

## ZONAL LABELING OF EDGE COMB PRODUCT OF GRAPHS

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**Abstract.** Let  $G = (V, E)$  be a plane graph. Zonal labeling of graph  $G$  is defined as an assignment of the two nonzero elements of the ring  $\mathbb{Z}_3$ , which are 1 and 2, to the vertices of  $G$  such that the sum of the labels of the vertices on the border of each region of the graph is  $0 \in \mathbb{Z}_3$ . A graph  $G$  that possesses such a labeling is termed as zonal graph. This paper will characterize edge comb product graphs that are zonal. The results show that  $P_m \succeq_e C_n$ ,  $C_n \succeq_e C_r$ ,  $S_p \succeq_e C_n$ , and  $S_p \succeq_e F_t$  are zonal in some cases, but not in others.

*Keywords:* Zonal Graph, Zonal Labeling, Edge Comb Product

### 1. Introduction

A graph  $G$  is defined as a pair of  $(V(G), E(G))$ , denoted by  $G = (V(G), E(G))$ , where  $E(G) \subseteq [V(G)]^2 = \{U \subseteq V(G) \mid |U| = 2\}$ , which means the elements of  $E(G)$  are 2-elements subsets of  $V(G)$  [1].  $E(G)$  and  $V(G)$  are respectively denoted edge set and vertex set of  $G$ . The number of vertices of graph  $G$  is denoted by  $|V(G)|$  and  $|E(G)|$  represents the number of edges of the graph. As described in [2], if each two edges of graph  $G$  is not crossing each other, then graph  $G$  is a **plane graph**. Furthermore, certain graphs definition such as cycle, path, star, and fan graph refer to [2] and [3]. According to [4], edge comb product of a graph  $G$  with another graph  $T$ , denoted by  $G \succeq_e T$ , is an operation that takes one copy of graph  $G$  and  $|E(G)|$  copy of graph  $T$ , then places the copies of  $T$  to each edge of graph  $G$ . Thus, the interior regions that bounded by  $V(G)$  and  $V(T)$  are referred as **interiors**, while the outer region that bounded by  $V(G \succeq_e T)$  is referred as **exterior**.

Zonal labeling on plane graphs was first introduced by Chartrand, Egan, and Zhang in [5]. The paper states that a **zonal labeling** of a plane graph  $G$  is an

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assignment of  $\{1, 2\} \subseteq \mathbb{Z}_3$  to the vertices of  $G$  such that the sum of the labels on the boundary of each region in  $G$  is  $0 \in \mathbb{Z}_3$ . Furthermore, graph  $G$  that admits such labeling is termed as **zonal graph**.

The concept of graphs that are nearly zonal has been studied in [6], [7], and [8] and known as **inner zonal**. A graph is classified as an inner zonal graph if the value (the sum of the labels on the boundary) of at least one region is not the zero element in  $\mathbb{Z}_3$ . In [9], graphs of cycle rank two have been described. Furthermore, a known theorem involving zonality of such graphs is provided in the paper. Subsequently, a further study was undertaken on defining a dual notion of cozonal labelings in [10].

The following results were established in [8] and [9].

**Theorem 1.1.** [8] *Every cycle and nontrivial tree is zonal graph.*

**Definition 1.2.** [9] *Let  $G$  be a graph of cycle rank 2 if  $G$  contains a subgraph  $H$ , where:*

- (1)  $H$  consists of two cycles,  $C$  and  $C'$ , with a joint vertex  $u$ , where  $u \in V(C)$  and  $u \in V(C')$  (Type 1),
- (2)  $H$  is constructed by adding a path  $P$  with length at least one that links  $C$  and  $C'$  where each end-vertices of  $P$  lies on  $C$  and  $C'$  respectively (Type 2), or
- (3)  $H$  is a subdivision of graph  $K_4 \setminus \{e\}$ , which means  $G$  consists of three distinct paths  $P_i, 1 \leq i \leq 3, i \in \mathbb{Z}$ , where at least two of the paths  $P_i, 1 \leq i \leq 3, i \in \mathbb{Z}$ , have length greater than or equal to 2 (Type 3).

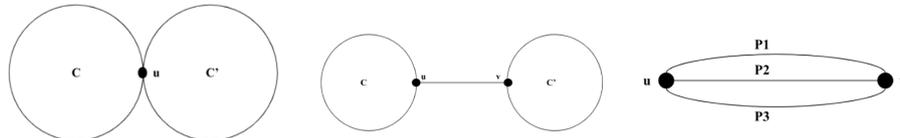


Figure. 1. Type 1, Type 2, and Type 3

Furthermore, graph  $G$  of cycle rank 2 is **minimal** if  $G$  is one of the graphs  $H$  mentioned in (1), (2), and (3).

**Theorem 1.3.** [9] *Let  $G$  be a type 1 graph with cycle rank 2. Graph  $G$  is zonal if and only if graph  $G$  is not minimal.*

The goal of this paper is to determine zonal labelings of edge comb product of graphs, which are edge comb product of path and cycle ( $P_m \supseteq_e C_n$ ), edge comb product of two cycles ( $C_n \supseteq_e C_r$ ), edge comb product of star and cycle ( $S_p \supseteq_e C_n$ ), and edge comb product of star and fan ( $S_p \supseteq_e F_t$ ).

**2. Main Results**

The following illustrations are zonal labeling of  $S_e \supseteq_e C_3$  and  $S_2 \supseteq_e F_4$  where the red colored numbers are the value of each region.

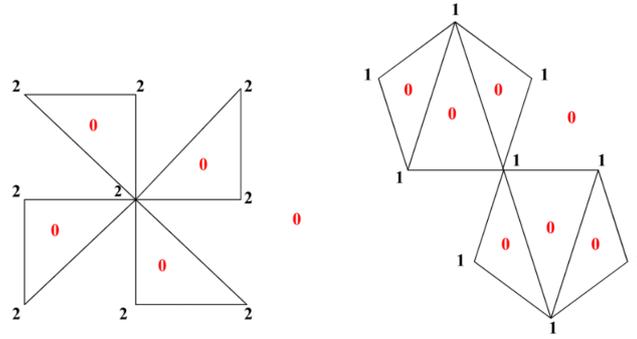


Figure 2.  $S_4 \supseteq_e C_3$  and  $S_2 \supseteq_e F_4$

**Theorem 2.1.** Graph  $C_n \supseteq_e C_r$  is zonal for every  $(n, r)$  except for  $(n, 3)$  where  $n \not\equiv 0 \pmod{3}$ .

**Proof.** Let  $z$  be the labeling of vertices of  $C_n \supseteq_e C_r$  with 1 or 2 in  $\mathbb{Z}_3$ .

(Case 1) Let  $r = 2k + 1$  and  $n = 2c + 1, \forall k \in \mathbb{Z}^+ \setminus \{1\}, \forall c \in \mathbb{Z}^+$ .

If labeling  $z$  assigns 1, 2  $\in \mathbb{Z}_3$  to the vertices on the boundary of the graph's interiors then label 2 can be assigned to  $k + 2$  and  $c + 2$  vertices of  $C_r$  and  $C_n$ , respectively. Hence, the remaining vertices of  $C_r$  and  $C_n$  can be labeled by 1. Consequently, we have:

$$\begin{aligned} z(C_r) &= \sum_{x \in V(C_r)} z(x), \\ &= 2(k + 2) + 1(k - 1), \\ &= 3k + 3 = 0 \in \mathbb{Z}_3, \\ z(C_n) &= \sum_{y \in V(C_n)} z(y), \\ &= 2(c + 2) + 1(c - 1), \\ &= 3c + 3 = 0 \in \mathbb{Z}_3. \end{aligned}$$

According to the definition of edge comb operation, we know that the number of interior  $C_r$  is  $n$ . Therefore, the value of the exterior of  $C_n \supseteq_e C_r$  is acquired as follows:

$$\begin{aligned} z(C_n \supseteq_e C_r) &= \sum_{u \in V(C_n \supseteq_e C_r)} z(u) = n \sum_{x \in V(C_n)} z(x) - \sum_{y \in V(C_n)} z(y), \\ &= (2c + 1)(3k + 3) - 3c - 3, \\ &= 6ck + 3c + 3k = 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 2)  $r = 2k + 1$  and  $n = 2c, \forall k \in \mathbb{Z}^+ \setminus \{1\}, \forall c \in \mathbb{Z}^+$ .

We have the value of the interiors as follows:

$$\begin{aligned} z(C_n) &= \sum_{y \in V(C_r)} z(y), \\ &= 1(c) + 2(c), \\ &= 3c = 0 \in \mathbb{Z}_3, \end{aligned}$$

where the value of  $C_r$  is the same as in the Case 1.

Thus, the exterior's value is as follows:

$$\begin{aligned} z(C_n \supseteq_e C_r) &= \sum_{u \in V(C_n \supseteq_e C_r)} z(u) = n \sum_{x \in V(C_r)} z(x) - \sum_{y \in V(C_n)} z(y), \\ &= (2c)(3k + 3) - 3c, \\ &= 6ck + 3c = 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 3)  $r = 2k$  and  $n = 2c + 1, \forall k \in \mathbb{Z}^+ \setminus \{1\}, c \in \mathbb{Z}^+$ .

Labels 1 and 2 can be assigned to the partition of  $C_r$  vertices, respectively.

Hence, we write:

$$\begin{aligned} z(C_r) &= \sum_{x \in V(C_r)} z(x), \\ &= 1(k) + 2(k), \\ &= 3k = 0 \in \mathbb{Z}_3. \end{aligned}$$

As for  $C_n$ , it has the exact same value as in the first case.

As a result, we acquire the exterior's value by the labeling of the vertices on the boundary of the exterior as follows:

$$\begin{aligned} z(C_n \supseteq_e C_r) &= \sum_{u \in V(C_n \supseteq_e C_r)} z(u) = n \sum_{x \in V(C_r)} z(x) - \sum_{y \in V(C_n)} z(y), \\ &= (2c + 1)(3k) - (3c + 3), \\ &= 6ck + 3k - 3c - 3 = 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 4)  $r = 2k$  and  $n = 2c$  for every  $c$  and  $k$  in  $\mathbb{Z}_3$ .

In this case, the labeling of the interiors' vertices, which include every vertex on the boundary of  $C_n$  and  $C_r$ , is identical to case 2 and 3, respectively.

Thus, the exterior's value is the following:

$$\begin{aligned} z(C_n \supseteq_e C_r) &= \sum_{u \in V(C_n \supseteq_e C_r)} z(u) = n \sum_{x \in V(C_r)} z(x) - \sum_{y \in V(C_n)} z(y), \\ &= 3k + 3c = 0 \in \mathbb{Z}_3. \end{aligned}$$

As a result, if  $r = 3$ , then each vertex of  $C_3$  can only be assigned by the same label (1 or 2), hence the value of each  $C_{r=3}$  of  $C_n \supseteq_e C_r$  is  $0 \in \mathbb{Z}_3$ . Let  $z$  assign label 1 to each vertex of  $C_3$ . Thus, the sum of the labels of the vertices on  $C_n$ , where  $n = 3k + 1, n = 3k + 2$ , or  $n = 3k$ , is  $z(C_{n=3k+1}) = 3k + 1 \neq 0 \in \mathbb{Z}_3, z(C_{n=3k+2}) = 3k + 2 \neq 0 \in \mathbb{Z}_3$ , or  $z(C_{n=3k}) = 3k = 0 \in \mathbb{Z}_3$ , respectively. Consequently, the values of interiors and the exterior of  $C_n \supseteq_e C_r$  are both  $0 \in \mathbb{Z}_3$  for each case, but not  $0 \in \mathbb{Z}_3$  when  $r = 3$ . Thus,  $z$  is a zonal labeling of the graph.  $\square$

**Theorem 2.2.** Graph  $P_m \supseteq_e C_n$  is zonal for every  $(m, n)$  except for  $(m, 3)$  where  $m \not\equiv 2 \pmod{3}$ .

**Proof.** It is obvious that  $P_2 \supseteq_e C_n = C_n$  is zonal by Theorem 1.1. By Definition 1.2, we know that  $P_3 \supseteq_e C_n$  is categorized as type (1) and has cycle rank 2. Hence,  $P_3 \supseteq_e C_n$  is minimal and so  $P_3 \supseteq_e C_n$  is not zonal by Theorem 1.3.

When  $m \geq 4$  and  $n \neq 3$ , the number of joint vertex of  $P_m \supseteq_e C_n$  is  $m - 2$  for every two copies of  $C_n$ .

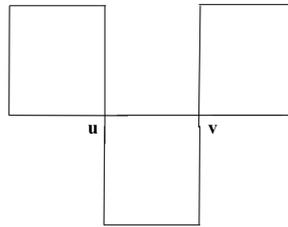


Figure 3.  $P_4 \supseteq_e C_4$  contains two joint vertices, which are  $u$  and  $v$

First, we define a vertex labeling  $z$  of  $P_m \supseteq_e C_n$ , where  $z$  assign label 1 and 2 to every vertex of the graph. We show that  $z$  is a zonal labeling for  $P_m \supseteq_e C_n$  when  $m \geq 4$  and  $n \neq 3$ . To see this, let us consider these four cases.

(Case 1)  $m = 2c$  and  $n = 2k + 1$  for  $c, k \in \mathbb{Z}^+ \setminus \{1\}$ .

Thus, we write the labeling of every  $x \in V(C_n)$  as follows:

$$\begin{aligned} z(C_n) &= \sum_{x \in V(C_n)} z(x), \\ &= 3k + 3 = 0 \in \mathbb{Z}_3. \end{aligned}$$

Since the number of joint vertices of  $P_m \supseteq_e C_n$  is  $2c - 2$ , then we must assign label 1 to its  $c - 1$  joint vertices and label 2 to the remaining joint vertices. Hence, we get the value of the exterior region is 0, where the labeling is the following:

$$\begin{aligned} z(P_m \supseteq_e C_n) &= (m - 1)z(C_n) - 2(c - 1) - 1(c - 1), \\ &= (2c - 1)(3k + 3) - 3c - 3 = 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 2)  $m = 2c$  and  $n = 2k$  for  $c \in \mathbb{Z}^+ \setminus \{1\}$  and  $k \in \mathbb{Z}^+$ .

Since  $n = 2$ , it follows from the proof of the preceding theorem that  $z(C_n) = 3k$  and its joint vertex labeling is the exact same as in the first case, therefore we get:

$$\begin{aligned} z(P_m \supseteq_e C_n) &= (m - 1)z(C_n) - 2(c - 1) - 1(c - 1), \\ &= (2c - 1)(3k) - 3c - 3 = 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 3)  $m = 2c + 1$  and  $n = 2k$  for  $c \in \mathbb{Z}^+ \setminus \{1\}$  and  $k \in \mathbb{Z}^+$ .

We know that each of its interior's value is  $z(C_n) = 3k$ .

Since  $m = 2c + 1$ , then the graph has  $2c - 1$  joint vertices. Thus, we get its exterior's value from the labeling of the exterior's vertices as follows:

$$\begin{aligned} z(P_m \supseteq_e C_n) &= (m - 1)z(C_n) - 2(c - 2) - 1(c + 1) \\ &= 6ck - 3c - 3 = 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 4)  $m = 2c + 1$  and  $n = 2k + 1$  for  $c, k \in \mathbb{Z}^+ \setminus \{1\}$ .

It is pretty straightforward to prove that the interiors' value of the graph, which are  $m$  copies of  $C_n$ , are  $0 \in \mathbb{Z}_3$ . Furthermore, for both Case 3 Case 4, the label of its joint vertices is identical. Hence, it is pretty straightforward to show that the exterior's value is  $0 \in \mathbb{Z}_3$ .

$$\begin{aligned} z(P_m \supseteq_e C_n) &= (m - 1)z(C_n) - 2(c - 2) - 1(c + 1), \\ &= 2c(3k + 3) - 3c - 3 = 0 \in \mathbb{Z}_3. \end{aligned}$$

Thus, we also show that  $P_m \supseteq_e C_3$  is not zonal if  $m \not\equiv 2 \pmod{3}$ . When  $m = 3c$  and  $m = 3c + 1$ , the vertices on the boundary of  $C_3$  can only have the same label, which is label 1 or 2, and so the region's value is  $0 \in \mathbb{Z}_3$ . Hence, it is impossible for the exterior's value to be  $0 \in \mathbb{Z}_3$ . On the other hand, if  $m = 3c + 1$ , then the exterior's value is  $0 \in \mathbb{Z}_3$  with an identical vertices labeling as the preceding cases. Consequently, the values of interiors and exterior of  $P_m \supseteq_e C_n$  are all 0 for each case, except for  $P_m \supseteq_e C_3$  when  $m \not\equiv 2 \pmod{3}$ , therefore the graph is zonal.  $\square$

**Theorem 2.3.** *Graph  $S_p \supseteq_e C_n$  is zonal if and only if  $p \equiv 1 \pmod{3}$ .*

**Proof.** We define a labeling  $z$  such that  $z : V(S_p \supseteq_e C_n) \rightarrow \{1, 2\}$ . Hence, we show that  $z$  is a zonal labeling of graph  $S_p \supseteq_e C_n$  if the number of vertices in the star graph is  $3c + 1$ . Consequently, suppose that  $p = 3c$  or  $p = 3c + 2$ , we show that  $S_p \supseteq_e C_n$  is not zonal.

(Case 1)  $p = 3c$  and  $n = 2k + 1$  for every  $c$  and  $k$  in  $\mathbb{Z}^+$ .

Let  $v$  denoted the joint vertex of the graph, hence label 1 or 2 can be assigned to the joint vertex and so  $z(v) = 1$  (or 2). Thus, the value of its exterior region is as follows:

$$\begin{aligned} z(S_p \supseteq_e C_n) &= p \sum_{x \in V(C_n)} z(x) - (p - 1)z(v), \\ &= 3c(3k + 3) - 3c + 1 \neq 0 \in \mathbb{Z}_3. \end{aligned}$$

It is straightforward to verify that there is no vertex labeling so that the exterior is  $0 \in \mathbb{Z}_3$  for  $z(v) = 2$ .

(Case 2)  $p = 3c$  and  $n = 2k$  for every  $c$  and  $k$  in  $\mathbb{Z}^+$ .

It follows from the preceding case that it is impossible for the exterior region of the graph to have zero value. To simply verify this we write the sum of its vertices labeling, where  $z(v) = 1$ , as follows:

$$\begin{aligned} z(S_p \supseteq_e C_n) &= p \sum_{x \in V(C_n)} z(x) - (p - 1)z(v), \\ &= (3c)(3k) - 3c + 1 \neq 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 3)  $p = 3c + 1$  and  $n = 2k$  for every  $c$  and  $k$  in  $\mathbb{Z}^+$ .

In this case, we know that all of the interior values are  $3k$ , thus the exterior value is the following if  $z(v) = 1$ .

$$\begin{aligned} z(S_p \supseteq_e C_n) &= p \sum_{x \in V(C_n)} z(x) - (p - 1)z(v), \\ &= (3k + 1)(3k) - 3c = 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 4)  $p = 3c + 1$  and  $n = 2k + 1$  for every  $c$  and  $k$  in  $\mathbb{Z}^+$ .

We define the labels of vertices in  $S_p \supseteq_e C_n$  and the sum of the labels as follows:

$$\begin{aligned} z(S_p \supseteq_e C_n) &= p \sum_{x \in V(C_n)} z(x) - (p - 1)z(v), \\ &= (3c + 1)(3k + 3) - 3k = 0 \in \mathbb{Z}_3. \end{aligned}$$

As shown above, the exterior's value,  $z(S_p \supseteq_e C_n)$  is  $0 \in \mathbb{Z}_3$ , where  $\sum_{x \in V(C_n)}$  denotes the value of all the graph's interiors.

(Case 5)  $p = 3c + 2$  and  $n = 2k$  for every  $c$  and  $k$  in  $\mathbb{Z}^+$ .

In this case, we show that  $S_p \supseteq_e C_n$  is not zonal since its exterior's value is not equal to zero in  $\mathbb{Z}_3$ , where:

$$\begin{aligned} z(S_p \supseteq_e C_n) &= p \sum_{x \in V(C_n)} z(x) - (p - 1)z(v), \\ &= (3c + 2)(3k) - 3c + 1 \neq 0 \in \mathbb{Z}_3. \end{aligned}$$

(Case 6)  $p = 3c + 2$  and  $n = 2k + 1$  for every  $c$  and  $k$  in  $\mathbb{Z}^+$ .

It follows from the previous case that every region value of  $S_p \supseteq_e C_n$  is  $0 \in \mathbb{Z}_3$ , except the exterior region. Therefore, we write:

$$\begin{aligned} z(S_p \supseteq_e C_n) &= p \sum_{x \in V(C_n)} z(x) - (p - 1)z(v), \\ &= (3c + 2)(3k + 3) - 3c + 1 \neq 0 \in \mathbb{Z}_3. \end{aligned}$$

Consequently,  $z$  is a zonal labeling of  $S_p \supseteq_e C_n$  when  $p = 3k + 1$ . □

**Theorem 2.4.** *Graph  $S_p \supseteq_e F_t$  if and only if  $|V(S_p \supseteq_e F_t)| = 3k$  for every  $k$  in  $\mathbb{Z}^+$ .*

**Proof.** Let  $z$  be a vertex labeling of  $S_p \supseteq_e F_t$ , which labels every vertex of the graph with label 1 or 2. We show that  $z$  is a zonal labeling of the graph if and only if the number of the vertices is  $3k$ , where  $k \in \mathbb{Z}^+$ .

Suppose that  $|V(S_p \supseteq_e F_t)| = 3k + 1$ . Since every interior of  $S_p \supseteq_e F_t$  is  $C_3$ , it follows that every vertex on the boundary of the interior can only be labeled with label 1 or 2, so that the interior region value is zero. As a result, the exterior's value will be  $3k + 1$  if label 1 is assigned or  $6k + 2$  if label 2 is assigned. Therefore,  $z$  is not a zonal labeling of  $S_p \supseteq_e F_t$  when  $|V(S_p \supseteq_e F_t)| = 3k + 1$ .

If  $|V(S_p \supseteq_e F_t)| = 3k + 2$ , then its exterior value is  $3k + 1$  (or  $6k + 1$ ), where each vertex is assigned by label 1 (or 2). Thus,  $z$  is not a zonal labeling of the graph when the number of its vertices is  $3k + 2$ .

Consequently, both of the exterior and interior values of  $S_p \supseteq_e F_t$  are  $3k$  (or  $6k$ ) if  $z$  assign label 1 (or 2) to every vertex on the region's boundary. Therefore, combining the results of the three cases,  $S_p \supseteq_e F_t$  is zonal if and only if  $|(S_p \supseteq_e F_t)| = 3k$  for every positive integer  $k$ .  $\square$

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