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A NOTE ON HAMILTONICITY CONDITIONS OF THE COPRIME AND NON-COPRIME GRAPHS OF A FINITE GROUP

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Abstract. Let G be a group. The coprime and non-coprime graphs of G are introduced by Ma et al. (2014) and Mansoori et al. (2016), respectively, when G is finite. By their definitions, which refer to coprime and non-coprime terms of two positive integers, those graphs must be related. We prove that they are closely related through their graph complement and preserve the isomorphism groups. Furthermore, according to Cayley's theorem, which states that any group G is isomorphic to a subgroup of the symmetric group on G, it implies that the studies of the coprime and non-coprime graphs of any group G (especially, when G is finite) can actually be represented by the coprime and non-coprime graphs of any subgroup of the symmetric group on G. This encourages us to specifically study the hamiltonicity of both kinds of graphs associated with G when G is isomorphic to the symmetric group on G.
Keywords: Coprime Graph, Non-coprime Graph, Hamiltonian Graph

1. Introduction

Throughout this paper, we assume that G and e_G are a finite group and its identity element, respectively. The order of G and element x of G are the number of elements of G and the smallest positive integer r such that $x^r = e_G$, respectively. In terms of a symmetric group on a given set X, denoted by $\mathfrak{S}_{|X|}$, we write:

$$\mathfrak{S}_{|X|} = \{f : X \longrightarrow X |\ f \text{ is a bijection (a permutation) on } X\}.$$

The element (or permutation) is represented as a cycle $(a_1\ a_2\ \cdots\ a_k)$ of length k. We refer to the following lemma that will be used later.

Lemma 1.1. *[1] Let $p_1, p_2, \cdots, p_N$ be prime numbers less than or equal to $|G|$. For any $m = \prod_{i=1}^{N} p_i^{\alpha_i}$ dividing $|G|!$ such that $\sum_{i=1}^{N} p_i^{\alpha_i} \leq |G|$, there exists $\sigma \in \mathfrak{S}_{|G|}$ whose order m.*

In addition, all graphs in this paper are assumed to be simple, undirected, and finite. Let Γ be a graph where $V(\Gamma)$ and $E(\Gamma)$ are the vertex set and the edge set of Γ, respectively. Specific Γ where $E(\Gamma) = \emptyset$ is called a null graph, denoted by

$N_{|V(\Gamma)|}$. A path and cycle in Γ are a list of m distinct vertices that are written as $v_{k_1} - v_{k_2} - \cdots - v_{k_m}$ and $v_{k_1} - v_{k_2} - \cdots - v_{k_m} - v_{k_1}$, respectively, where $v_{k_i} \in V(\Gamma)$ for all $i \in \{1, 2, \cdots, m\}$ and the symbol '$-$' between two vertices describes that those vertices are adjacent. The length of a path and cycle is the number of edges (or symbol '$-$') of the path and cycle. Graph Γ is said to be connected if there exists a path joining any two distinct vertices in Γ. Also, Γ is said to be Hamiltonian if Γ contains a Hamiltonian cycle, which is a cycle of length $|V(\Gamma)|$. The complement of Γ, denoted by $\overline{\Gamma}$, is a graph where $V(\overline{\Gamma}) = V(\Gamma)$ and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. A join graph of Γ_1 and Γ_2, denoted by $\Gamma_1 + \Gamma_2$, is a graph where $V(\Gamma_1 + \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and all vertices of Γ_1 are adjacent to all vertices of Γ_2. Let Γ' be a subgraph of Γ. A removal graph Γ' of Γ, denoted by $\Gamma - \Gamma'$, is a graph where $V(\Gamma - \Gamma') = V(\Gamma) \backslash V(\Gamma')$ and $E(\Gamma - \Gamma') = E(\Gamma) \backslash (E(\Gamma') \cup \{uv \mid u \in V(\Gamma'), v \in V(\Gamma) \backslash V(\Gamma')\})$. The independence number of Γ, denoted by $\alpha(\Gamma)$, is the maximum number of vertices of Γ where they are not adjacent to each other. The clique number of Γ, denoted by $\omega(\Gamma)$, is the independence number of $\overline{\Gamma}$.

In basic number theory, two positive integers are coprime if the greatest common divisor of them equals 1. Otherwise, they are non-coprime (not coprime) if the greatest common divisor of them is greater than 1. Paul Erdös and Gabor N. Sarkozy [2] studied cycles in a coprime graph of the set of integers. This graph continues being studied by some researchers such as [3], [4]. Its terminology has been developed by changing the set of integers to a finite group. A coprime graph of G is a graph where G is its vertex set and any two vertices are adjacent if and only if their orders are coprime, denoted by $C(G)$. It was introduced by Ma et al. in [5] who also studied structural algebras earned from this graph. Some research regarding coprime graphs and the variations can be seen for example [6], [7]. The authors of [5] prove that $C(D_{2n})$ is Hamiltonian if n is odd and $C(D_{2n})$ is not Hamiltonian if n is even. Algebraically, one has studied that D_{2n} is isomorphic to $\langle x, y \rangle \leq \mathfrak{S}_n$ where $x = (1\ 2\ 3\ \cdots\ n)$ and y is a product of $\lfloor \frac{n}{2} \rfloor$ transpositions. In this discussion, we will prove the hamiltonicity conditions of $C(\mathfrak{S}_n)$, which is provided in Theorem 2.3 and Theorem 2.7.

In the reverse case, a non-coprime graph of G is a graph where $G \backslash \{e_G\}$ is its vertex set and any two vertices are adjacent if and only if their orders are non-coprime, denoted by $NC(G)$. This graph was introduced by Mansoori et al. in [8] and studied by some researchers, see for example [1]. The authors of [9] generalized this graph by respecting it to a fixed subgroup of G. Associated with $C(G)$, we realize that the number of vertices of $C(G)$ is one more than the number of vertices of $NC(G)$. By their definitions, we can guess that they associate with each other through the complement form and it is not in a full number of vertices. We will prove it in Theorem 2.1 with a simple argument.

Theorem 1.2. *[5] For given finite groups G and H, if $G \simeq H$, then $C(G) \simeq C(H)$.*

Recall the following Cayley's Theorem about any group G'.

Theorem 1.3. *[10] Any group G' is isomorphic to a subgroup of $\mathfrak{S}_{|G'|}$.*

By Theorem 1.2 and Theorem 1.3, we can study the coprime graphs of any finite G via the coprime graphs of any subgroup of the symmetric group on G. Not only Theorem 1.2, but also we will prove in the further discussion section (which is Theorem 2.2), that non-coprime graphs apply the same notion.

We refer to the following theorem that will be used later.

Theorem 1.4. *[1] Let p be the largest prime number less than or equal to $|G|$. It follows that $NC(\mathfrak{S}_{|G|})$ is connected if and only if $|G| > p+1$.*

From Theorem 1.4, the connectedness of $NC(\mathfrak{S}_{|G|})$ with such condition of $|G|$ motivates us to prove the existence of a Hamiltonian cycle in $NC(\mathfrak{S}_{|G|})$. It will be proven in Theorem 2.9.

2. Main Results

We first investigate the relationship between coprime and non-coprime graphs of finite groups. Consider the following theorem.

Theorem 2.1. $C(G) = \overline{NC(G)} + N_1$ *and* $NC(G) = \overline{C(G)} - N_1$.

Proof. By definitions, clearly $V(C(G)) = V(\overline{NC(G)}) \cup \{e_G\}$ and $V(NC(G)) = V(\overline{C(G)}) \backslash \{e_G\}$. Take any two distinct non-identity elements u, v of G. If their orders are coprime, then u, v are adjacent in $C(G)$ or in $\overline{NC(G)}$. It follows that $E(C(G)) = E(\overline{NC(G)}) \cup \Big(V(\overline{NC(G)}) \times \{e_G\}\Big)$. Therefore, $C(G) = \overline{NC(G)} + N_1$. With a similar way, if the orders of u and v are not coprime, then u and v are adjacent in $NC(G)$ or in $\overline{C(G)}$. It follows that $E(NC(G)) = E(\overline{C(G)}) \backslash \{ue_G \mid u \in V(NC(G))\}$. Therefore, $NC(G) = \overline{C(G)} - N_1$. □

Theorem 2.2. *For given finite groups G and H, if $G \simeq H$, then $NC(G) \simeq NC(H)$.*

Proof. The proof is straightforward from Theorem 1.2 and Theorem 2.1. □

Theorem 2.1 proves that $C(G)$ is closely related to $NC(G)$ through $\overline{(NC(G))}$ and also that $NC(G)$ is closely related to $C(G)$ through $\overline{(C(G))}$. Consequently, not only $C(G)$, but $NC(G)$ also preserves the isomorphism of groups. From Theorem 2.1, we know that $C(G)$ is always connected since all vertices (non-identity elements) are joined by an edge to e_G, whereas $NC(G)$ is not (see in [1]). By simple observation, we obtain the following result.

Theorem 2.3. $C(\mathfrak{S}_3)$ *is Hamiltonian.*

Proof. Simply we see a Hamiltonian cycle as follows.

$$e_{\mathfrak{S}_3} - (1\ 2) - (1\ 2\ 3) - (1\ 3) - (1\ 3\ 2) - (2\ 3) - e_{\mathfrak{S}_3}.$$ □

Consider the following property that deals with a Hamiltonian graph.

Lemma 2.4. *If Γ is Hamiltonian on k vertices, then $\alpha(\Gamma) \leq \left\lfloor \frac{k}{2} \right\rfloor$.*

Proof. Recall that a Hamiltonian graph on k vertices contains a cycle of length k and the independence number of a graph that contains the cycle of length k (let's say, it is denoted by C_k) is equal to $\lfloor \frac{k}{2} \rfloor$. Let A and B be the independent set of Γ and C_k, respectively. Since every subset having 2 elements of A is not an edge of Γ then it is not an edge of C_k. Therefore, $A \subseteq B$ and it follows that $\alpha(\Gamma) = |A| \leq |B| = \lfloor \frac{k}{2} \rfloor$. □

Lemma 2.5. *For all $n > 3$, the number of elements of even order in $\mathfrak{S}_n$ is greater than $\frac{n!}{2}$.*

Proof. Recall that the number of odd permutations of $\mathfrak{S}_n$ is $\frac{n!}{2}$ (from the complement of an alternating group A_n). Those permutations have even length so that they have even order. In fact, there are some even permutations that are written as the multiplication of cycles of even length for any $n > 3$. Thus, the number of permutations of even order in $\mathfrak{S}_n$ is greater than $\frac{n!}{2}$ in this case. □

Proposition 2.6. *For all $n > 3$, it follows that*

$$\alpha\left(C(\mathfrak{S}_n)\right), \omega\left(NC(\mathfrak{S}_n)\right) > \frac{n!}{2}.$$

Proof. Clearly, all elements whose even order in $\mathfrak{S}_n$ are not adjacent to each other in $C(\mathfrak{S}_n)$ but they are adjacent to each other in $NC(\mathfrak{S}_n)$. Therefore, the proof is complete by Lemma 2.5. □

By Theorem 2.3, Lemma 2.4, and Proposition 2.6, we derive the following theorem.

Theorem 2.7. *$C(\mathfrak{S}_n)$ is Hamiltonian if and only if $n = 3$.*

Proof. The left-hand side has proved in Theorem 2.3. Conversely, let $C(\mathfrak{S}_n)$ is Hamiltonian. Suppose that $n > 3$. From Proposition 2.6, we obtain $\alpha(C(\mathfrak{S}_n)) > \frac{n!}{2}$. It follows by Lemma 2.4, $C(\mathfrak{S}_n)$ is not Hamiltonian. It is a contradiction with the hypothesis. □

Consider the following lemma which asserts that the number of non-identity elements of $\mathfrak{S}_n$ with a given order, is greater than 1.

Lemma 2.8. *The number of non-identity elements of order d in $\mathfrak{S}_n$ is greater than 1.*

Proof. Recall that the conjugacy class of k-cycles ($k > 1$) in $\mathfrak{S}_n$ consists of the elements of the same cycle form and two elements of $\mathfrak{S}_n$ with the same cycle form have the same order. These imply that the cardinality of the conjugacy class of $x \in \mathfrak{S}_n$ determines the least number of elements whose the same order with x. Furthermore, the cardinality of the conjugacy class of $x \in G$ is equal to 1 if and only if the center of G only consists of e_G (whose order 1). Since the center of $\mathfrak{S}_n$

is not only consist of $e_{\mathfrak{S}_n}$ then the cardinality of the conjugacy class of an element of order $d > 1$ is not equal to 1. □

Using Theorem 1.4, Lemma 2.8, and Lemma 1.1, we obtain the following theorem that provides hamiltonicity of connected non-coprime graph of $\mathfrak{S}_n$.

Theorem 2.9. *The connected $NC(\mathfrak{S}_n)$ is Hamiltonian.*

Proof. Let $p_{\max}$ be the largest prime number less than or equal to n. By Theorem 1.4, $NC(\mathfrak{S}_n)$ is connected if and only if $n > p_{\max} + 1$. Clearly, n must be greater than or equal to 9.

Let p be any prime number less than n. Define H_p to be the set of all elements having order divisible by p. Since for any p satisfies p divides $n!$ then by Lemma 1.1, there exists $\sigma \in \mathfrak{S}_n$ whose order p and thus $\sigma \in H_p$. In other words, $H_p \neq \emptyset$ for all p. Now, define $\mathbb{H}_2 = H_2$ and

$$\mathbb{H}_p = H_p \setminus \bigcup_{\substack{\text{prime } p' \\ 2 \le p' < p}} \mathbb{H}_{p'}.$$

Let explicitly $2, p_1, p_2, \cdots, p_{\max}$ be prime numbers less than n. Since $n > p_{\max} + 1$ then $2 + p_i \le n$ for all $p_i \in \{2, p_1, p_2, \cdots, p_{\max}\}$. By Lemma 2.8 and Lemma 1.1, there exist positive integers $\gamma, \gamma_1, \gamma_2, \cdots, \gamma_{\max} > 1$ which represent the number of elements in $\mathfrak{S}_n$ of order $2, 2p_1, 2p_2, \cdots, 2p_{\max}$, respectively. By definition of the non-coprime graph of $\mathfrak{S}_n$, all of those elements are adjacent to each other. Consider the following algorithm to construct a Hamiltonian cycle in $NC(\mathfrak{S}_n)$.

(i) Relabel the elements of $\mathfrak{S}_n$ based on their order such as $x_j^{(i)}$ meaning an ith element of order j in $\mathfrak{S}_n$. We set a path π starting at $x_2^{(1)}$ and then goes to $x_{2p_1}^{(1)}$.

(ii) From $x_{2p_1}^{(1)}$, π goes to traverse all elements in $\mathbb{H}_{p_1}$ and then it goes back to $x_{2p_1}^{(2)}$ in $\mathbb{H}_2$.

(iii) From $x_{2p_1}^{(2)}$, π goes to $x_{2p_2}^{(1)}$.

(iv) From $x_{2p_2}^{(1)}$, π goes to traverse all elements in $\mathbb{H}_{p_2}$ and then it goes back to $x_{2p_2}^{(2)}$ in $\mathbb{H}_2$.

(v) From $x_{2p_2}^{(2)}$, π goes to $x_{2p_3}^{(1)}$.

(vi) From $x_{2p_3}^{(1)}$, π goes to traverse all elements in $\mathbb{H}_{p_3}$ and then it goes back to $x_{2p_3}^{(2)}$ in $\mathbb{H}_2$.

(vii) Do the same procedures as above until π traverse all elements in $\mathbb{H}_{p_{\max}}$ and then goes back to $x_{2p_{\max}}^{(2)}$ in $\mathbb{H}_2$.

(viii) From $x_{2p_{\max}}^{(2)}$, π goes to traverse the remaining elements in $\mathbb{H}_2$ such as $x_{2p_i}^{(3)}, x_{2p_i}^{(4)}, \cdots, x_{2p_i}^{(\gamma_i)}$ for each i and then it ends at $x_2^{(1)}$.

(ix) Therefore, we obtain π as a Hamiltonian cycle in $NC(\mathfrak{S}_n)$.

The proof is completed. □

We collect the results from the previous discussions into the following theorem.

Theorem 2.10. *For any finite group G such that $G \simeq \mathfrak{S}_{|G|}$, we have the following assertions:*

(i) *$C(G)$ is Hamiltonian if and only if $|G| = 3$, and*

(ii) *Let p is the largest prime number less than or equal to $|G|$. It follows that $NC(G)$ is Hamiltonian if and only if $|G| > p + 1$.*

3. Conclusion

Especially for any finite group G that is exactly isomorphic to the symmetric group on G, we have proved the hamiltonicity condition of $C(G)$ and $NC(G)$ in Theorem 2.10. Looking back to the introduction, the hamiltonicity condition of the coprime and non-coprime graphs of any finite group is completely proved after one can tackle the remaining cases i.e. hamiltonicity conditions of $C(G)$ and $NC(G)$ with $G \simeq H$ where H be a proper subgroup of the symmetric group on G.

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