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NUMERICAL SOLUTION OF EUROPEAN PUT OPTION FOR BLACK-SCHOLES MODEL USING KELLER BOX SCHEME

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Abstract. *In this study, we propose numerically determining option pricing using the Black-Scholes model. The Keller box method, a numerical method with a box-shaped implicit scheme, is chosen to solve the problem of pricing stock options, especially European-type put options. This option pricing involves several parameters, such as stock price volatility, risk-free interest rate, and strike price. The numerical stability of the method is checked using Von Neumann stability before the simulation is conducted. The influence of interest rates, volatility, and strike price on the option price states that the higher the value of the interest rate parameter, the lower the option price value, while the greater the value of stock price volatility and strike price, the higher the option price.*

Kata Kunci: Black-Scholes model, European put option, Keller-box method, option pricing, Von Neumann stability

1. Introduction

An option is a financial contract that gives the option holder the right to buy or sell an underlying asset at a specified price at maturity. The problem with options is how to determine the value or price of the option. Besides the use of options is growing rapidly, it can also be used as a hedging tool in investing. The Black-Scholes-Merton model commonly used in option pricing, especially European-type options [1,2]. The model is known for its simplicity in providing an estimation of the option price. In partial differential equation form, the Black-Scholes model can be written as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.1}$$

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where V represents the option value, S represents the underlying asset, t represents time, T represents the maturity date of option, r represents a constant of risk free interest rate, and σ represents a volatility constant.

Analytical solution of Black-Scholes model has been achieved by solving partial differential equation [1,2]. The solution of the model is also can be determined by series solution such as Adomian decomposition method [3,4] and homotopy perturbation method [5,6], transformation method [7], numerical method using explicit and implicit scheme [8] or Crank-Nicolson [9].

Numerical solution implicit scheme is well known because of its stability solution. The Keller-box scheme is a famous scheme to solve differential equation in physics [10,11,12]. However, this method is rare to be applied in finance problem especially solving Black-Scholes differential equation [13]. Nevertheless, it delivers to high error on strike price like result of its payoff.

In this study, we propose to solve European put option using Black-Scholes differential equation using Keller-box method [12,14,15]. Numerical stability analysis of the model previously is tested before the simulation. The result of solution using Keller-box scheme is compared with the result of solution using Crank-Nicolson scheme. Further, the effect of some parameters such as interest rate, volatility and strike price are also studied.

2. Methodology

The Keller-box method is a numerical approach with an implicit scheme that was introduced by Keller [14]. This method provides several features to solve parabolic partial differential equation such as Black-Scholes model. Some properties of the Keller-box method that are expected in option pricing are efficient, unconditionally stable and second-order accuracy [15].

The general steps to solve the problem using the Keller-box method are:

(1) Reduce second-order equations to be first-order equations.
(2) Discretize the first-order equation using center difference based on Keller-box scheme as in Figure 1.
(3) Linearize the discretized equation and write it in the form of a matrix equation.
(4) Solve the system of linear equations using block elimination technique.

Before conversion ODEs, Black-Scholes differential equation is converted under transformation $\tau = T - t$ so that Equation 1.1 is written as:

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0, \tag{2.1}$$

with payoff $V(S,0) = \max(K - S, 0)$ and boundary conditions:

$$V(0,\tau) = Ke^{-r\tau},$$
$$V(S,\tau) = 0 \quad \text{as} \quad S \to \infty.$$

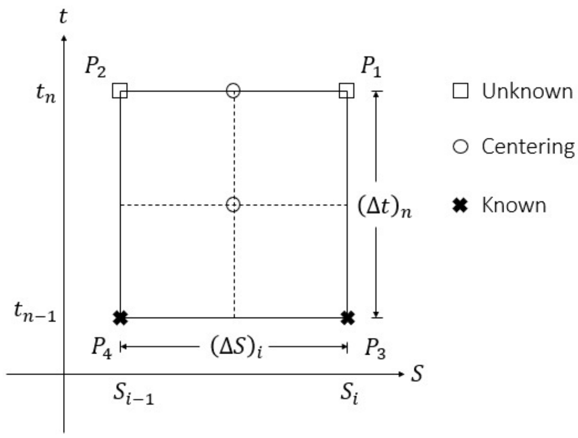

Figure. 1. Box scheme

2.1. *Conversion ODEs*

The first step of this method requires that higher-order systems are converted to first-order systems, which yields:

$$W = \frac{\partial V}{\partial S}, \tag{2.2}$$

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial W}{\partial S} - \frac{1}{4} r S^2 W + rV = 0. \tag{2.3}$$

2.2. *Discretization*

In order to create discretization, Equation 2.2 uses the centre of a line segment P_1P_2, while Equation 2.3 uses the centre of a box $P_1P_2P_3P_4$ as shown in Figure 1. The systems of ODEs in Equation 2.2 – Equation 2.3 are converted to be:

$$W^n_{i-\frac{1}{2}} = \frac{V^n_i - V^n_{i-1}}{\Delta S} = \frac{W^n_i + W^n_{i-1}}{2} \tag{2.4}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\left(V^n_i - V^n_{i-1}\right)}{\Delta \tau} - \frac{1}{4}\sigma^2 S^2_{i-\frac{1}{2}} \frac{\left(W^n_i - W^n_{i-1}\right)}{\Delta S} - \frac{1}{4} r S_{i-\frac{1}{2}}\left(W^n_i + W^n_{i-1}\right) + \frac{1}{4} r \left(V^n_i - V^n_{i-1}\right) \\
=&\frac{1}{2}\frac{\left(V^{n-1}_i - V^{n-1}_{i-1}\right)}{\Delta \tau} + \frac{1}{4}\sigma^2 S^2_{i-\frac{1}{2}} \frac{\left(W^{n-1}_i - W^{n-1}_{i-1}\right)}{\Delta S} + \frac{1}{4} r S_{i-\frac{1}{2}}\left(W^{n-1}_i + W^{n-1}_{i-1}\right) \\
&- \frac{1}{4} r \left(V^{n-1}_i - V^{n-1}_{i-1}\right).
\end{aligned} \tag{2.5}$$

2.3. *Numerical Stability*

In order to analyze the stability of Keller-box method, we apply Von Neumann stability to Equation 2.4 and Equation 2.5. Assume that:

$$V^n_i = \tilde{V}^n e^{I\theta i}, \qquad V^{n-1}_i = \tilde{V}^{n-1} e^{I\theta i},$$
$$W^n_i = \tilde{W}^n e^{I\theta i}, \qquad W^{n-1}_i = \tilde{W}^{n-1} e^{I\theta i},$$

then Equation 2.4 and Equation 2.5 can be written as:

$$\tilde{A}\tilde{\mathbf{V}}^n = \tilde{B}\tilde{\mathbf{V}}^{n-1}, \tag{2.6}$$

where

$$\tilde{A} = \begin{bmatrix} 1 - e^{-I\theta i} & \frac{\Delta S}{2}\left(1 - e^{-I\theta i}\right) \\ (s_1)_i\left(1 + e^{-I\theta i}\right) & (s_3)_i + (s_4)_i e^{-I\theta i} \end{bmatrix}, \; \tilde{\mathbf{V}}^n = \begin{bmatrix} \tilde{V}_n \\ \tilde{W}_n \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0 & 0 \\ (s_2)_i\left(1 + e^{-I\theta i}\right) & -(s_3)_i - (s_4)_i e^{-I\theta i} \end{bmatrix}, \; \tilde{\mathbf{V}}^{n-1} = \begin{bmatrix} \tilde{V}_{n-1} \\ \tilde{W}_{n-1} \end{bmatrix}.$$

and:

$$\begin{aligned} (s_1)_i &= 1 + \frac{r}{2}\Delta\tau, \\ (s_2)_i &= 1 - \frac{r}{2}\Delta\tau, \\ (s_3)_i &= -\frac{\sigma^2}{2}\frac{\Delta\tau}{\Delta S}\left(S_{i-\frac{1}{2}}\right)^2 - \frac{r}{2}S_{i-\frac{1}{2}}\Delta\tau, \\ (s_4)_i &= \frac{\sigma^2}{2}\frac{\Delta\tau}{\Delta S}\left(S_{i-\frac{1}{2}}\right)^2 + \frac{r}{2}S_{i-\frac{1}{2}}\Delta\tau. \end{aligned}$$

By assuming the amplification factor (G):

$$G = \begin{bmatrix} 1 - e^{-I\theta i} & \frac{\Delta S}{2}\left(1 - e^{-I\theta i}\right) \\ (s_1)_i\left(1 + e^{-I\theta i}\right) & (s_3)_i + (s_4)_i e^{-I\theta i} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ (s_2)_i\left(1 + e^{-I\theta i}\right) & -(s_3)_i - (s_4)_i e^{-I\theta i} \end{bmatrix}, \tag{2.7}$$

then Equation 2.6 can be formulated as $G = \tilde{U}^n/\tilde{U}^{n-1}$ where $\tilde{U}^n = (\tilde{V}^n, \tilde{W}^n)$. Furthermore, the eigenvalues of G are $\lambda_1 = 0$ and:

$$\lambda_2 = \frac{\sigma^2 S^2 \Delta\tau(1 - \cos\theta) - i \cdot r\Delta S \sin\theta}{-\sigma^2 S^2 \Delta\tau(1 - \cos\theta) - \Delta S^2(1 - \cos\theta) - \frac{1}{2}\Delta S^2 \Delta\tau(1 + \cos\theta) + i \cdot r\Delta S \sin\theta}.$$

This value shows that $|G| \le 1$ causing system in Equation 2.6 is unconditionally stable.

2.4. *Linearization*

For Equation 2.4 and Equation 2.5, the i iterate can be written as

$$V_i^n = V_i^{n-1} + \delta V_i^n, \tag{2.8}$$

$$W_i^n = W_i^{n-1} + \delta W_i^n. \tag{2.9}$$

By substituting Equation 2.8 and Equation 2.9 into Equation 2.4 – Equation 2.5, we obtain:

$$\delta V_i^n + \delta V_{i-1}^n - \frac{\Delta S}{2}\left(\delta W_i^n + \delta W_{i-1}^n\right) = (r_2)_i, \tag{2.10}$$

$$(s_1)_i \delta V_i^n + (s_1)_i \delta V_{i-1}^n + (s_3)_i \delta W_i^n + (s_4)_i \delta W_{i-1}^n = (r_1)_i, \tag{2.11}$$

where:

$$\begin{aligned} (r_1)_i &= -\frac{r}{2}\Delta t\left(V_i^{n-1} + V_{i-1}^{n-1}\right) - 2(s_3)_i W_i^{n-1} - 2(s_4)_i W_{i-1}^{n-1}, \\ (r_2)_i &= -V_i^{n-1} - V_{i-1}^{n-1} + \frac{\Delta S}{2}\left(W_i^{n-1} + W_{i-1}^{n-1}\right). \end{aligned}$$

2.5. *The block tridiagonal matrix*

Domain discretization for Equation 2.10 – Equation 2.11 is $S_0 = 0$, $S_i = S_{i-1} + \Delta S$ for $i = 1, 2, \cdots, I-1$, $S_I = S_\infty$ where $i = 0$ is the condition when the stock price is 0 ($S = 0$), while $i = I$ is the condition when the stock price is maximum ($S = S_{\max}$). As the result, the changes of boundary condition are:

$$\begin{aligned}\delta V_0 &= (r_1)_0 = K(e^{-r\tau_1} - e^{-r\tau_0}),\\ \delta V_I &= (r_1)_I = 0,\end{aligned}$$

and the system can be written as a matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -\frac{\Delta S}{2} & 1 & -\frac{\Delta S}{2} & 0 & 0 \\ (s_2)_i & (s_4)_i & (s_1)_i & (s_3)_i & 0 & 0 \\ 0 & 0 & -1 & -\frac{\Delta S}{2} & 1 & -\frac{\Delta S}{2} \\ 0 & 0 & (s_2)_i & (s_4)_i & (s_1)_i & (s_3)_i \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \delta V_0 \\ \delta W_0 \end{bmatrix} \\ \begin{bmatrix} \delta V_i \\ \delta W_i \end{bmatrix} \\ \begin{bmatrix} \delta V_I \\ \delta W_I \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} (r_1)_0 \\ (r_2)_0 \end{bmatrix} \\ \begin{bmatrix} (r_1)_i \\ (r_2)_i \end{bmatrix} \\ \begin{bmatrix} (r_1)_I \\ (r_2)_I \end{bmatrix} \end{bmatrix}. \tag{2.12}$$

The system of equations (2.12) can be written as:

$$\mathbf{A}\delta = \mathbf{r} \tag{2.13}$$

where:

$$\mathbf{A} = \begin{bmatrix} [A_0] & [C_0] & & & & & \\ [B_1] & [A_1] & [C_1] & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & [B_i] & [A_i] & [C_i] & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & [B_{l-1}] & [A_{l-1}] & [C_{l-1}] \\ & & & & & [B_l] & [A_l] \end{bmatrix}, \delta = \begin{bmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_i \\ \vdots \\ \delta_{I-1} \\ \delta_I \end{bmatrix}, \mathbf{r} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_i \\ \vdots r_{I-1} \\ r_I \end{bmatrix}$$

$$\delta_i = \begin{bmatrix} \delta V_i \\ \delta W_i \end{bmatrix}, r_i = \begin{bmatrix} (r_1)_i \\ (r_2)_i \end{bmatrix}$$

It can be seen that matrix $\mathbf{A}$ in Equation 2.13 is a block matrix tridiagonal where $[A_i]$, $[B_i]$, $[C_i]$ are 2×2 matrix and the other entries are 2×2 zero matrix where

$[A_i]$, $[B_i]$, $[C_i]$ are defined as

$$[A_0] = \begin{bmatrix} 1 & 0 \\ -1 & -\frac{\Delta S}{2} \end{bmatrix}, \quad [A_i] = \begin{bmatrix} (S_1)_i & (S_3)_i \\ -1 & -\frac{\Delta S}{2} \end{bmatrix}, \quad 1 \leq i \leq I-1,$$

$$[A_I] = \begin{bmatrix} (S_1)_I & (S_3)_I \\ 1 & 0 \end{bmatrix}, \quad [B_I] = \begin{bmatrix} (S_1)_i & (S_4)_i \\ 0 & 0 \end{bmatrix}, \quad 1 \leq i \leq I,$$

$$[C_i] = \begin{bmatrix} 0 & 0 \\ 1 & -\frac{\Delta S}{2} \end{bmatrix}, \quad 1 \leq i \leq I-1.$$

An $\mathbf{A}$ matrix is an $I \times I$ block tridiagonal matrix with each block of size 2×2, while δ and $\mathbf{r}$ are $I \times 1$ column vectors. The matrix $\mathbf{A}$ is decomposed to be a lower and an upper triangular matrix, well known as LU decomposition method, to find the solution of δ. When $\mathbf{A}\delta = \mathbf{r}$, the block tridiagonal matrix $\mathbf{A}$ acts on the δ vector to produce another vector $\mathbf{r}$. The block tridiagonal matrix $\mathbf{A}$ is further divided into the lower and upper parts of the triangular matrix. Thus, forward sweep is used at this step and $\mathbf{A}= LU$ can be written as $LU\delta = \mathbf{r}$. By letting $U\delta = y$, it yields $Ly = \mathbf{r}$, which gives the solution y. It will feeds into $U\delta = y$ to complete δ. Since we are dealing with a triangular matrix, backward sweep is the recommended method for the next step.

3. Results and Discussions

As an illustration in calculating a European put option, some parameters used are $S_{max} = 300$, $r = 0.05$, $\sigma = 0.05$ $K = 50$, $\Delta S = 1$, $\Delta\tau = 0.001$ and $T = 1$ in order to compare two numerical scheme. The comparison of the option values calculated by the Keller-box method and the Crank-Nicolson method with the analytical solution can be seen in Figure 2. The option value generated by the Keller-box method is more similar to the analytical solution than the option value of the Crank-Nicolson method. It also can be justified from Table 1 that Mean Absolut Error due to the Keller-box method is smaller when compared to the Crank-Nicolson method at $K = 50$. It can be said that the Keller-box method is better in determining the value of European put options.

Various parameters are used to see the influence of interest rate, volatility, and strike price on the behavior of a put option value. The value of the European put option for various interest rate are presented in Figure 3 for $K = 50$, $\sigma = 0.1$, and $T = 1$. It can be seen that the various value of interest rate influence to price of European put option. The higher interest rate given, the lower price of European put option. The value of interest rate also affects the error. It increases at the left side and decreases on the right side of strike price when interest rate increases.

In addition, the influence on the price of European put options is also the change in volatility as in Figure 4 for various $\sigma = 0.1, 0.3, 0.5$ at $r = 0.1$, $K = 50$ and $T = 1$. It shows that the higher volatility delivers to the higher price of European put option. Turning to the error affected by volatility, the highest total error is given by the lowest volatility $\sigma = 0.05$ excluding the error at around $S = 0$. Then, it will decay when interest rate increase.

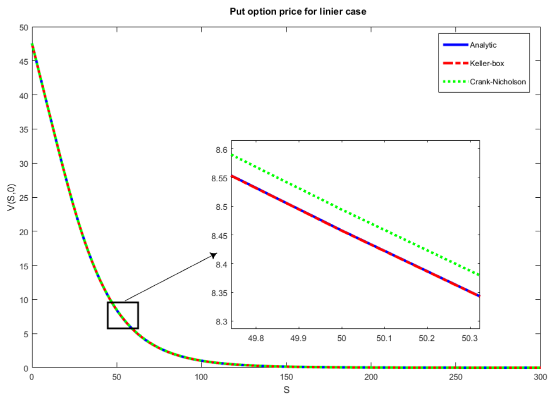

Figure. 2. Comparison between analytic solution, Keller-box method and Crank-Nicolson method

Table 1. *Mean Absolut Error* and an option value comparison between Keller-box and Crank-Nicolson scheme

Interest rate	Volatilty	*Mean Absolut Error*		Option value at $S = 50$		
		Keller-box	Crank-Nicolson	Analytical	Keller-box	Crank-Nicolson
$r = 0.01$	$\sigma = 0.1$	2.94×10^{-4}	1.40×10^{-3}	1.74510	1.74491	1.77400
	$\sigma = 0.3$	3.93×10^{-4}	1.47×10^{-3}	5.68662	5.68626	5.69661
	$\sigma = 0.5$	8.90×10^{-4}	1.80×10^{-3}	9.57469	9.57389	9.58078
$r = 0.03$	$\sigma = 0.1$	3.33×10^{-4}	4.61×10^{-3}	1.31321	1.31292	1.41329
	$\sigma = 0.3$	3.60×10^{-4}	4.81×10^{-3}	5.16393	5.16353	5.19961
	$\sigma = 0.5$	8.20×10^{-4}	5.37×10^{-3}	9.00304	9.00222	9.02456
$r = 0.05$	$\sigma = 0.1$	3.78×10^{-4}	7.63×10^{-3}	0.96395	0.96359	1.11983
	$\sigma = 0.3$	3.38×10^{-4}	7.96×10^{-3}	4.67709	4.67667	4.73733
	$\sigma = 0.5$	7.57×10^{-4}	8.74×10^{-3}	8.45777	8.45693	8.49427
$r = 0.07$	$\sigma = 0.1$	4.32×10^{-4}	1.04×10^{-2}	0.68933	0.68893	0.88326
	$\sigma = 0.3$	3.51×10^{-4}	1.09×10^{-2}	4.22494	4.22449	4.30831
	$\sigma = 0.5$	6.99×10^{-4}	1.19×10^{-2}	7.93815	7.93730	7.98913

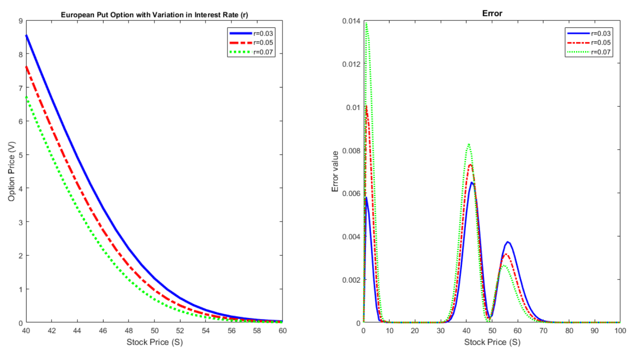

Figure. 3. Influence of interest rate on option price

Meanwhile, the effect of strike price over the price of European put option is also analysed for several $K = 50, 70, 90$ at $r = 0.2$, $\sigma = 0.1$ and $T = 1$ as in Figure

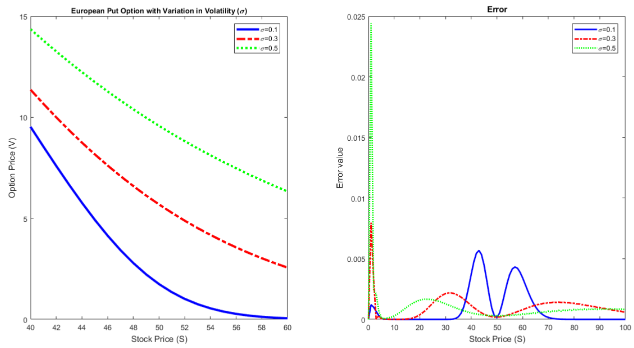

Figure. 4. Influence of volatility on option price

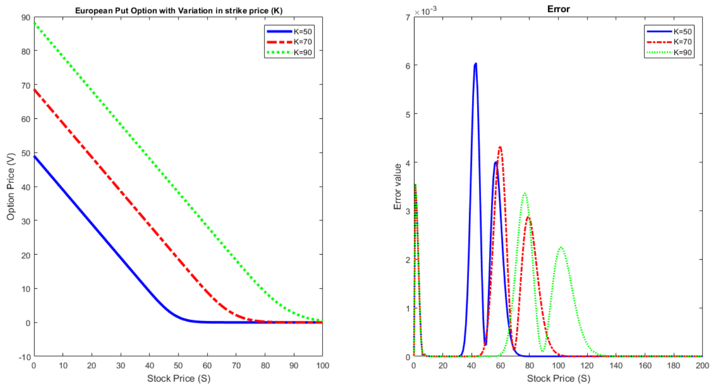

Figure. 5. Influence of strike price on option price

5. The increase of strike price leads to shift the option price higher. Further, it also affects the shifting lowest point of error based on the strike price and the decrease of the total error which the lowest total error is at strike price $K = 90$. Overall, the calculation of European put option prices using the Keller-box method yields a relatively small error.

4. Conclusion

Based on the results obtained, the Keller-box method is very well applied in option pricing through the Black–Scholes model. The method is unconditionally stable numerically when it is tested in the model. It also produces option values that are very closer to the analytical solution with a higher small error around the strike price. The results of analysis on the influence several parameters can be concluded that the higher the interest rate (r) lead to the lower the European put option price, while the higher the strike price (K), the higher the European put option price. This method can be extended into non-linear or higher dimensional problem because it delivers to relatively small error of the solution on linear one. Nonetheless, the difficulty is how to arrange discretization of equation system such that it produces good result in simulation process. Hopefully, this study could be extended into non-linear or higher dimensional cases.

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