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STABILITY ANALYSIS OF DRUG ABUSE TRANSMISSION DYNAMICS

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Abstract. *This research introduced a nonlinear deterministic model known as SLARS (Susceptible, Light Users, Addicted, and Reformed Users) along with a stability analysis to investigate the dynamics of drug abuse transmission. The basic reproduction number (R_0) was calculated using the next-generation matrix method. Following this, the study examined the local stability of both the drug abuse-free and endemic equilibrium points. The findings indicated that when R_0 is below 1, the drug abuse-free equilibrium point is locally stable. In contrast, when R_0 is greater than 1, the endemic equilibrium point shows local stability. Additionally, simulation results supported the analytical conclusions of the study.*

Keywords: Mathematical Modelling, Drug Abuse Transmission, SLARS Model

1. Introduction

Drugs are powerful substances essential for maintaining good health and are commonly used for therapeutic purposes. However, they are often misused, particularly by young people. This misuse, also known as substance abuse, refers to the harmful consumption of drugs in ways or quantities that pose risks to the individual, those around them, or both [1]. Addiction, on the other hand, is a chronic and recurring brain condition characterized by an irresistible compulsion to seek and use drugs, despite experiencing negative consequences. Similar to other illnesses like heart disease, addiction disrupts the typical, healthy operation of the affected organ, leading to severe adverse outcomes. Both conditions are preventable and treatable, but without intervention, they can endure throughout a person's lifetime [2].

The abuse of drugs is almost widespread throughout the entire region of Indonesia. Based on the survey results in 2019, the prevalence rate of drug abuse at

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the national level in the past year was 1.80% of the entire population of Indonesia aged 15 to 64 years. The research findings in 2018 also indicate that the prevalence trend of drug abuse among students and university students in Indonesia is quite high at 3.2%. Similarly, the prevalence rate in the working sector is 2.1% [3]. The prevalence and ratio of drug abuse in Indonesia are currently lower than the global average. In 2018, about 269 million people worldwide, out of 4.98 billion aged 15 to 64, had used drugs at least once in the previous year. However, there is a potential for a significant increase in the future without effective prevention efforts. If such an increase occurs, it could have worrisome impacts on the physical, mental, and social health of the next generation. Indonesia has shifted from being a 'transit country' to a 'destination country' in the global illegal drug trade [4].

The effects of drug abuse vary based on factors such as the drug's mechanism of action, the quantity consumed, and the user's history, among others. Nonetheless, studies have shown that, on a global scale, drug abuse negatively impacts the health, economic stability, and overall well-being of nations [5]. As outlined by Machhi et al. [6], in general, people begin taking drugs for a variety of reasons. That is: 1) The recurrent mentions of drugs in public media can spark curiosity, leading individuals to seek personal experiences with drugs; 2) Peer pressure from friends engaged in drug abuse, coupled with the constant escalation of drug experiences, may entice others to initiate drug use; 3) Individuals facing frustration and depression may turn to drugs as a means of seeking relief from their emotional challenges; 4) The need for extended periods of wakefulness, as seen in students preparing for exams, may drive occasional drug use to stay awake throughout the night; 5) A desire for a different or altered reality can tempt young people to start using drugs, under the false impression that substances can open up new perspectives; 6) Prolonged use of prescription pain-relieving drugs, initially prescribed by a doctor, may lead to addiction as individuals seek continued relief from pain; 7) Family history, including exposure to drug use among elders, may contribute to a higher likelihood of children in the family engaging in drug consumption; 8) The pursuit of excitement and adventure can motivate young individuals to experiment with drugs to satisfy their instinct for thrill and excitement.

White and Comiskey [7] developed a model to describe heroin use within a fixed community, categorizing individuals into three groups: susceptible individuals, drug users, and drug users undergoing treatment. Their study revealed a stable balance and pointed out two possible situations: one in which heroin use is eliminated and another in which it continues to be a persistent problem. Mushayabasa and Tapedzesa [8] developed an innovative mathematical modeling framework designed to explore the effects of illegal drug use on a community.The model illustrated how transmission occurs through a social "contact" mechanism between individuals who are susceptible and those who are involved in illegal drug use. The research included analyses of both epidemic and endemic situations, concentrating on threshold dynamics determined by the basic reproduction number. It provided example numerical findings, featuring a case study of communities in Cape Town, Gauteng, Mpumalanga, and Durban in South Africa. Furthermore, the model was enhanced to include intervention strategies that vary over time.

This study utilizes the model created by Fantaye and Birhanu [9] to analyze drug abuse cases in Indonesia. under the assumption that the total population was divided into susceptible *(S(t))*, light drug user *(L(t))*, drug addicts *(A(t))*, and reformed drug users *(R(t))* classes. Notably, the model considered the possibility of light drug users and drug addicts transitioning to the reformed class. This transition was attributed to interventions targeting risk groups, such as boosting self-confidence and equipping young drug users with the skills to resist peer pressure.

2. Model Formulation

The overall population is categorized into four groups: those who are susceptible ($S(t)$), individuals who use drugs lightly ($L(t)$), drug addicts ($A(t)$), and those who have reformed from drug use ($R(t)$). We assume that:

(1) Susceptible population increased by the present recruitment rate.
(2) The increase in drug use can be primarily linked to the effective contact or interactions between vulnerable individuals and those who are already addicted.
(3) There is a natural death rate for all sub-populations.
(4) There is a migration from susceptible to light drug users because of the effective contact rate between susceptible and addicted sub-populations.
(5) Individuals from the light drug users sub-population can become drug addicts or reformed drug users
(6) There is a drug-induced disease death rate for drug addicts sub-populations.
(7) Individuals from the drug addicts sub-population can become reformed drug users.
(8) Reformed drug users can become susceptible.

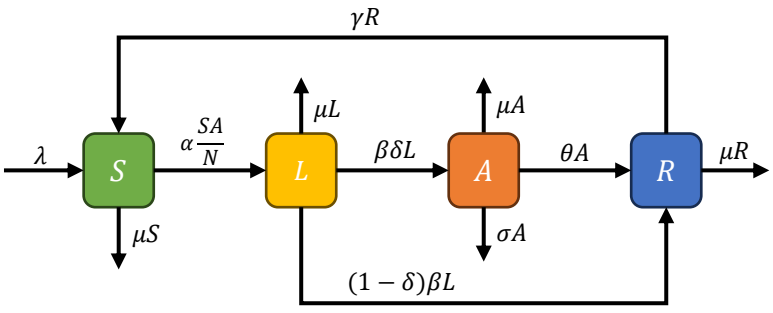


Figure. 1: Drug abuse $SLARS$ model

With these assumptions, the $SLARS$ model of drug abuse is shown below.

$$\begin{aligned}
\frac{dS}{dt} &= \lambda - \alpha\frac{SA}{N} + \gamma R - \mu S,\\
\frac{dL}{dt} &= \alpha\frac{SA}{N} - (\beta+\mu)L,\\
\frac{dA}{dt} &= \beta\delta L - (\sigma+\theta+\mu)A,\\
\frac{dR}{dt} &= \theta A + (1-\delta)\beta L - (\gamma+\mu)R,
\end{aligned} \tag{2.1}$$

with:

$$S(0) > 0, E(0) \geq 0, I(0) \geq 0, R(0) \geq 0.$$

Model parameters and their descriptions are given in Table 1.

Table 1: Model Parameters.

Parameter	**Description**
λ	Recruitment Rate
α	Rate of effective contact between Susceptible and Addicted
β	Rate at which light drug user ($E(t)$) become *Addicted* ($I(t)$) or *Quitters* ($R(t)$)
δ	The proportion of individuals that move from light drug users sub-population to the addicted sub-population.
θ	Recovery rate
γ	Rate when Reformed become Susceptible
σ	Drug-induced decease death rate
μ	Natural death rate

All feasible solutions to the system in Equation (2.1) are:

$$\Omega = \{(S,L,A,R) \in R_+^4; 0 \leq S(t)+L(t)+A(t)+R(t) \leq \frac{\lambda}{\mu}\}.$$

This can be demonstrated by deriving the total population over time in the model Equation (2.1). The total population is:

$$\begin{aligned}
N(t) &= S(t)+L(t)+A(t)+R(t),\\
\frac{dN}{dt} &= \frac{dS}{dt}+\frac{dL}{dt}+\frac{dA}{dt}+\frac{dR}{dt},\\
&= \lambda - \alpha\frac{SA}{N} + \gamma R - \mu S + \alpha\frac{SA}{N} - (\beta+\mu)L + \beta\delta L - (\sigma+\theta+\mu)A\\
&+ \ \theta A + (1-\delta)\beta L - (\gamma+\mu)R,\\
&= \lambda - \mu N - \sigma A.
\end{aligned} \tag{2.2}$$

Equation (2.2) is a first-order ordinary differential equation, which can be addressed using the integrating factor method.

$$\frac{dN}{dt} = \lambda - \mu N(t) - \sigma A(t). \tag{2.3}$$

This implies that:

$$\frac{dN}{dt} \leq \lambda - \mu N(t). \tag{2.4}$$

By multiplying each term of Equation (2.4) by the integrating factor $e^{\mu t}$, then we have:

$$N(t) \leq \frac{\lambda}{\mu} + Ce^{-\mu t}. \tag{2.5}$$

By using N_0 as the initial value of the total overall population, we obtain:

$$C = N_0 - \frac{\lambda}{\mu}. \tag{2.6}$$

Substituting the value of C from Equation (2.6) into inequality (2.5), we obtain:

$$N(t) \leq \frac{\lambda}{\mu} + N_0 e^{-\mu t}. \tag{2.7}$$

From equation (2.7), we get:

$$\lim_{t\to\infty} \sup N(t) \leq \frac{\lambda}{\mu}. \tag{2.8}$$

Consequently, Ω is positively invariant. It means that all solutions of the system (2.1) with positive initial values will remain as solutions of the system for $t \geq 0$.

3. Equilibrium Point and Basic Reproduction Number ($\Re_0$)

There are two categories of equilibrium points: the drug abuse-free equilibrium point and the drug abuse endemic equilibrium point. To identify these equilibrium points, each equation in the system (2.1) is set to zero. The equilibrium points can be represented as follows:

$$\lambda - \alpha\frac{SI}{N} + \gamma R - \mu S = 0, \tag{3.1}$$

$$\alpha\frac{SI}{N} - (\beta + \mu)E = 0, \tag{3.2}$$

$$\beta\delta E - (\sigma + \theta + \mu)I = 0, \tag{3.3}$$

$$\theta I + (1 - \delta)\beta E - (\gamma + \mu)R = 0. \tag{3.4}$$

3.1. *Drug Abuse-Free Equilibrium Point*

To determine the equilibrium point for drug-free users, assume that $S = S^0$, $L = L^0$, $A = A^0$, and $R = R^0$. The drug abuse-free equilibrium point is a stable state that signifies a condition in which there is no drug abuse within the population, resulting in no individuals occasionally using drugs ($L^0 = 0$), no individuals addicted to using drugs ($A^0 = 0$), and no individuals quitting using drug ($R^0 = 0$). We get the drug abuse-free equilibrium point

$$E_0 = (S^0, E^0, I^0, R^0) = (\frac{\lambda}{\mu}, 0, 0, 0). \tag{3.5}$$

3.2. *Drug Abuse Endemic Equilibrium Point*

The endemic equilibrium point for drug abuse occurs when drug users persist within the community. Let $S = S^*$, $E = E^*$, $I = I^*$, and $R = R^*$, the system (2.1) can be restated as follows:

$$\lambda - \alpha\frac{S^*A^*}{N} + \gamma R^* - \mu S^* = 0, \tag{3.6}$$

$$\alpha\frac{S^*A^*}{N} - (\beta + \mu)L^* = 0, \tag{3.7}$$

$$\beta\delta L^* - (\sigma + \theta + \mu)A^* = 0, \tag{3.8}$$

$$\theta A^* + (1 - \delta)\beta L^* - (\gamma + \mu)R^* = 0. \tag{3.9}$$

Using equation (3.7) dan (3.8) we obtain:

$$S^* = \frac{\lambda(\sigma + \theta + \mu)(\beta + \mu)}{\mu\alpha\beta\delta}. \tag{3.10}$$

By adding (3.6) with (3.7), we get:

$$\lambda + \gamma R^* - \mu S^* - (\beta + \mu)L^* = 0. \tag{3.11}$$

Then substitute Equation (3.10) into Equation (3.11):

$$R^* = \frac{\lambda}{\gamma}\left(\frac{(\sigma + \theta + \mu)(\beta + \mu)}{\alpha\beta\delta} - 1\right) + \frac{(\beta + \mu)L^*}{\gamma}. \tag{3.12}$$

From Equation (3.8) we have:

$$A^* = \frac{\beta\delta L^*}{(\sigma + \theta + \mu)}. \tag{3.13}$$

Substitute Equation (3.13) into Equation (3.9), we get:

$$R^* = \frac{\theta\beta\delta L^*}{(\gamma + \mu)(\sigma + \theta + \mu)} + \frac{(1 - \delta)\beta L^*}{(\gamma + \mu)}. \tag{3.14}$$

Next, by equating Equation (3.12) and Equation (3.14), we obtain:

$$L^* = \frac{(\sigma + \theta + \mu)(\gamma + \mu)\lambda\left(1 - \dfrac{(\sigma + \theta + \mu)(\beta + \mu)}{\alpha\beta\delta}\right)}{\mu(\beta + \gamma + \mu)(\sigma + \theta + \mu) + \beta\delta\gamma(\sigma + \mu)}. \tag{3.15}$$

By substituting Equation (3.15) into Equation (3.13), we get:

$$A^* = \frac{\beta\delta(\gamma + \mu)\lambda\left(1 - \dfrac{(\sigma + \theta + \mu)(\beta + \mu)}{\alpha\beta\delta}\right)}{\mu(\beta + \gamma + \mu)(\sigma + \theta + \mu) + \beta\delta\gamma(\sigma + \mu)}. \tag{3.16}$$

Next, substitute Equation (3.15) into Equation (3.14):

$$R^* = \frac{(\beta\delta\theta + (1 - \delta)\beta)\lambda\left(1 - \dfrac{(\sigma + \theta + \mu)(\beta + \mu)}{\alpha\beta\delta}\right)}{\beta\delta\gamma(\sigma + \mu) + \mu(\beta + \gamma + \mu)(\sigma + \theta + \mu)}. \tag{3.17}$$

Therefore, the endemic equilibrium point for drug abuse is given by:

$$
\begin{aligned}
E_* &= (S^*, L^*, A^*, R^*) \\
&= \left(\frac{\lambda(\sigma+\theta+\mu)(\beta+\mu)}{\mu\alpha\beta\delta}, \frac{(\sigma+\theta+\mu)(\gamma+\mu)\lambda\left(1-\dfrac{(\sigma+\theta+\mu)(\beta+\mu)}{\alpha\beta\delta}\right)}{\mu(\beta+\gamma+\mu)(\sigma+\theta+\mu)+\beta\delta\gamma(\sigma+\mu)}, \right. \\
&\quad \frac{\beta\delta(\gamma+\mu)\lambda\left(1-\dfrac{(\sigma+\theta+\mu)(\beta+\mu)}{\alpha\beta\delta}\right)}{\mu(\beta+\gamma+\mu)(\sigma+\theta+\mu)+\beta\delta\gamma(\sigma+\mu)}, \\
&\quad \left. \frac{(\beta\delta\theta+(1-\delta)\beta(\sigma+\theta+\mu))\lambda\left(1-\dfrac{(\sigma+\theta+\mu)(\beta+\mu)}{\alpha\beta\delta}\right)}{\beta\delta\gamma(\sigma+\mu)+\mu(\beta+\gamma+\mu)(\sigma+\theta+\mu)} \right).
\end{aligned} \tag{3.18}
$$

3.3. *Basic Reproduction Number ($\mathfrak{R}_0$)*

The Basic Reproduction Number ($\mathfrak{R}_0$) measures the estimated secondary drug abusers number generated by one addicted individual entering the susceptible subpopulation. The $\mathfrak{R}_0$ for the system (2.1) can be determined by applying the next-generation matrix method. Let F^* be the matrix representing the rate of new drug abusers and V^* be the matrix representing the movement of individuals between subpopulations, then we obtain a new subsystem for infected and transferred individuals, designated as $E(t)$ and $I(t)$, respectively.

$$
\begin{aligned}
\frac{dL}{dt} &= \alpha\frac{SA}{N} - (\beta+\mu)L, \\
\frac{dA}{dt} &= \beta\delta L - (\sigma+\theta+\mu)A,
\end{aligned} \tag{3.19}
$$

that is,

$$
F^* = \begin{bmatrix} \alpha\frac{SA}{N} \\ 0 \end{bmatrix}, \text{ and } V^* = \begin{bmatrix} (\beta+\mu)L \\ -\beta\delta L + (\sigma+\theta+\mu)A \end{bmatrix}.
$$

Next, F and V are calculated based on partial derivatives of F^* and V^* with respect to variables L and A. Then, substitute the values of the drug abuse-free equilibrium (3.1) into F and V to obtain:

$$
F = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \text{ and } V = \begin{bmatrix} (\beta+\mu) & 0 \\ -\beta\delta & (\sigma+\theta+\mu) \end{bmatrix}. \tag{3.20}
$$

Then, FV^{-1} becomes:

$$
FV^{-1} = \begin{bmatrix} \dfrac{\alpha\beta\delta}{(\beta+\mu)(\sigma+\theta+\mu)} & \dfrac{\alpha}{\sigma+\theta+\mu} \\ 0 & 0 \end{bmatrix}. \tag{3.21}
$$

The largest eigenvalue can be determined by solving the characteristic equation $det(FV^{-1}-\lambda I)=0$. We obtained the eigen values for matrix FV^{-1} are $\lambda = 0$ and

$\lambda = \alpha\beta\delta/(\beta+\mu)(\sigma+\theta+\mu)$. Since $\Re_0$ is the largest eigenvalue of the matrix FV^{-1}, then the basic reproduction number $\Re_0$ is given by:

$$\Re_0 = \frac{\alpha\beta\delta}{(\beta+\mu)(\sigma+\theta+\mu)}. \tag{3.22}$$

4. Local Stability Analysis of Equilibrium Point

The stability of the system's equilibrium points in model (2.1) is assessed through linearization with the Jacobian matrix.

$$J_{S,L,A,R} = \begin{bmatrix} -\alpha\frac{A}{N} - \mu & 0 & -\alpha\frac{S}{N} & \gamma \\ \alpha\frac{A}{N} & -(\beta+\mu) & \alpha\frac{S}{N} & 0 \\ 0 & \beta\delta & -(\sigma+\theta+\mu) & 0 \\ 0 & (1-\delta)\beta & \theta & -(\gamma+\mu) \end{bmatrix}. \tag{4.1}$$

The stability of both the drug abuse-free equilibrium point and the endemic equilibrium point will be assessed through the following analysis.

4.1. *Local Stability Analysis of Drug Abuse-Free Equilibrium Point*

Theorem 4.1. *The equilibrium solution E_0 of the system of Equation (5.2) is asymptotically stable if $\Re_0 < 1$.*

According to Equation (4.1), the Jacobian matrix at the equilibrium point where there is no drug abuse is expressed as:

$$J(E_0) = \begin{bmatrix} -\mu & 0 & -\alpha & \gamma \\ 0 & -(\beta+\mu) & \alpha & 0 \\ 0 & \beta\delta & -(\sigma+\theta+\mu) & 0 \\ 0 & (1-\delta)\beta & \theta & -(\gamma+\mu) \end{bmatrix}. \tag{4.2}$$

The characteristic equation of the matrix $J(E_0)$ is derived using the identity matrix I and is given by:

$$\left| \begin{bmatrix} -\mu-\lambda & 0 & -\alpha & \gamma \\ 0 & -(\beta+\mu+\lambda) & \alpha & 0 \\ 0 & \beta\delta & -(\sigma+\theta+\mu+\lambda) & 0 \\ 0 & (1-\delta)\beta & \theta & -(\gamma+\mu+\lambda) \end{bmatrix} \right| = 0$$

$$(\mu+\gamma+\lambda)(\mu+\lambda)(\lambda^2 + (2\mu+\beta+\sigma+\theta)\lambda + (\beta+\mu)(\sigma+\theta+\mu) - \alpha\beta\delta) = 0. \tag{4.3}$$

From Equation (4.3), we obtain:

$$\lambda_1 = -(\gamma+\mu),$$
$$\lambda_2 = -\mu.$$

Here, $\lambda_1 < 0$ and $\lambda_2 < 0$. By applying the Routh-Hurwitz criterion, the third part of Equation (4.3) will have negative real parts when:

$$2\mu+\beta+\sigma+\theta > 0, \text{ and } \alpha\beta\delta\left(\frac{1}{R_0} - 1\right) > 0 \text{ if } R_0 < 1. \tag{4.4}$$

Therefore, with $\Re_0 < 1$, all eigenvalue of (4.2) has real negative part, and consequently, the drug abuse-free equilibrium point E_0 is asymptotically stable.

4.2. *Local Stability Analysis of Drug Abuse Endemic Equilibrium Point*

The stability of the endemic equilibrium point is determined based on the value of its basic reproduction number, $\Re_0$, as stated in the following theorem.

Theorem 4.2. *The equilibrium solution E_* of the system of Equation (3.18) is asymptotically stable if $\Re_0 > 1$.*

From Equation (4.1), the Jacobian matrix at the drug abuse endemic equilibrium point is expressed as:

$$J(E_*) = \begin{bmatrix} -\alpha\frac{I^*}{N} - \mu & 0 & -\alpha\frac{S^*}{N} & \gamma \\ \alpha\frac{I^*}{N} & -(\beta+\mu) & \alpha\frac{S^*}{N} & 0 \\ 0 & \beta\delta & -(\sigma+\theta+\mu) & 0 \\ 0 & (1-\delta)\beta & \theta & -(\gamma+\mu) \end{bmatrix}. \tag{4.5}$$

Then, the characteristic equation of the matrix $J(E_*)$ is obtained using the identity matrix I and is given by:

$$\begin{aligned} 0 &= \det(J(E_*) - I\lambda), \\ &= \left| \begin{bmatrix} -\alpha\frac{I^*}{N} - \mu - \lambda & 0 & -\alpha\frac{S^*}{N} & \gamma \\ \alpha\frac{I^*}{N} & -(\beta+\mu) - \lambda & \alpha\frac{S^*}{N} & 0 \\ 0 & \beta\delta & -(\sigma+\theta+\mu) - \lambda & 0 \\ 0 & (1-\delta)\beta & \theta & -(\gamma+\mu) - \lambda \end{bmatrix} \right|. \end{aligned} \tag{4.6}$$

From Equation (4.6), we get:

$$P(\lambda) = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \tag{4.7}$$

where:

$$\begin{aligned} a_4 &= 1, \\ a_3 &= 4\mu + \sigma + \theta + \beta + \gamma + \frac{\mu\alpha\beta\delta(\gamma+\mu)(R_0-1)}{[\beta\delta\gamma(\sigma+\mu) + \mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0} > 0, \\ a_2 &= (3\mu + \sigma + \theta + \beta + \gamma)\mu + (2\mu + \sigma + \theta + \beta)(\gamma+\mu) \\ &\quad + \frac{\mu\alpha\beta\delta(3\mu + \sigma + \theta + \beta + \gamma)(\gamma+\mu)(R_0-1)}{[\beta\delta\gamma(\sigma+\mu) + \mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0} > 0, \\ a_1 &= (2\mu + \sigma + \theta + \beta)(\gamma+\mu) + \frac{\mu\alpha\beta\delta(2\mu + \sigma + \theta + \beta)(\gamma+\mu)^2(R_0-1)}{[\beta\delta\gamma(\sigma+\mu) + \mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0} \\ &\quad + \frac{\mu\alpha\beta^2\delta(\gamma+\mu)(\delta\alpha + \gamma(1-\delta))(R_0-1)}{[\beta\delta\gamma(\sigma+\mu) + \mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0} > 0, \\ a_0 &= \frac{\mu\alpha\beta^2\delta[\alpha\delta(\gamma+\mu) + \delta\gamma(\sigma+\mu) + \gamma(\sigma+\theta+\mu)](\gamma+\mu)(R_0-1)}{[\beta\delta\gamma(\sigma+\mu) + \mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0} > 0, \end{aligned}$$

$$
\begin{aligned}
a_3a_2 - a_1 &= (3\mu+\sigma+\theta+\beta+\gamma)((4\mu+\sigma+\theta+\beta+\gamma)\mu+(\gamma+\mu)(2\mu+\sigma+\theta+\beta)) \\
&+ \frac{\mu\alpha\beta\delta(\gamma+\mu)[(3\mu+\sigma+\theta+\beta)(4\mu+\sigma+\theta+\beta+\gamma)(\delta\alpha+\gamma(1-\delta)\beta](R_0-1)}{[\beta\delta\gamma(\sigma+\mu)+\mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0} \\
&+ \frac{(3\mu+\sigma+\theta+\beta+\gamma)(\mu\alpha\beta\delta(\gamma+\mu)(R_0-1))^2}{([\beta\delta\gamma(\sigma+\mu)+\mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0)^2} > 0, \\
a_1a_2a_3 - (a_0a_3^2 + a_1^2) &= (4\mu+\sigma+\theta+\beta+\gamma)((2\mu+\sigma+\theta+\beta)(\gamma+\mu)\mu) \\
&+ (3\mu+\sigma+\theta+\beta+\gamma)\mu + (2\mu+\sigma+\theta+\beta)(\gamma+\mu) \\
&+ \frac{[(3\mu+\sigma+\theta+\beta)\mu+(2\mu+\sigma+\theta+\beta)(\gamma+\mu)]\mu\alpha\beta\delta(\gamma+\mu)(R_0-1)}{[\beta\delta\gamma(\sigma+\mu)+\mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0} \\
&+ \frac{(\mu\alpha\beta\delta)^2(2\mu+\sigma+\theta+\beta)(3\mu+\sigma+\theta+\beta+\gamma)(\gamma+\mu)^3(R_0-1)^2}{([\beta\delta\gamma(\sigma+\mu)+\mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0)^2} \\
&+ \frac{((\mu\alpha\beta\delta)(R_0-1))^2[\alpha\sigma(\sigma+\mu)+\gamma(\sigma+\theta+\mu)+\sigma\gamma(\gamma+\mu)]}{([\beta\delta\gamma(\sigma+\mu)+\mu(\sigma+\theta+\mu)(\beta+\gamma+\mu)]R_0)^2} > 0.
\end{aligned}
$$

The Routh-Hurwitz criterion indicates that all roots of the polynomial equation (4.6) will have negative real parts if the following conditions are met for $R_0 > 1$: $a_4 > 0$, $a_3 > 0$, $a_2 > 0$, $a_1 > 0$, $a_0 > 0$, $a_3a_2 - a_1 > 0$, and $a_1a_2a_3 - (a_0a_3^2 + a_1^2) > 0$. Therefore, the endemic equilibrium point E_* related to drug abuse is asymptotically stable.

5. Numerical Simulation

The model simulation is conducted using MAPLE, utilizing the initial variable values from Table 2 and the parameter values from Table 3.

Table 2: Variable Values.

Variable	**Estimated Value**	**Source**
$S(0)$	0.93011391	[10]
$L(0)$	0.0257304	[10]
$A(0)$	0.01953271	[10]
$R(0)$	0.02459826	Estimated

Table 3: Parameter Values.

Parameter	**Estimated Value**	**Source**
Λ	0.00299	Assumed
α_1	0.01	Assumed
α_2	0.06	Assumed
β	0.08	Assumed
δ	0.3	Assumed
θ	0.0056	Assumed
γ	0.0012	Assumed

Parameter	**Estimated Value**	**Source**
σ	0.0049	[10]
μ	0.003	[11]

By substituting the parameter values from Table 3 into System (2.1) and Equation (3.22), the equilibrium points and the basic reproduction number of the SLARS model are obtained as follows.

$$\begin{aligned}
\frac{dS}{dt} &= 0.00299 - 1.003344482\ \alpha\ SA + 0.0012\ R - 0.003S,\\
\frac{dL}{dt} &= 1.003344482\ \alpha\ SA - 0.083\ L,\\
\frac{dA}{dt} &= 0.024\ L - 0.0135\ A,\\
\frac{dR}{dt} &= 0.0056\ A + 0.056\ L - 0.0042\ R,
\end{aligned} \tag{5.1}$$

and using Equation (3.22), the basic reproduction number is calculated and presented in Table 4.

Table 4: Simulation Result of $\Re_0$.

Simulation	**Effective Contact Rate Parameter Value**	$\Re_0$
1	$\alpha_1 = 0.01$	$\Re_0 = 0.214190094 < 1$
2	$\alpha_2 = 0.06$	$\Re_0 = 1.285140562 > 1$

Using Equation (5.2) for Simulation 1, We derive the drug abuse-free equilibrium point as follows.

$$E_0 = (S^0, E^0, I^0, R^0) = (0.9966666667, 0, 0, 0), \tag{5.2}$$

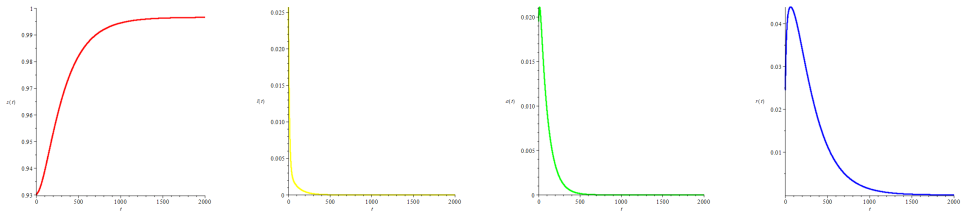
and for Equation (4.3), we get:

$$\begin{aligned}
\lambda_1 = -0.0042 &< 0,\\
\lambda_2 = -0.003 &< 0,\\
2\mu + \beta + \sigma + \theta = 0.0965 &> 0,\\
(\beta + \mu)(\sigma + \theta + \mu) - \alpha\beta\delta = 0.0008805 &> 0.
\end{aligned}$$

Here, $\lambda_1 < 0$, $\lambda_2 < 0$, and because of $2\mu+\beta+\sigma+\theta > 0$ and $(\beta+\mu)(\sigma+\theta+\mu)-\alpha\beta\delta > 0$. Using the Routh-Hurwitz criterion, we can determine that the third equation from Equation (4.3) will exhibit negative real parts. Given that when $\Re_0 < 1$ all eigenvalues also have negative real parts, we conclude that the drug abuse-free equilibrium point is asymptotically stable.

The numerical simulation results for the drug abuse model are shown in Figure 2.

In Figure 2, we observe that the populations of susceptible and reformed drug users are steadily increasing towards a drug abuse-free equilibrium, while the numbers of light drug users and drug addicts are gradually decreasing at the same rate



(a) *Susceptible* (b) *Light User* (c) *Addicted* (d) *Reformed*

Figure. 2: The Solution Graph of *SLARS* Model for Drug Abuse-Free

towards this equilibrium point. This phenomenon occurs because $\Re_0 = 0.214190094$, which is less than one. This supports the theorem that indicates the stability of the drug abuse-free equilibrium when $\Re_0 < 1$. To put it more simply, if $\Re_0$ is under 1, it suggests that each addicted person, on average, leads to fewer than one new drug abuser during their time of substance use.

Using Equation (3.18) for Simulation 2, we have endemic equilibrium point:

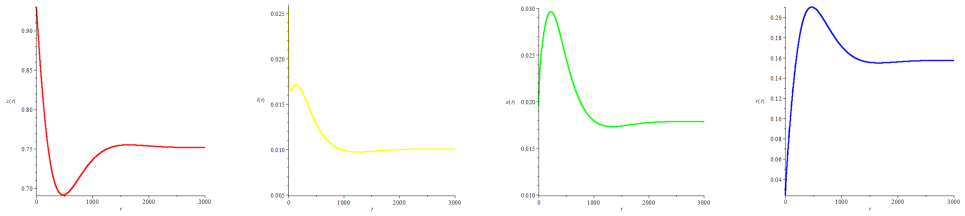
$$\begin{aligned} E_* &= (S^*, L^*, A^*, R^*) \\ &= (0.7521875000, 0.01002933408, 0.01782992726, 0.1574976908), \end{aligned}$$

and from Equation (4.7) we obtain:

$$\begin{aligned} a_4 &= 1, \\ a_3 &= 0.1008249825 > 0, \\ a_2 &= 0.0015350242 > 0, \\ a_1 &= 0.0000487428 > 0, \\ a_0 &= 0.00000000455 > 0, \\ a_3a_2 - a_1 &= 0.0001498945110 > 0, \\ a_1a_2a_3 - (a_0a_3^2 + a_1^2) &= 0.00000000726 > 0. \end{aligned} \tag{5.3}$$

Since $a_4 > 0, a_3 > 0, a_2 > 0, a_1 > 0, a_0 > 0, a_3a_2 - a_1 > 0, a_1a_2a_3 - (a_0a_3^2 + a_1^2) > 0$, based on Routh-Hurwitz criteria, drug abuse equilibrium point (E_*) is asymptotically stable for $\Re_0 > 1$. The outcomes of the numerical simulation for the model of drug abuse are depicted in Figure 3.

According to Figure 3, the populations of susceptible individuals, light drug users, drug addicts, and reformed drug users are stabilizing towards the endemic equilibrium point of drug abuse. This situation arises because $\Re_0 = 1.285140562$, which exceeds one. This aligns with the theorem that states that the drug abuse endemic equilibrium point is stable when $\Re_0$ is greater than 1. In other words, when $\Re_0$ is above 1, it indicates that each light drug user and drug addict, on average, leads to more than one new person becoming a drug abuser. As a result, the presence of drug abusers can potentially amplify the spread of drug use within the community.



(a) *Susceptible* (b) *Light User* (c) *Addicted* (d) *Reformed*

Figure. 3: The Solution Graph of *SLARS* Model for Endemic Drug Abuse

6. Conclusion

This study examines a deterministic mathematical model to investigate the dynamics of drug abuse transmission. We established the model's epidemiological and mathematical validity within a specified domain. We calculated essential parameters, including the basic reproduction number ($\Re_0$), the drug abuse-free equilibrium point (E_0), and the endemic equilibrium point (E_*). Our findings indicated that when $\Re_0 < 1$, the drug abuse-free equilibrium point (E_0) is locally asymptotically stable. In contrast, when $\Re_0 > 1$, there is an endemic equilibrium point (E_*) that is also locally asymptotically stable.

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