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THE LIE ALGEBRA $\mathfrak{su}(3)$ REPRESENTATION WITH RESPECT TO ITS BASIS

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Abstract. *The eight-dimensional Lie algebra of* 3×3 *anti-Hermitian matrices with its traces equal to zero is denoted by* $\mathfrak{su}(3)$ *whose Lie group is denoted by* $\mathrm{SU}(3)$. *The research aims to provide all representations of* $\mathfrak{su}(3)$ *with respect to its basis which is realized on the three complex variables homogeneous polynomials* $\mathbb{P}_1$ *of degree three. The first step is to construct representations of* $\mathrm{SU}(3)$ *on the space* $\mathcal{H}$ *and the second step is to find all derived representations of* $\mathrm{SU}(3)$. *The obtained results are eight explicit formulas of representations* $\mathfrak{su}(3) \curvearrowright \mathbb{P}_1$.
Keywords: Anti-Hermitian Matrices, Derived Representation, SU(3), $\mathfrak{su}(3)$

1. Introduction

Representation theory of Lie groups and Lie algebras are widely applied in both mathematics and physics ([1], [2], [3]). In mathematics, Lie groups and Lie algebras contribute in many branches of mathematics such as in biological mathematics and finance mathematics ([4], [5]). Moreover, representations of Lie algebras are vast studied by many researchers ([6]-[14]). In addition, matrix Lie groups can be used as a model for understanding representation theory.

The set of all $n \times n$ matrices whose their adjoints are equal to their inverse and their determinant are equals one is denoted by $\mathrm{SU}(n)$. In other words, $\mathrm{SU}(n)$ is the collections of $n \times n$ unitary matrices with determinant equal to one. We write it as $\mathrm{SU}(n) = \{S \in \mathrm{U}(n); \det(S) = 1\}$. The dimension of $\mathrm{SU}(n)$ is $n^2 - 1$. In case of $n = 2$, the unitary-irreducible representation of $\mathrm{SU}(2) \curvearrowright \mathbb{P}_n$ where $\mathbb{P}_n$ is homogeneous polynomials of two-complex variables can be found in ([15], [16]).

In this research, we shall take for case $n = 3$ in which the representation of the matrix Lie group $\mathrm{SU}(3)$ can be applied to the field of particle theory [16]. Different from previous results, we give explicit formulas for representations of $\mathrm{SU}(3)$. The

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derived representations of SU(3) arise to the notion of representation of the Lie algebra of SU(3) which is denoted by $\mathfrak{su}(3)$.

It is well known that the Lie algebra $\mathfrak{su}(3)$ consists of 3×3 anti-Hermitian matrices whose traces of them equals zero, namely, $\mathfrak{su}(3) = \{P \in \mathrm{M}(3, \mathbb{C}); P^* + P = O, \mathrm{tr}(P) = 0\}$. Roughly speaking, it is also well known that $\mathfrak{su}(3)$ contains all smooth left-invariant vector fields of SU(3). Furthermore, the representations of the Lie algebra $\mathfrak{su}(3)$ is important since it gives the best model in studying representation of matrix Lie groups. The obtained result of representation of the Lie algebra $\mathfrak{su}(3)$ in this research can be developed to the general case $\mathfrak{su}(n) \curvearrowright \mathbb{P}_n$ where $\mathbb{P}_n$ is homogeneous polynomials of n-complex variables.

We organize this paper as follows: In part 1, we introduce the motivation and state of art of our research, in part 2, we deliver some of the relevant backgrounds: The Lie group SU(3), the Lie algebra $\mathfrak{su}(3)$ and its basis, representation of matrix Lie groups, and derived representations, part 3, we state the main result, particularly, in Proposition 3.2 equipped with complete proofs.

2. Preliminaries

In this section, we shall briefly review some of the relevant materials which shall be used in main results: $\mathfrak{su}(3)$ notion and its basis, representation theory, derived representations. As mentioned before the Lie algebra $\mathfrak{su}(3)$ can be written in the following form:

$$\mathfrak{su}(3) = \{P \in \mathrm{M}(3, \mathbb{C}); P^* + P = O, \mathrm{tr}(P) = 0\}. \tag{2.1}$$

Definition 2.1. *[17] The Lie algebra $\mathfrak{su}(3)$ has the basis $\mathcal{A} = \{\tau_k = -\frac{i}{2}\zeta_k, \quad i^2 = -1$ and $k = 1, 2, \cdots, 8\}$ where ζ_k are given in the following 3×3 Hermitian matrices:*

$$\zeta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \zeta_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \zeta_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \zeta_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\zeta_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \zeta_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \zeta_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \zeta_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We recall that the matrix Lie group H is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ and closed subgroup as well. A representation of H is given by the following definition

Definition 2.2. *[16] A finite complex representation of a matrix Lie group H on a finite complex vector space $\mathcal{V}$ or $H \curvearrowright \mathcal{V}$ is a Lie group homomorphism given by*

$$\Phi : H \to \mathrm{GL}(\mathcal{V}). \tag{2.2}$$

In the context of representation theory, Definition 2.2 is equivalent to the following definition:

Definition 2.3. *[16] A finite complex representation of a matrix Lie group H on a finite complex vector space $\mathcal{V}$ or $H \curvearrowright \mathcal{V}$ is a linear action of H on $\mathcal{V}$. Namely, for all $x \in H$ we have*

$$\Phi(g) : \mathcal{V} \ni v \mapsto \Phi(g)v := g.v \in \mathcal{V}. \tag{2.3}$$

Proposition 2.4. *Let $\Phi : H \curvearrowright \mathcal{V}$ be a representation of a matrix Lie group on the complex vector space $\mathcal{V}$. Then there exists a unique $\pi : \mathfrak{h} \curvearrowright \mathcal{V}$*

$$\pi(A) = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta A})|_{\delta=0} \tag{2.4}$$

where $\Phi(e^{\delta A}) = e^{\pi(A)}$.

The representation $\pi : \mathfrak{h} \curvearrowright \mathcal{V}$ is named by *derived representation* associated to $\Phi : H \curvearrowright \mathcal{V}$. Indeed, if H is simply connected then the representation $\pi : \mathfrak{h} \curvearrowright \mathcal{V}$ can be obtained from the representation $\Phi : H \curvearrowright \mathcal{V}$. In our case, SU(3) is topologically simply connected. Therefore, we can obtain the $\pi : \mathfrak{su}(3) \curvearrowright \mathbb{P}_1$ from $\Phi : \mathrm{SU}(3) \curvearrowright \mathbb{P}_1$ where $\mathbb{P}_1$ is three complex variable homogeneous polynoms.

3. Results and Discussion

Firstly, we compute all exponential matrices in the basis $\mathcal{A} = \{\tau_k = -\frac{i}{2}\zeta_k, \quad i^2 = -1 \text{ and } k = 1, 2, \cdots, 8\}$ of $\mathfrak{su}(3)$ as written in Definition 2.1. All computations we state in the following proposition.

Proposition 3.1. *Let $\mathcal{A} = \{\tau_k = -\frac{i}{2}\zeta_k, \quad i^2 = -1$ and $k = 1, 2, \cdots, 8\}$ be a basis of $\mathfrak{su}(3)$ as written in Definition 2.1. Then the exponential matrices of elements of $\mathcal{A}$ can be listed in the following forms:*

$$e^{\delta\tau_1} = \begin{pmatrix} \cos\frac{\delta}{2} & -i\sin\frac{\delta}{2} & 0 \\ -i\sin\frac{\delta}{2} & \cos\frac{\delta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ e^{\delta\tau_2} = \begin{pmatrix} 0\cos\frac{\delta}{2} & -\sin\frac{\delta}{2} & 0 \\ \sin\frac{\delta}{2} & \cos\frac{\delta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e^{\delta\tau_3} = \begin{pmatrix} e^{-i/2} & 0 & 0 \\ 0 & e^{i/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ e^{\delta\tau_4} = \begin{pmatrix} \cos\frac{\delta}{2} & 0 & -i\sin\frac{\delta}{2} \\ 0 & 1 & 0 \\ -i\sin\frac{\delta}{2} & 0 & \cos\frac{\delta}{2} \end{pmatrix},$$

$$e^{\delta\tau_5} = \begin{pmatrix} \cos\frac{\delta}{2} & 0 & -i\sin\frac{\delta}{2} \\ 0 & 1 & 0 \\ \sin\frac{\delta}{2} & 0 & \cos\frac{\delta}{2} \end{pmatrix}, \ e^{\delta\tau_6} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\delta}{2} & -i\sin\frac{\delta}{2} \\ 0 & -i\sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix},$$

$$e^{\delta\tau_7} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\delta}{2} & -\sin\frac{\delta}{2} \\ 0 & \sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix}, \ e^{\delta\tau_8} = \begin{pmatrix} e^{\frac{-i}{2\sqrt{3}}} & 0 & 0 \\ 0 & e^{\frac{-i}{2\sqrt{3}}} & 0 \\ 0 & 0 & e^{\frac{i}{\sqrt{3}}} \end{pmatrix}.$$

Proof. We just compute exponential matrices for $\delta\tau_1$. The exponential matrices of other elements can be computed in similar ways. We can observe that the matrix τ_1 has three linear independent eigenvectors:

$$s_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, s_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \tag{3.1}$$

corresponding to the eigenvalues $\lambda_1 = 0, \lambda_2 = i/2$, and $\lambda_3 = -i/2$. By using Proposition 2.3 in [16], $e^{\delta\tau_1}$ can be computed as $e^{\delta\tau_1} = Pe^D P^{-1}$ where P is an invertible matrix contained all linearly independent eigenvectors of τ_1 and D is the diagonal matrix contained all δ-multiples of eigenvalues of τ_1. In other words, we get:

$$e^{\delta\tau_1} = Pe^D P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i/2} & 0 \\ 0 & 0 & e^{-i/2} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1}.$$

Then we have the exponential matrix of $\delta\tau_1$ or:

$$e^{\delta\tau_1} = \begin{pmatrix} \cos\frac{\delta}{2} & -i\sin\frac{\delta}{2} & 0 \\ -i\sin\frac{\delta}{2} & \cos\frac{\delta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

as required. □

Secondly, we let SU(3) acts on the space $\mathbb{P}_1$ by Φ. Thirdly, derived representation of Φ is the representation π of $\mathfrak{su}(3)$ realized on the space $\mathbb{P}_1$. Now we are ready to state our main result in the following proposition.

Proposition 3.2. *Let $\mathcal{A} = \{\tau_k = -\frac{i}{2}\zeta_k, \quad i^2 = -1$ and $k = 1, 2, \cdots, 8\}$ be a basis for $\mathfrak{su}(3)$ as written in Definition 2.1 and let*

$$\begin{aligned} \mathcal{B} = \{v_1 = x^3, v_2 = y^3, v_3 = z^3, v_4 = x^2y, v_5 = x^2z, v_6 = xy^2, v_7 = xz^2,\\ v_8 = y^2z, v_9 = yz^2, v_{10} = xyz\} \end{aligned} \tag{3.2}$$

be a basis for three-complex homogeneous polynomial space $\mathbb{P}_1$ of degree 3. Then the representations of $\pi : \mathfrak{su}(3) \curvearrowright \mathbb{P}_1$ with respect to the basis of $\mathcal{A}$ are of the forms:

$$\hat{\tau}_1 v_1 = \pi(\tau_1)v_1 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_1})v_1|_{\delta=0} = \frac{3}{2}iv_4 \tag{3.3}$$

$$\hat{\tau}_2 v_2 = \pi(\tau_2)v_2 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_2})v_2|_{\delta=0} = -\frac{3}{2}v_6 \tag{3.4}$$

$$\hat{\tau}_3 v_3 = \pi(\tau_3)v_3 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_3})v_3|_{\delta=0} = 0 \tag{3.5}$$

$$\hat{\tau}_4 v_4 = \pi(\tau_4)v_4 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_4})v_4|_{\delta=0} = iv_{10} \tag{3.6}$$

$$\hat{\tau}_5 v_5 = \pi(\tau_5)v_5 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_5})v_5|_{\delta=0} = v_{10} - \frac{1}{2}v_1 \tag{3.7}$$

$$\hat{\tau}_6 v_6 = \pi(\tau_6)v_6 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_6})v_6|_{\delta=0} = iv_{10} \tag{3.8}$$

$$\hat{\tau}_7 v_7 = \pi(\tau_7)v_7 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_7})v_7|_{\delta=0} = -v_{10} \tag{3.9}$$

$$\hat{\tau}_8 v_{10} = \pi(\tau_8)v_{10} = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_8})v_{10}|_{\delta=0} = 0. \tag{3.10}$$

Proof. Let x be any element of SU(3) and v be element of $\mathbb{P}_1$. The map $[\Phi(x)v](z) = v(x^{-1}z), z \in \mathbb{C}^3$ is a actually a representation. To see that, let us compute:

$$\begin{aligned} \Phi(x)[\Phi(y)v](z) &= [\Phi(y)v](x^{-1}z), \\ &= v(y^{-1}x^{-1}z), \\ &= \Phi(xy)v(z). \end{aligned}$$

We just compute for the cases $\hat{\tau}_1 v_1$, $\hat{\tau}_2 v_2$, and the last $\hat{\tau}_2 v_3$. Another case can be computed in the similar ways. Firstly, from Proposition 3.1 we have:

$$e^{\delta\tau_1} = \begin{pmatrix} \cos\frac{\delta}{2} & -i\sin\frac{\delta}{2} & 0 \\ -i\sin\frac{\delta}{2} & \cos\frac{\delta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By direct computation, we have the inverse of $e^{\delta\tau_1}$ is of the form:

$$e^{\delta\tau_1} = \begin{pmatrix} \cos\frac{\delta}{2} & i\sin\frac{\delta}{2} & 0 \\ i\sin\frac{\delta}{2} & \cos\frac{\delta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In addition, we can compute the representations $\Phi(e^{\delta\tau_1})v_1$ as follows:

$$\begin{aligned} \Phi(e^{\delta\tau_1})v_1(z) &= v_1(e^{-\delta\tau_1}z), \\ &= v_1\left[\begin{pmatrix} \cos\frac{\delta}{2} & i\sin\frac{\delta}{2} & 0 \\ i\sin\frac{\delta}{2} & \cos\frac{\delta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right], \\ &= v_1\left[\begin{pmatrix} x\cos\frac{\delta}{2} + iy\sin\frac{\delta}{2} \\ xi\sin\frac{\delta}{2} + y\cos\frac{\delta}{2} \\ z \end{pmatrix}\right], \\ &= (x\cos\frac{\delta}{2} + iy\sin\frac{\delta}{2})^3. \end{aligned}$$

The derived representation of $\Phi(e^{\delta\tau_1})v_1$ is given by $\hat{\tau}_1 v_1$ which is nothing but the representation π of $\mathfrak{su}(3)$ represented on $\mathbb{P}_1$. By direct computations, then we get an explicit formula in the following form:

$$\begin{aligned} \hat{\tau}_1 v_1 = \pi(\tau_1)v_1 &= \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_1})v_1|_{\delta=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\delta}(x\cos\frac{\delta}{2} + iy\sin\frac{\delta}{2})^3|_{\delta=0} \\ &= 3(x\cos\frac{\delta}{2} + iy\sin\frac{\delta}{2})^2(-x/2\sin\frac{\delta}{2} + iy/2\cos\frac{\delta}{2})|_{\delta=0} \\ &= 3(x)^2(iy/2) = \frac{3}{2}ix^2y \\ &= \frac{3}{2}iv_4. \end{aligned}$$

Secondly, we can compute inverse of $e^{\delta\tau_2}$, namely we have:

$$e^{-\delta\tau_2} = \begin{pmatrix} 0\cos\frac{\delta}{2} & \sin\frac{\delta}{2} & 0 \\ -\sin\frac{\delta}{2} & \cos\frac{\delta}{2} & 0 \\ 0 & & 0\ 1 \end{pmatrix}.$$

We can observe that:

$$\Phi(e^{\delta\tau_2})v_2(z) = v_2(e^{-\delta\tau_2}z) = (-x\sin\frac{\delta}{2} + y\cos\frac{\delta}{2})^3.$$

Similarly we have:

$$\begin{aligned}
\hat{\tau}_2 v_2 = \pi(\tau_2)v_2 &= \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_1})v_1|_{\delta=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\delta}(-x\sin\frac{\delta}{2} + y\cos\frac{\delta}{2})^3|_{\delta=0} \\
&= 3(x\cos\frac{\delta}{2} + iy\sin\frac{\delta}{2})^2(-x/2\sin\frac{\delta}{2} + iy/2\cos\frac{\delta}{2})|_{\delta=0} \\
&= 3(y)^2(-x/2) = -\frac{3}{2}ixy^2 \\
&= -\frac{3}{2}v_6.
\end{aligned}$$

Thirdly, since $e^{\delta\tau_2}$ is the diagonal matrix then the inverse of $e^{\delta\tau_2}$ is easy to consider. Then we have the representation $\hat{\tau}_3 v_3 = 0$ as desired. □

4. Conclusion

We concluded that the representations of $\pi : \mathfrak{su}(3) \curvearrowright \mathbb{P}_1$ with respect to the basis of $\mathcal{A}$ are of the forms:

$$\hat{\tau}_1 v_1 = \pi(\tau_1)v_1 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_1})v_1|_{\delta=0} = \frac{3}{2}iv_4 \tag{4.1}$$

$$\hat{\tau}_2 v_2 = \pi(\tau_2)v_2 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_2})v_2|_{\delta=0} = -\frac{3}{2}v_6 \tag{4.2}$$

$$\hat{\tau}_3 v_3 = \pi(\tau_3)v_3 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_3})v_3|_{\delta=0} = 0 \tag{4.3}$$

$$\hat{\tau}_4 v_4 = \pi(\tau_4)v_4 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_4})v_4|_{\delta=0} = iv_{10} \tag{4.4}$$

$$\hat{\tau}_5 v_5 = \pi(\tau_5)v_5 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_5})v_5|_{\delta=0} = v_{10} - \frac{1}{2}v_1 \tag{4.5}$$

$$\hat{\tau}_6 v_6 = \pi(\tau_6)v_6 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_6})v_6|_{\delta=0} = iv_{10} \tag{4.6}$$

$$\hat{\tau}_7 v_7 = \pi(\tau_7)v_7 = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_7})v_7|_{\delta=0} = -v_{10} \tag{4.7}$$

$$\hat{\tau}_8 v_{10} = \pi(\tau_8)v_{10} = \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi(e^{\delta\tau_8})v_{10}|_{\delta=0} = 0. \tag{4.8}$$

We can generalize to the case representations of $\pi : \mathfrak{su}(N) \curvearrowright \mathbb{P}_k$ where $\mathbb{P}_k$ is N complex variables of degree N.

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