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UNRAVELING THE IMPACT OF THE MEMORY, THE COMPETITION, AND THE LINEAR HARVESTING ON A LOTKA-VOLTERRA MODEL

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Abstract. *The harvesting of population has a dominant influence in balancing the ecosystem. In this manuscript, the impact of harvesting in addition to competition, and memory effect on a prey-predator interaction following the Lotka-Volterra model is studied. The mathematical validation is provided by proofing that all solutions of the model are always exist, non-negative, and bounded. Obeying Matignon condition, Lyapunov function, and generalized LaSalle invariance principle, the local and global stability are investigated. To complete the analytical results, some numerical simulations are given to show the occurrence of forward bifurcation and the impact of the memory index. All results state that three possible circumstances may occur namely the extinction of both populations, the prey-only population, and the co-existence of both populations.*

Keywords: Caputo fractional derivative, Harvesting, Lotka-Volterra

1. Introduction

In recent decades, studying the interaction between prey and its predator has gotten extraordinary attention from many researchers, especially mathematicians [1–4]. Most of them employ mathematical modeling with a deterministic approach using first-order derivative as the operator. However, it has been observed that these traditional models may not accurately capture the complex dynamics and behaviors of predator-prey systems in certain scenarios. For example, in some ecosystems, the memory of past interactions between predator and prey can significantly influence their current dynamics [5–7]. This has led to the development of fractional-order models that incorporate memory with power-law fading, allowing for a more accurate representation of the system's behavior. One of the famous fractional-order

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operators is given by the Caputo fractional derivative, which considers memory with power-law fading [8, 9]. This operator is regarded as a powerful tool in studying predator-prey interaction due to the completeness of the analytical and numerical theories [10–12].

In this paper, we study the predator-prey model which is adopted from the classical ones provided by Lotka [13] and Volterra [14]. This model is given by a first-order differential equation as follows [15].

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy, \\ \frac{dy}{dt} &= cxy - dy, \end{aligned} \tag{1.1}$$

where a, b, c, and d respectively define the intrinsic growth rate of the prey population, the predation rate, the birth rate of the predator based on the predation process, and the natural death rate of the predator. For each population, competition naturally exists in nature to get food. Thus the model (1.1) becomes:

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy - px^2, \\ \frac{dy}{dt} &= cxy - dy - qy^2, \end{aligned} \tag{1.2}$$

where p is the prey death rate due to competition and q is the predator death rate due to competition. We let there exist harvesting on both populations due to the existence of humans as the top position in the food chain [16, 17]. Thus we have linear harvesting on model (1.4) such that:

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy - px^2 - h_1 x, \\ \frac{dy}{dt} &= cxy - dy - qy^2 - h_2 y, \end{aligned} \tag{1.3}$$

where h_1 and h_2 are the harvesting rates on prey and predator, respectively. Now, to include the memory effect, we replace the first-order derivative at the left-hand side of model (1.3) with the fractional-order derivative ${}^C\mathcal{D}_t^\alpha$ such that:

$$\begin{aligned} {}^C\mathcal{D}_t^\alpha x &= ax - bxy - px^2 - h_1 x = F(x, y), \\ {}^C\mathcal{D}_t^\alpha y &= cxy - dy - qy^2 - h_2 y = G(x, y), \end{aligned} \tag{1.4}$$

where ${}^C\mathcal{D}_t^\alpha$ is the Caputo fractional-order derivative defined by:

$${}^C\mathcal{D}_t^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} z'(t) ds,$$

with $\alpha \in (0, 1]$ is the order of the derivative, $t \geq 0$, $z \in C^n([0, +\infty), \mathbb{R})$ and Γ is the Euler Gamma function [8]. The model (1.4) is our focus in this work.

To explore the model (1.4), we provide four mathematical findings that become our novelty and key contributions of our works as follows.

(i) Most of the predator-prey model are given without combine the competition, propotional harvesting, and the memory effect. Therefore, we investigate the dynamical behaviors of a Lotka-Volterra model by including those biological component which never been founded in other articles.

(ii) The mathematical validity of the model is proven by showing the existence, uniqueness, non-negativity, and boundedness of the solution.

(iii) Dynamical behaviors around the equilibrium points are shown completely such as local dynamics using Matignon's condition and local dynamics by using Volterra-Linear Lyapunov functions along with the generalized invariance principle.

(iv) Some numerical simulations are demonstrated not only to confirm the analytical findings but also to show the existence of two forward bifurcations in a small neighborhood value of the harvesting rate on prey.

To facilitate these results, we organize our paper as follows. In Section 2, we present the analytical results including the existence, uniqueness, non-negativity, boundedness, feasible equilibria, local dynamics, and global stability. We explore more the dynamical behaviors of model (1.4) by giving Section 3 which has significant results namely two sequential forward bifurcations driven by the prey harvesting rate followed by the phase portraits for each condition when bifurcation occurs. The impact of the memory is also given in the numerical section to show the relationship between memory index and convergence rate. We end our works by giving significant conclusions in Section 4.

2. Analytical Results

2.1. *Existence and Uniqueness*

We give the following theorem to show that for each initial condition of model (1.4) in $\mathbb{R}^2_+ := \{(x,y)|x \geq 0,\ y \geq 0\}$ has a unique solution.

Theorem 2.1. *Let* $\Omega := \left\{(x,y) \in \mathbb{R}^2 | \max\left(|x|, |y|\right) \leq \sigma\right\}$. *The solution of model* (1.4) *start from* $\mathbb{R}^2_+$ *always exists and is unique in* Ω.

Proof. We denote $H = (x,y)$ and $\tilde{H} = (\tilde{x},\tilde{y})$. For each mapping $H(X) = (F(H), G(H))$ where $H, \tilde{H} \in \Omega$, we obtain:

$$\begin{aligned}
\left|H(X) - H(\tilde{X})\right| &= \left|F(X) - F(\tilde{X})\right| + \left|G(X) - G(\tilde{X})\right|, \\
&= \left|\left(ax - bxy - px^2 - h_1 x\right) - \left(a\tilde{x} - b\tilde{x}\tilde{y} - p\tilde{x}^2 - h_1\tilde{x}\right)\right| \\
&\quad + \left|\left(cxy - dy - qy^2 - h_2 y\right) - \left(c\tilde{x}\tilde{y} - d\tilde{y} - q\tilde{y}^2 - h_2\tilde{y}\right)\right|, \\
&= \left|a\left(x - \tilde{x}\right) - b\left(xy - \tilde{x}\tilde{y}\right) - p\left(x^2 - \tilde{x}^2\right) - h_1\left(x - \tilde{x}\right)\right| \\
&\quad + \left|c\left(xy - \tilde{x}\tilde{y}\right) - d\left(y - \tilde{y}\right) - q\left(y^2 - \tilde{y}^2\right) - h_2\left(y - \tilde{y}\right)\right|, \\
&= \left|\left(a - by - p\left(x + \tilde{x}\right) - h_1\right)\left(x - \tilde{x}\right) - b\tilde{x}\left(y - \tilde{y}\right)\right| \\
&\quad + \left|cy\left(x - \tilde{x}\right) + \left(c\tilde{x} - d - q\left(y + \tilde{y}\right) - h_2\right)\left(y - \tilde{y}\right)\right|, \\
&\leq \left(a + h_1 + (b + 2p)\sigma\right)\left|x - \tilde{x}\right| + b\tilde{x}\left|y - \tilde{y}\right| \\
&\quad + c\sigma\left|x - \tilde{x}\right| + \left(d + h_2 + (c + 2q)\sigma\right)\left|y - \tilde{y}\right|, \\
&= \zeta_a\left|x - \tilde{x}\right| + \zeta_b\left|y - \tilde{y}\right|, \\
&= \zeta\left|X - \tilde{X}\right|,
\end{aligned}$$

where $\zeta_a = a + h_1 + (b+c+2p)\sigma$, $\zeta_b = d + h_2 + (b+c+2q)\sigma$, and $\zeta = \max\{\zeta_a, \zeta_b\}$. From [18], we achieve $F(H)$ satisfies the locally Lipschitz condition. Thus, based on Theorem 3.4 in [8], the model (1.4) has a unique solution in Ω. □

2.2. *Non-Negativity and Boundedness*

In this subsection, the non-negativity and boundedness of each solution will be confirmed. Therefore, we have the following theorem.

Theorem 2.2. *For each non-negative initial condition in* $\mathbb{R}^2_+$*, model* (1.4) *has the non-negative and bounded solution.*

Proof. We start by showing the non-negativity of solution. For $\alpha = 1$, we have the first-order derivative as the operator. Thus the model (1.4) can be rewritten as:

$$\begin{aligned}\frac{dx}{dt} &= (a - by - px - h_1)\,x,\\ \frac{dy}{dt} &= (cx - d - qy - h_2)\,y,\end{aligned}$$

where the solutions are

$$\begin{aligned}x(t) &= x(0)\exp\left(\int_0^t (a - by - px - h_1)\,d\tau\right),\\ y(t) &= y(0)\exp\left(\int_0^t (cx - d - qy - h_2)\,d\tau\right),\end{aligned}$$

which can be easily confirmed that $x(t) \geq 0$ and $y(t) \geq 0$ for each $x(0) \geq 0$ and $y(0) \geq 0$. Since $x(t)$ and $y(t)$ non-negative for $\alpha = 1$, following Theorem 6 in [19], the Caputo fractional-order model (1.4) also always non-negative. This completes the proof of the non-negativity of solution.

Next, the boundedness of the solution will be ensured. We let a non-negative linear function as follows.

$$\mathcal{N} = x + \frac{by}{c}. \tag{2.1}$$

Let ω be a positive real number. By computing the Caputo fractional-order derivative, we have the following inequality:

$$\begin{aligned}{}^C\mathcal{D}_t^\alpha \mathcal{N} + \omega\mathcal{N} &= {}^C\mathcal{D}_t^\alpha x + \frac{b}{c}{}^C\mathcal{D}_t^\alpha y + \omega\left(x + \frac{by}{c}\right),\\ &= \left(ax - bxy - px^2 - h_1x\right) + \frac{b}{c}\left(cxy - dy - qy^2 - h_2y\right) + \omega\left(x + \frac{by}{c}\right),\\ &= (a + \omega - h_1)\,x - px^2 + (\omega - d - h_2)\,\frac{by}{c} - \frac{bqy^2}{c}.\end{aligned}$$

We set $0 < \omega \leq \min\{h_1, h_2\}$. Hence:

$$\begin{aligned}{}^C\mathcal{D}_t^\alpha \mathcal{N} + \omega\mathcal{N} &\leq ax - px^2 - \frac{by}{c} - \frac{bqy^2}{c},\\ &= \frac{a^2}{4p} - p\left(x - \frac{a}{2p}\right)^2 - \frac{by}{c} - \frac{bqy^2}{c},\end{aligned}$$

$$\leq \frac{a^2}{4p}.$$

Based on Lemma 2.5 in [20], we have:

$$\mathcal{N}(t) \leq \left(\mathcal{N}(0) - \frac{a^2\omega}{4p}\right) E_\alpha\left[-\omega t^\alpha\right] + \frac{a^2\omega}{4p}.$$

Obeying the convergence of Mittag-Leffler function given by Lemma 5 in [21], we acquire $\mathcal{N}(t) \to \dfrac{a^2\omega}{4p}$ as $t \to \infty$. This ensure that each solution of model (1.4) tends to the region:

$$\mathcal{M} := \left\{(x, y) \in \mathbb{R}^2 | \mathcal{N} \leq \frac{a^2\omega}{4p} + \varepsilon = \sigma, \varepsilon > 0\right\}.$$

Since all solutions will enter to $\mathcal{M}$, each solution of model (1.4) is bounded by σ. This ends the proof. □

2.3. *Feasible Equilibria*

We identify the feasible equilibria by solving the $F(x, y) = G(x, y) = 0$. Thus, we have

$$\begin{aligned}[a - by - px - h_1]\, x &= 0,\\ [cx - d - qy - h_2]\, y &= 0,\end{aligned}$$

which gives three equilibrium points as follows.

(i) The extinction of both populations point (EBPP) denotes by $\mathcal{K}_1 = (0, 0)$. The EBPP always exists which describes the condition when both prey and predator disappear from the ecosystem.

(ii) The prey-only point (POP) denotes by $\mathcal{K}_2 = (\bar{x}, 0)$ where $\bar{x} = \dfrac{a - h_1}{p}$. The POP exists when $h_1 < a$ which states the condition when the predator is extinct and only the prey exists in the ecosystem.

(iii) The prey-predator co-existence point (PPCP) given by $\mathcal{K}_3 = (\hat{x}, \hat{y})$ where $\hat{x} = \dfrac{d + q\hat{y} + h_2}{c}$ and $\hat{y} = \dfrac{ac - ((d + h_2)p + ch_1)}{bc + pq}$. The PPCP exists when $c > \dfrac{((d + h_2)p + ch_1)}{a}$, which represents the condition when both prey and predator exist in the ecosystem.

In the next subsection, we give the stability properties of model (1.4).

2.4. *Local and Global Dynamics*

In this subsection, we present some theorems including their analytical proofs to show the behaviors of solutions of each equilibrium point both locally and globally. To simplify our work, we provide the following abbreviation: LAS for locally asymptotically stable and GAS for globally asymptotically stable.

Theorem 2.3. *The EBPP* $\mathcal{K}_1 = (0, 0)$ *is both LAS and GAS when* $h_1 > a$. *Otherwise, it is a saddle point.*

Proof. We linearize the model (1.4) around EBPP to obtain the following Jacobian matrix.

$$\mathcal{J}(x,y)|_{\mathcal{K}_1} = \begin{bmatrix} a-h_1 & 0 \\ 0 & -(d+h_2) \end{bmatrix}.$$

Therefore, we have eigenvalues $\lambda_1 = a - h_1$ and $\lambda_2 = -(d+h_2)$. We obtain $\lambda_2 < 0$ and hence $|\arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$. Furthermore, $\lambda_1 < 0$ when $h_1 > a$ which implies $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. When $h_1 < a$, we have $\lambda_2 > 0$ and hence $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$. Following Matignon condition [22], we confirm the LAS condition holds when $h_1 > a$.

Now, we will show that the GAS condition holds similar properties. We define a positive definite Lyapunov function as follows.

$$\mathcal{V}_1(x,y) = x + \frac{by}{c}. \tag{2.2}$$

By computing the Caputo fractional-derivative of eq. (2.2), we acquire:

$$\begin{aligned} {}^{C}\mathcal{D}_t^{\alpha}\mathcal{V}_1(x,y) &= {}^{C}\mathcal{D}_t^{\alpha}x + \frac{b}{c}{}^{C}\mathcal{D}_t^{\alpha}y, \\ &= \left(ax - bxy - px^2 - h_1x\right) + \frac{b}{c}\left(cxy - dy - qy^2 - h_2y\right), \\ &= -(h_1-a)\,x - px^2 - (d+h_2)\,\frac{by}{c} - \frac{b}{c}qy^2, \\ &\le -(h_1-a)\,x - (d+h_2)\,\frac{by}{c}. \end{aligned}$$

This confirms that ${}^{C}\mathcal{D}_t^{\alpha}\mathcal{V}_1(x,y) \le 0$ only when $h_1 > a$ for every $(x,y) \in \mathbb{R}_+^2$ and ${}^{C}\mathcal{D}_t^{\alpha}\mathcal{V}_1(x,y) = 0$ only when $(x,y) = (0,0)$. By employing the generalized LaSalle invariance principle (Lemma 4.6 in [23]), the proof of GAS of EBPP completes. □

Theorem 2.4. *The POP $\mathcal{K}_2 = (\bar{x}, 0)$ is LAS if $\bar{x} < \frac{h_2+d}{c}$ and GAS if $\bar{x} < \frac{h_2}{c}$. The POP is a saddle point if $\bar{x} > \frac{h_2+d}{c}$.*

Proof. We provide the following Jacobian matrix by doing linearization around $\mathcal{K}_2 = (\bar{x}, 0)$.

$$\mathcal{J}(x,y)|_{\mathcal{K}_2} = \begin{bmatrix} -p\bar{x} & -bx \\ 0 & c\bar{x} - (h_2+d) \end{bmatrix}.$$

Thus, the eigenvalues become $\lambda_1 = -p\bar{x}$ and $\lambda_2 = c\bar{x} - (h_2+d)$. Clearly, $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ since $\lambda_1 < 0$. Thus LAS condition of POP depends on the sign of λ_2. It is confirmed that λ_2 is negative for $\bar{x} < \frac{h_2+d}{c}$ and positive for $\bar{x} > \frac{h_2+d}{c}$ which ensure $|\arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ and $|\arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$, respectively. Obeying Matignon condition [22], the LAS and saddle point properties are justified.

Next, we study the global dynamics of the POP. We first rewrite the model (1.4) into the following system.

$$\begin{aligned} \frac{dx}{dt} &= (-p(x-\bar{x}) - by)\, x, \\ \frac{dy}{dt} &= cxy - dy - qy^2 - h_2 y. \end{aligned} \tag{2.3}$$

Now, we define a positive Volterra-Linear Lyapunov function as follows.

$$\mathcal{V}_2(x,y) = \left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}\right) + \frac{by}{c}. \tag{2.4}$$

By obeying Lemma 3.1 in [24] along with the model (2.3), we compute the Caputo fractional-derivative of eq. (2.4). The following inequality holds.

$$\begin{aligned} {}^C\mathcal{D}_t^\alpha \mathcal{V}_2(x,y) \le & \left(\frac{x-\bar{x}}{x}\right) {}^C\mathcal{D}_t^\alpha x + \frac{b}{c} {}^C\mathcal{D}_t^\alpha y, \\ = & \left(\frac{x-\bar{x}}{x}\right) (-p(x-\bar{x}) - by)\, x + \frac{b}{c}\left(cxy - dy - qy^2 - h_2 y\right), \\ = & -p\,(x-\bar{x})^2 + b\bar{x}y - \frac{b}{c}dy - \frac{b}{c}qy^2 - \frac{b}{c}h_2 y, \\ \le & -p\,(x-\bar{x})^2 - (h_2 - c\bar{x})\,\frac{by}{c}. \end{aligned}$$

Since ${}^C\mathcal{D}_t^\alpha \mathcal{V}_2(x,y) \le 0$ only when $\bar{x} < \dfrac{h_2}{c}$ for every $(x,y) \in \mathbb{R}_+^2$ and ${}^C\mathcal{D}_t^\alpha \mathcal{V}_2(x,y) = 0$ only when $(x,y) = (\bar{x}, 0)$, the generalized LaSalle invariance principle (Lemma 4.6 in [23]) says that th POP $\mathcal{K}_2 = (\bar{x}, 0)$ is GAS when $\bar{x} < \dfrac{h_2}{c}$. □

Theorem 2.5. *The PPCP $\mathcal{K}_3 = (\hat{x}, \hat{y})$ is always LAS and GAS.*

Proof. The Jacobian matrix of a linearized model at $\mathcal{K}_3 = (\hat{x}, \hat{y})$ is given by

$$\mathcal{J}(x,y)|_{\mathcal{K}_3} = \begin{bmatrix} -p\hat{x} & -b\hat{x} \\ c\hat{y} & -q\hat{y} \end{bmatrix},$$

which give a pair of eigenvalues as follows.

$$\lambda_{1,2} = -\frac{1}{2}\left(p\hat{x} + q\hat{y} \pm \sqrt{(p\hat{x} - q\hat{y})^2 - 4bc\hat{x}\hat{y}}\right).$$

For $(p\hat{x} - q\hat{y})^2 < 4bc\hat{x}\hat{y}$, the eigenvalues $\lambda_{1,2} \in \mathbb{C}^2$ with the real parts are negative. Therefore, $|\arg(\lambda_{1,2})| > \dfrac{\alpha\pi}{2}$. For $(p\hat{x} - q\hat{y})^2 \ge 4bc\hat{x}\hat{y}$, we can easily proof that the eigenvalues $\lambda_{1,2}$ always negative and hence $|\arg(\lambda_{1,2})| = \pi > \dfrac{\alpha\pi}{2}$. This means, the PPCP always LAS when the existence condition holds. Now, we will show the global stability of PPCP. We first rewrite model (1.4) into

$$\begin{aligned} \frac{dx}{dt} &= -\left[p\,(x-\hat{x}) + b\,(y-\hat{y})\right] x, \\ \frac{dy}{dt} &= \left[c\,(x-\hat{x}) - q\,(y-\hat{y})\right] y. \end{aligned} \tag{2.5}$$

Next, we define a Volterra Lyapunov function as follows.

$$\mathcal{V}_3(x,y) = \left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + \frac{b}{c}\left(y - \hat{y} - \hat{y}\ln\frac{y}{\hat{y}}\right). \tag{2.6}$$

Using Lemma 3.1 in [24] and eq. (2.5), the eq. (2.4) has the following Caputo fractional-order derivative.

$$\begin{aligned}
{}^C\mathcal{D}_t^\alpha \mathcal{V}_3(x,y) \le & \left(\frac{x-\hat{x}}{x}\right) {}^C\mathcal{D}_t^\alpha x + \frac{b}{c}\left(\frac{y-\hat{y}}{y}\right) {}^C\mathcal{D}_t^\alpha y, \\
= & -\left(\frac{x-\hat{x}}{x}\right)\left[p\left(x-\hat{x}\right) + b\left(y-\hat{y}\right)\right]x, \\
& + \frac{b}{c}\left(\frac{y-\hat{y}}{y}\right)\left[c\left(x-\hat{x}\right) - q\left(y-\hat{y}\right)\right]y, \\
= & -p\left(x-\hat{x}\right)^2 - \frac{bq}{c}\left(y-\hat{y}\right)^2.
\end{aligned}$$

Since ${}^C\mathcal{D}_t^\alpha \mathcal{V}_3(x,y)$ always negative or zero and ${}^C\mathcal{D}_t^\alpha \mathcal{V}_3(x,y) = 0$ only when $(x,y) = (\hat{x},\hat{y})$, we confirm from the generalized LaSalle invariance principle (Lemma 4.6 in [23]) that the PPCP $\mathcal{K}_3 = (\hat{x},\hat{y})$ is always GAS. □

3. Numerical Results

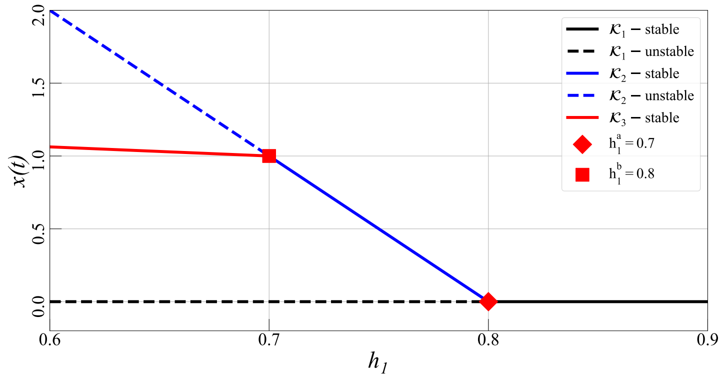
We provide some numerical simulations to show the dynamical behaviors of model (1.4) via phase portraits and time series. The numerical solution of the Caputo fractional-order model is developed based on the generalized Adam-Bashforth-Moulton scheme presented by Diethelm et al. [25]. In this subsection, we focus on studying the numerical simulations by observing the impact of the harvesting along with its memory effect on the dynamical behaviors of model (1.4). From the previous study, we have confirmed that the harvesting rate of prey plays a role in the existence of stability conditions of equilibria.

Thus, in this section, we only focus on the impact of the harvesting on prey. To facilitate our work, we use hypothetical values for the parameters by considering the non-availability of the real data. We set them as follows.

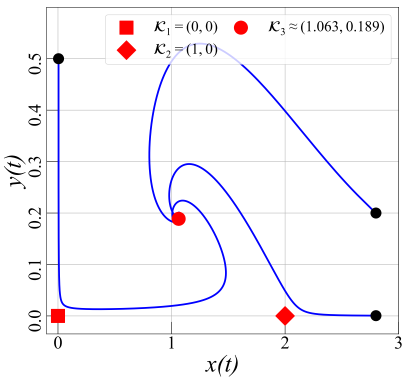
$$a = 0.8,\ b = 0.5,\ c = 0.3,\ d = 0.1,\ p = 0.1,\ q = 0.1,\ h_2 = 0.2,\ \text{and } \alpha = 0.9. \tag{3.1}$$

To show the impact of the harvesting on prey, we vary the value of h_1 in the interval $0.6 \le h_1 \le 0.9$. As a result, a bifurcation diagram is given in Figure 1(a).

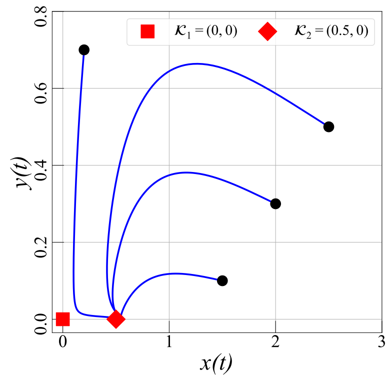
When $0.6 \le h_1 < h_1^a = 0.7$, we have a LAS PPCP $\mathcal{K}_3$ and a pair of unstable EBPP $\mathcal{K}_1$ and POP $\mathcal{K}_2$. When h_1 exceeds h_1^a, the PPCP merges with POP and becomes LAS via forward bifurcation while the EBPP is still unstable. This condition is still maintained for $h_1^a < h_1 < h_1^b = 0.8$. When h_1 crosses h_1^b, the LAS POP disappears and merges with EBPP and becomes LAS via forward bifurcation. Thus, TBPP $\mathcal{K}_1$ is the only equilibrium point on model (1.4). We confirm this condition by showing three phase portraits on those three intervals by setting $h_1 = 0.6, 0.75, 0.9$ which is given by Figure 1(b,c,d). In Figure 1(b), both EBPP and POP are saddle



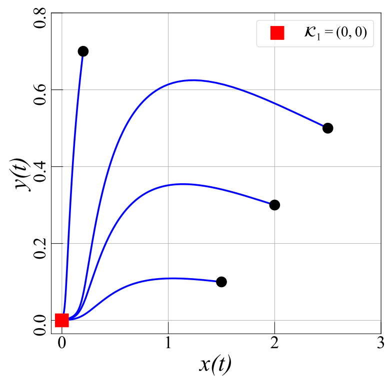
(a) Bifurcation diagram



(b) $h_1 = 0.6$



(c) $h_1 = 0.75$

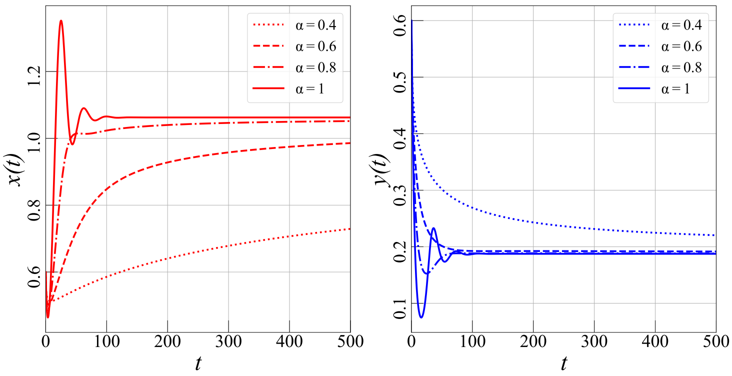


(d) $h_1 = 0.9$

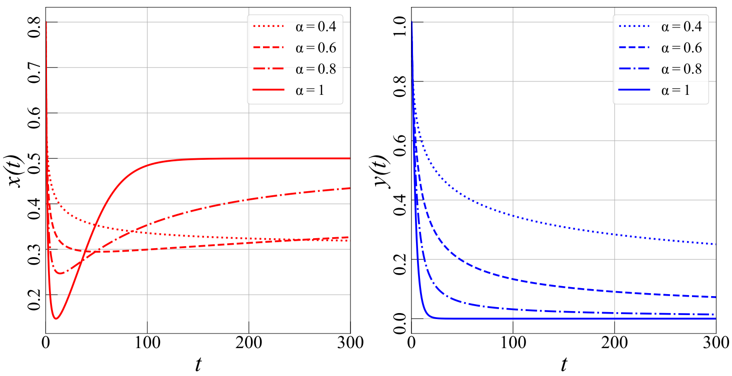
Figure. 1. The occurrence of forward bifurcation driven by the harvesting on prey parameter (h_1) using parameter values: $a = 0.8$, $b = 0.5$, $c = 0.3$, $d = 0.1$, $p = 0.1$, $q = 0.1$, $h_2 = 0.2$, and $\alpha = 0.9$

points and PPCP becomes GAS. In Figure 1(c), the EBPP is a saddle point and POP is GAS.

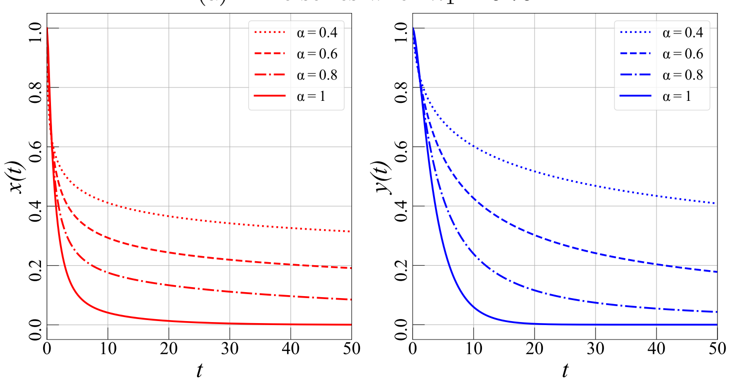
The POP becomes unique and the GAS equilibrium point on Figure 1(d). These phenomena show that there are three possible existence conditions in this interaction: (i) both prey and predator coexist in balance, (ii) predator becomes extinct and



(a) Time-series when $h_1 = 0.6$



(a) Time-series when $h_1 = 0.75$



(a) Time-series when $h_1 = 0.9$

Figure. 2. The memory effect on model (1.4) shown by the time-series using parameter values: $a = 0.8$, $b = 0.5$, $c = 0.3$, $d = 0.1$, $p = 0.1$, $q = 0.1$, and $h_2 = 0.2$

prey still exists, and (iii) both prey and predator become extinct in the ecosystem.

Now, we will show how the memory affects the dynamical behaviors of the model. By using parameter values on eq. (3.1), we obtain Figure 2.

For h_1, we confirm from Theorem 2.5 that $\mathcal{K}_3$ is both LAS and GAS. By computing the numerical solution for $0 \leq t \leq 500$ and $\alpha = 0.4, 0.6, 0.8, 1$, we have

Figure 2(a). Although all solutions converge to PPCP, the convergence rate reduces when the order-α decreases. We also ensure that when α increases, the peak of population for prey increases while for predator decreases. This confirms that when the memory effect strengthens, the population density of prey increases which indicates that their memory about the predation affects their ability to survive. A similar condition is also shown by Figure 2(b).

When $h_1 = 0.75$, although Theorem 2.4 says that the POP is asymptotically stable, the convergence rate and the peak of populations are changed when the order-α is varied. The peak of the prey population rises when α increases which is inversely proportional to the density of the predator. The different phenomenon occurs for $h_1 = 0.9$ which is given by Figure 2(c). When the memory effect weakened, both prey and predator accelerated towards extinction which was indicated by the increasing convergence rate.

4. Conclusion

The dynamical behaviors of a Lotka-Volterra predator-prey model incorporating competition, proportional harvesting, and memory effect have been investigated and studied completely. The validity of the model has been ensured by showing the existence and uniqueness as well as the non-negativity and boundedness of the solution. The feasibility of the equilibrium point has been provided along with the local and global dynamics using the Matignon condition and Lyapunov function, respectively. The numerical simulations are given to explore the dynamical behaviors using phase portraits and time series. We have found that three possible conditions may occur according to the value of the harvesting rate on prey namely the extinction of both prey and predator, the existence of prey-only, and the coexistence of both prey and predator. The impact of the memory has been shown that when the memory effect strengthens, the convergence rate also increases while the peak of the population also changes.

Credit Authorship Contribution Statement

H.S. Panigoro: Conceptualization, Methodology, Formal analysis, Discussion, Writing–Original Draft, Visualization, Writing–review and editing, Funding acquisition, Supervision. **Emli Rahmi:** Conceptualization, Methodology, Formal analysis, Data, Discussion, Writing–review and editing, Supervision. **Dian Savitri:** Conceptualization, Methodology, Writing–Original Draft, Writing–review and editing, Discussion, Supervision. **Lazarus Kalvein Beay:** Conceptualization, Methodology, Writing–Original Draft, Writing–review and editing, Discussion, Supervision. All authors discussed the results and contributed to the final manuscript.

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The authors state that none of the work described in this manuscript appears to have been influenced by any known competing financial interests or personal ties.

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