,4}^T & O_{1\times 4} & O_{1\times 3} \\
O_{1\times 3} & O_{1\times 3} & O_{1\times 2} & e_{1,3}^T & O_{1\times 4} \\
O_{1\times 4} & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,2}^T \\
\hline
O_{1\times 1} & O_{1\times 2} & e_{1,5}^T & O_{1\times 4} & O_{1\times 3} \\
O_{1\times 2} & O_{1\times 3} & O_{1\times 2} & e_{1,4}^T & O_{1\times 4} \\
O_{1\times 3} & O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & e_{1,3}^T \\
\hline
O_{1\times 1} & O_{1\times 2} & O_{1\times 3} & e_{1,5}^T & O_{1\times 4} \\
O_{1\times 2} & O_{1\times 3} & O_{1\times 4} & O_{1\times 2} & e_{1,4}^T \\
\hline
O_{1\times 1} & O_{1\times 2} & O_{1\times 3} & O_{1\times 4} & e_{1,5}^T
\end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \\ E_3^{*(d)} \\ E_4^{*(d)} \\ E_5^{*(d)} \end{bmatrix}.$$

It can be seen that the matrix $B_n^{*(d)}$, $n = 2, 3, 4, 5$ forms a pattern for the matrix $B_n^{*(d)}$. Thus, the general form of the $B_n^{*(d)}$ matrix is obtained as follows.

$$B_n^{*(d)} = \left[\begin{array}{cccccc}
e_{1,n}^T & O_{1\times(n-1)} & O_{1\times(n-2)} & \cdots & \cdots & O_{1\times 1} \\
O_{1\times 2} & e_{1,(n-1)}^T & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times 2} \\
\vdots & O_{1\times 2} & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & O_{1\times 2} & \ddots & \vdots & \vdots \\
O_{1\times(n-1)} & \vdots & \vdots & O_{1\times 2} & e_{1,2}^T & O_{1\times(n-1)} \\
O_{1\times n} & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times 2} & e_{1,1}^T \\
\hline
O_{1\times 1} & e_{1,n}^T & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times 2} \\
O_{1\times 2} & O_{1\times 2} & e_{1,(n-1)}^T & O_{1\times(n-1)} & \cdots & O_{1\times 3} \\
\vdots & \vdots & O_{1\times 2} & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & O_{1\times 2} & \ddots & O_{1\times(n-1)} \\
O_{1\times(n-1)} & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times 2} & e_{1,2}^T \\
\hline
O_{1\times 1} & O_{1\times 2} & e_{1,n}^T & O_{1\times(n-1)} & \cdots & O_{1\times 3} \\
\vdots & O_{1\times 3} & O_{1\times 2} & e_{1,(n-1)}^T & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & O_{1\times(n-1)} \\
O_{1\times(n-2)} & O_{1\times(n-1)} & \cdots & \cdots & O_{1\times 2} & e_{1,3}^T \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
O_{1\times 1} & O_{1\times 2} & \cdots & O_{1\times(n-2)} & e_{1,n}^T & O_{1\times(n-1)} \\
O_{1\times 2} & O_{1\times 3} & \cdots & O_{1\times(n-1)} & O_{1\times 2} & e_{1,(n-1)}^T \\
\hline
O_{1\times 1} & O_{1\times 2} & \cdots & \cdots & O_{1\times(n-1)} & e_{1,n}^T
\end{array}\right] = \begin{bmatrix} E_1^{*(d)} \\ E_2^{*(d)} \\ \vdots \\ E_n^{*(d)} \end{bmatrix}$$

where $E_n^{*(d)}$ is a partition matrix that assigns the diagonal entries of A in $vech^*(A)$ to the same entry in $vecd^*(A)$.

Next, the properties associated with $B_n^{*(d)}$ are as follows:

Theorem 3.3. *The $B_n^{*(d)}$ is a permutation matrix.*

Proof. Based on Definitions 2.6, $B_n^{*(d)}$ is a permutation matrix. □

Corollary 3.4. *The $B_n^{*(d)}$ is an orthogonal matrix.*

Proof. Based on Theorem 3.3, $B_n^{*(d)}$ is a permutation matrix. Then, based on Theorem 2.12, the $B_n^{*(d)}$ is an orthogonal matrix. □

Theorem 3.5. *Let $B_n^{*(d)}$ be an matrix that transforms $vech^*(A)$ to $vecd^*(A)$. Then*
$$\det\left(B_n^{*(d)}\right) = \begin{cases} -1, & \text{if } B_n^{*(d)} \text{ is an odd permutation,} \\ 1, & \text{if } B_n^{*(d)} \text{ is an even permutation.} \end{cases}$$

Proof. Based on Definition 3.3, the entry in a column or row of a permutation matrix is 1, and the other entries are 0, and this matrix comes from an identity matrix in which rows are swapped (an odd or even number). The number of inversions is also related to the number of line swaps. So, the determinant for $B_n^{*(d)}$ can be determined by the number of inversions. By using row reduction, the determination of the matrix $B_n^{*(d)}$ can be seen using odd or even permutations. Based on Definitions 2.8 and 2.9, if $B_n^{*(d)}$, the number of inversions is odd, then the permutation is odd, so that $|B_n^{*(d)}| = -1$ if the number of inversions is even, so the permutation is even, and then $|B_n^{*(d)}| = 1$. □

Let A be an $n \times n$ symmetric matrix. In [12], it is stated that there is an $n \times n$ matrix D_n that transforms $vech(A)$ to $vec(A)$, i.e: $D_n vech(A) = vec(A)$. Then, we will define a matrix that is analogous to the duplicate matrix which transforms $vecd^*(A)$ to $vec(A)$, symbolized by $D_n^{*(d)}$, where it explicitly assigns each $vecd^*(A)$ entry exactly one to the $vecd(A)$ entry, i.e.,

$$D_n^{*(d)} vecd^*(A) = vecd(A). \tag{3.3}$$

Given $D_n^{*(d)}$ for $n = 2, 3, 4$.

$$D_2^{*(d)} = \begin{bmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,0\,1 \\ 0\,1\,0 \end{bmatrix}, \quad D_3^{*(d)} = \begin{bmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,1\,0\,0 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,1\,0\,0\,0 \end{bmatrix}, \text{ and}$$

$$D_4^{*(d)} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \end{bmatrix}.$$

Next, $D_n^{*(d)}$ for $n = 2, 3, 4$ can be written as follows.

$$D_2^{*(d)} = \left[\begin{array}{c|c} \left[\, e_{1,2} \otimes e_{1,2}^T \,\right] & \left[\, e_{2,2} \otimes e_{1,1}^T \,\right] \\ \hline \left[\, e_{2,2} \otimes e_{2,2}^T \,\right] & \left[\, e_{1,2} \otimes e_{1,1}^T \,\right] \end{array}\right],$$

$$D_3^{*(d)} = \left[\begin{array}{c|c|c} \left[\, e_{1,3} \otimes e_{1,3}^T \,\right] & \left[\, e_{2,3} \otimes e_{1,2}^T \,\right] & \left[\, e_{3,3} \otimes e_{1,1}^T \,\right] \\ \hline \left[\, e_{2,3} \otimes e_{2,3}^T \,\right] & \begin{bmatrix} e_{1,2} \otimes e_{1,2}^T \\ e_{1,1} \otimes e_{2,2}^T \end{bmatrix} & \left[\, O_{3\times 1} \,\right] \\ \hline \left[\, e_{3,3} \otimes e_{3,3}^T \,\right] & \left[\, e_{2,3} \otimes e_{2,2}^T \,\right] & \left[\, e_{1,3} \otimes e_{1,1}^T \,\right] \end{array}\right],$$

$$D_4^{*(d)} = \left[\begin{array}{c|c|c|c} \left[\, e_{1,4} \otimes e_{1,4}^T \,\right] & \left[\, e_{2,4} \otimes e_{1,3}^T \,\right] & \left[\, e_{3,4} \otimes e_{1,2}^T \,\right] & \left[\, e_{4,4} \otimes e_{1,1}^T \,\right] \\ \hline \left[\, e_{2,4} \otimes e_{2,4}^T \,\right] & \begin{bmatrix} e_{1,2} \otimes e_{1,3}^T \\ e_{1,2} \otimes e_{2,3}^T \end{bmatrix} & \left[\, e_{4,4} \otimes e_{2,2}^T \,\right] & \left[\, O_{4\times 1} \,\right] \\ \hline \left[\, e_{3,4} \otimes e_{3,4}^T \,\right] & \begin{bmatrix} e_{2,3} \otimes e_{2,3}^T \\ e_{1,1} \otimes e_{3,3}^T \end{bmatrix} & \begin{bmatrix} e_{1,3} \otimes e_{1,2}^T \\ O_{1\times 2} \end{bmatrix} & \left[\, O_{4\times 1} \,\right] \\ \hline \left[\, e_{4,4} \otimes e_{4,4}^T \,\right] & \left[\, e_{3,4} \otimes e_{3,3}^T \,\right] & \left[\, e_{2,4} \otimes e_{2,2}^T \,\right] & \left[\, e_{1,4} \otimes e_{1,1}^T \,\right] \end{array}\right].$$

Thus, we have a general form of $D_n^{*(d)}$:

$$\text{For } i = 1 : \left[\, \boldsymbol{e}_{j,n} \otimes \boldsymbol{e}_{1,n-(j-1)}^T \,\right],$$

$$\text{For } j = 1 : \left[\, \boldsymbol{e}_{i,n} \otimes \boldsymbol{e}_{i,n}^T \,\right],$$

$$\text{For } i < j, i = 2, 3, \cdots, n-1 : \begin{bmatrix} \boldsymbol{e}_{j+1,n} \otimes \boldsymbol{e}_{i,n-(j-1)}^T \\ O_{n\times n-(j-1)} \end{bmatrix},$$

$$\text{For } i \geq j, j = 2, 3, \cdots, n : \begin{bmatrix} \boldsymbol{e}_{i-(j-1),i} \otimes \boldsymbol{e}_{i-(j-1),n-(j-1)}^T \\ \boldsymbol{e}_{j-1,n-i} \otimes \boldsymbol{e}_{i,n-(j-1)}^T \\ O_{(n-i)\times n-(j-1)} \end{bmatrix}.$$

Let $D_n^{*(d)}$ have a Moore-Penrose inverse, which $D_n^{*(d)+}$. Next denotes; we will give several properties of the inverse Moore-Penrose of $D_n^{*(d)}$.

Theorem 3.6. *Let $D_n^{*(d)}$ be the $n^2 \times \dfrac{n(n+1)}{2}$ is analogous to the duplication matrix and $D_n^{*(d)+}$ be its Moore-Penrose inverse. Then:*

(a) $rank(D_n^{*(d)}) = \dfrac{n(n+1)}{2}$,
(b) $D_n^{*(d)+} = \left(D_n^{*(d)T} D_n^{*(d)}\right)^{-1} D_n^{*(d)T}$,
(c) $D_n^{*(d)+} D_n^{*(d)} = I_{\frac{n(n+1)}{2}}$,
(d) $D_n^{*(d)+} vec(A) = vecd^*(A)$ *for every $n \times n$ symmetric matrix A.*

Proof.

(a) Clearly, for every $\dfrac{n(n+1)}{2} \times 1$ vector $\mathbf{x}$ an $n \times n$ symmetric matrix A exist, such that $\mathbf{x} = vecd^*(A)$. However, if for some symmetric A, $D_n^{*(d)} vecd^*(A) = 0$, then from the definition of $D_n^{*(d)}$, $vec(A) = 0$, which then implies that $vecd^*(A) = 0$. Thus, $D_n^{*(d)}\mathbf{x} = 0$, and so $D_n^{*(d)}$ has full column rank.
(b),(c) Based on Theorem 3.6 (a) and Theorem 2.15.
(d) Obtained by pre-multiplying 3.3 and applying 3.6 (c). □

Theorem 3.7. *Let $A = [a_{ij}]$ be an $n \times n$ symmetric matrix, then*

a) $D_n^{*(h)} = D_n^{*(d)} B_n^{*(d)}$ *and* $D_n^{*(d)} = D_n^{*(h)} B_n^{*(d)}$,
b) $B_n^{*(d)} D_n^{*(d)+} = D_n^{*(h)+}$ *and* $B_n^{*(d)T} D_n^{*(h)+} = D_n^{*(d)+}$.

Proof.

a) Substituting Equation 3.2 into 3.3, we have $vec(A) = D_n^{*(d)} B_n^{*(d)} vech^*(A)$. Next, substituting $vec(A) = D_n^{*(h)} vech^*(A)$ into $vec(A) = D_n^{*(d)} B_n^{*(d)}$ $vech^*(A)$, we have $D_n^{*(h)} vech^*(A) = D_n^{*(d)} B_n^{*(d)} vech^*(A)$. So, it is implied that $D_n^{*(h)} = D_n^{*(d)} B_n^{*(d)}$ to the first result of part a). Since $B_n^{*(d)}$ is a non-singular matrix, then $B_n^{*(d)}$ has an inverse. The second result is obtained by post-multiplying both sides of $D_n^{*(h)} = D_n^{*(d)} B_n^{*(d)}$ by $B_n^{*(d)-1}$.
b) Subtituting $D_n^{*(h)} = D_n^{*(d)} B_n^{*(d)}$ into Theorem 3.6 (b), we have $D_n^{*(h)+} = B_n^{*(d)-1} (D_n^{*(d)T} D_n^{*(d)})^{-1} (B_n^{*(d)T})^{-1} B_n^{*(d)T} D_n^{*(d)T}$. Since $B_n^{*(d)}$ is an orthogonal matrix, that is $B_n^{*(d)-1} = B_n^{*(d)T}$, by Theorem 3.6 (b), the first result is obtained $B_n^{*(d)} D_n^{*(d)+} = D_n^{*(h)+}$. Next, substituting $D_n^{*(d)} = D_n^{*(h)} B_n^{*(d)}$ into Theorem 3.6 (b), we have $D_n^{*(d)+} = (B_n^{*(d)T})^T (D_n^{*(h)T} D_n^{*(h)})^{-1} D_n^{*(h)T}$, by Theorem 2.15, we have $B_n^{*(d)T} D_n^{*(h)+} = D_n^{*(d)+}$. □

4. Conclusions

This article provides new operator, i.e., $vech^*$ and $vecd^*$. From these definitions, we have transformation matrix, which transforms $vech^*$ to $vecd^*$ operator for arbitrary matrix $n \times n$ where $n \in N$. The transformed matrices $B_n^{*(d)}$ are permutation and orthogonal matrix, and the matrix has determinant 1 or -1. Then, it provides a matrix analogous to the duplication matrix related to the *vecd** and finds its properties.

5. Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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