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ON METRIC DIMENSION OF EDGE COMB PRODUCT OF SYMMETRIC GRAPHS

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Abstract. *Consider a finite graph G that is simple, undirected, and connected. Let W be an ordered set of vertices with $|W| = k$. The representation of a vertex v is defined as an ordered k-tuple that consists of the distances from vertex v to each vertex in W. The set W is called a resolving set for G if the k-tuples for any two vertices in G are distinct. The metric dimension of G, denoted by $\dim(G)$, is the smallest possible size of such a set W. In this paper, we determine the metric dimension of edge comb product of trees with complete multipartites or Petersen graphs.*

Keywords: Edge Comb Product, Symmetric, Metric Dimension

1. Introduction

In a connected graph G, the distance $d(u, v)$ between any two vertices u and v is the length of the shortest path connecting them. Consider an ordered set of vertices $W = \{w_1, w_2, \cdots, w_k\}$ in G and a vertex v in G. The representation $r(v|W)$ of v with respect to W is given by the k-tuple $(d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$. If each vertex in G has a unique representation with respect to W, then W is called a resolving set for G. The resolving set with the smallest size is called a basis for G, and its cardinality is referred to as the metric dimension of G, denoted by $\dim(G)$.

It is known that the concept of metric dimension was first introduced independently by Slater [1] and by Harary and Melter [2]. They provided characterizations of the metric dimension for tree graphs, which were later proven using a different approach by Chartrand et al. [3]. Studies have been conducted to consider the metric dimension of regular graphs [4], unicyclic graphs [5], fullerene graphs ([6], [7]),

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zero divisor graphs [8], identified graphs [9], and prime ideal graphs [10]. Several applications of metric dimension can be seen in [11] and [12].

In this paper, we determine the metric dimension of an edge comb product of symmetric graphs. In particular, we consider tree combined with either complete multipartites or Petersen graph which are known to be symmetric.

2. Known Results

A complete multipartite graphs $K_{m\times n}$ is a graph which is obtained by taking m set each containing n vertices such that two vertices in the same set are not adjacent and two vertices in different sets are adjacent. Moreover, a generalized Petersen graph $P(n,m)$ is a graph with a vertex set:

$$V(P(n,m)) = \{u_i, v_i \mid i \in [1,n]\},$$

and an edge set:

$$E(P(n,m)) = \{u_i u_{i+1}, v_i v_{i+m} \mid i \in [1,n]\},$$

where the indices are taken modulo n. The metric dimension of generalized Petersen graph $P(n,2)$ has been determined by Javaid et al. [4].

Theorem 2.1. *[4] Let $P(n,2)$ be the generalized Petersen graph. Then $dim(P(n,2)) = 3$ for $n \geq 5$.*

An *automorphism* is a permutation of the vertices in $V(G)$ that preserves adjacency. A graph is *vertex-transitive* if, for any two distinct vertices v_1 and v_2 in $V(G)$, there exists an automorphism that maps v_1 to v_2. Similarly, a graph is *edge-transitive* if, for any two distinct edges e_1 and e_2 in $E(G)$, there exists an automorphism that maps e_1 to e_2. Furthermore, a graph is called *symmetric* if given any two ordered pairs of adjacent vertices (u_1, v_1) and (u_2, v_2) in G, then there is an automorphism $f : V(G) \to V(G)$ such that

$$f(u_1) = u_2, \; f(v_1) = v_2.$$

Note that symmetric graphs are always vertex-transitive and edge-transitive, hence symmetric is a stronger condition. In particular, Petersen graph $P(5,2)$ is well known for its symmetry, which poses as an interesting graph.

The edge comb product of two graphs G_1 and G_2, denoted by $G_1 \rhd_e G_2$, is constructed by taking one copy of G_1 and $|E(G_1)|$ copies of G_2, and attaching the i-th copy of G_2 to the i-th edge of G_1 [13]. Most of the time, the choice of an edge in the product might alter the result of the graph. However, if the graph is symmetric, then the choice of the edge does not matter. Recently, the metric dimension of an edge comb product of vertex-transitive graphs was determined [14]. One of the results is written below.

Theorem 2.2. *[14] Let T be a tree graph whose size is m, H be a vertex-transitive graph, and p be the cardinality of pendants in T. For any edge $e \in E(H)$, we have:*

$$dim(T \rhd_e H) \geq m(dim(H) - 2) + p.$$

3. Main Results

First, we will consider an edge comb product of tree graph with complete multipartites $K_{m\times n}$ for some positive integers m and n.

Theorem 3.1. *Let $m \geq 2$ and $n \geq 2$ such that $K_{m\times n} \ncong C_4$. Let T be a tree graph with size $h \geq 2$ and p be the number of pendants in G. For any edge $e \in E(K_{m\times n})$, we have:*

$$dim(T \rhd_e K_{m\times n}) = h(m(n-1)-2)+p.$$

Proof. Let $K^e_{m\times n}$ be a complete multipartite graph that is attached to the edge e. This graph has the vertex set

$$V(K^e_{m\times n}) = \{v^e_{i,j} \mid i \in [1,m], j \in [1,n]\},$$

and the edge set

$$E(K^e_{m\times n}) = \{v^e_{i,j}v^e_{i',j'} \mid i,i' \in [1,m], i \neq i', j,j' \in [1,n]\}.$$

Since the metric dimension of $K_{m\times n}$ is $m(n-1)$, then by Theorem 2.2, we have

$$dim(T \rhd_e K_{m\times n}) \geq h(m(n-1)-2)+p.$$

To prove $dim(T \rhd_e K_{m\times n}) \leq h(m(n-1)-2)+p$, we need to set a proper vertex set that can be uniquely determined by most of other vertices. Without loss of generality, for edges $e \in E(T)$ containing pendant let $v^e_{1,2} \in K^e_{m\times n}$ be the vertex identified to other $K^{e'}_{m\times n}$ for some $e' \in E(T)$. Likewise, for edges $e \in E(T)$ which does not contain pendant let $v^e_{1,2}, v^e_{2,2} \in K^e_{m\times n}$ be the vertices identified to other $K^{e'}_{m\times n}$ for some $e' \in E(T)$. Let Q be the vertex set that has $v^e_{1,2}$ for every pendant edges e and $v^e_{1,2}, v^e_{2,2}$ for every non-pendant edges. Next, let

$$S = Q \bigcup_{\substack{e\in E(T)\\ i\in[1,m]}} v_{i,1}.$$

Set $W = V(G \rhd_e K_{m\times n}) \setminus S$. It is obvious that vertices in W has a unique representation. Hence, we can only consider vertices in S. Let x and y be vertices in S. We split the cases as follows.

Case 1. Let $n \geq 3$ or e be a pendant edge. In particular, if $n = 2$, then $m \geq 3$.

Subcase 1.1. Let $x = v^e_{i,k}$ and $y = v^e_{j,l}$ for $i < j$ and $k,l \in \{1,2\}$. We have

$$d(v^e_{i,k}, v^e_{j,n}) = 1 \neq 2 = d(v^e_{j,l}, v^e_{j,n}) \implies r(v^e_{i,k} \mid W) \neq r(v^e_{j,l} \mid W).$$

Subcase 1.2. Let $x = v^e_{i,k}$ and $y \in \{v^{e'}_{j,1}, v^{e'}_{l,2}\}$ with e and e' are adjacent such that $v^{e'}_{i',2} = v^e_{1,2}$ for some $i',k,l \in \{1,2\}$ and $j \geq 2$. If $i \leq m-1$ then

$$d(v^e_{i,k}, v^e_{m,n}) = 1 \neq 2 = d(y, v^e_{m,n}) \implies r(v^e_{i,k} \mid W) \neq r(y \mid W).$$

Else, let $i = m$. This implies

$$d(v^e_{i,k}, v^e_{m-1,n}) = 1 \neq 2 = d(y, v^e_{m-1,n}) \implies r(v^e_{i,k} \mid W) \neq r(y \mid W).$$

Subcase 1.3. Let $x = v^e_{i,k}$ and $y \in \{v^{e'}_{j,1}, v^{e'}_{l,2}\}$ with e and e' are not adjacent, $k, l \in \{1, 2\}$ and $j \geq 2$. This implies

$$d(v^e_{i,k}, v^e_{m,2}) \leq 2 \neq 3 \leq d(y, v^e_{m,2}) \implies r(v^e_{i,k} \mid W) \neq r(y \mid W).$$

Case 2. Let e be a non-pendant edge and $n = 2$. Consequently, $m \geq 3$.
Subcase 2.1. Let $x = v^e_{i,k}$ and $y = v^e_{j,l}$ for $i < j$ and $k, l \in \{1, 2\}$. If $j \geq 3$, then

$$d(v^e_{i,k}, v^e_{j,2}) = 1 \neq 2 = d(v^e_{j,l}, v^e_{j,2}) \implies r(v^e_{i,k} \mid W) \neq r(v^e_{j,1} \mid W).$$

Else, let $j = 2$ which implies $i = 1$. There exists an edge $e^* \in E(T)$ such that $v^{e^*}_{i',2} = v^e_{1,2}$ and $v^{e^*}_{m,2} \in W$. If $k = 1$ then,

$$d(v^e_{1,1}, v^{e^*}_{m,2}) = 1 \neq 2 = d(v^e_{j,l}, v^{e^*}_{m,2}) \implies r(v^e_{1,1} \mid W) \neq r(v^e_{j,l} \mid W).$$

Otherwise, $k = 2$ implying

$$d(v^e_{1,2}, v^{e^*}_{m,2}) = 3 \neq 2 = d(v^e_{j,l}, v^{e^*}_{m,2}) \implies r(v^e_{1,2} \mid W) \neq r(v^e_{j,l} \mid W).$$

Subcase 2.2. Let $x = v^e_{i,k}$ and $y \in \{v^{e'}_{j,1}, v^{e'}_{l,2}\}$ with e and e' are adjacent such that $v^{e'}_{i',2} = v^e_{1,2}$ for some $i', k, l \in \{1, 2\}$ and $j \geq 2$. If $i \leq m - 1$ then

$$d(v^e_{i,k}, v^e_{m,2}) = 1 \neq 2 = d(y, v^e_{m,2}) \implies r(v^e_{i,k} \mid W) \neq r(y \mid W).$$

Else, let $i = m$. There exists an edge $e^* \in E(T)$ such that $v^{e^*}_{i^*,2} = v^e_{1,2}$ for $i^* \in \{1, 2\}$ and $v^{e^*}_{m,2} \in W$. Since T is a tree, then $V(H^{e^*}) \cap V(H^{e'}) = \varnothing$. Therefore,

$$d(v^e_{i,k}, v^{e^*}_{m,2}) = 2 \neq 3 = d(y, v^{e^*}_{m,2}) \implies r(v^e_{i,k} \mid W) \neq r(y \mid W).$$

Subcase 2.3. Let $x = v^e_{i,k}$ and $y \in \{v^{e'}_{j,1}, v^{e'}_{l,2}\}$ with e and e' are not adjacent, $k, l \in \{1, 2\}$ and $j \geq 2$. This implies

$$d(v^e_{i,k}, v^e_{m,2}) \leq 2 \neq 3 \leq d(y, v^e_{m,2}) \implies r(v^e_{i,k} \mid W) \neq r(y \mid W).$$

It can be seen that every representation of vertices in S are distinct. Therefore, $dim(G \rhd_e K_{m \times n}) = h(m(n-1) - 2) + p$. □

An example of a resolving set in an edge comb product of a tree and $K_{3 \times 2}$ is given in Figure 1.

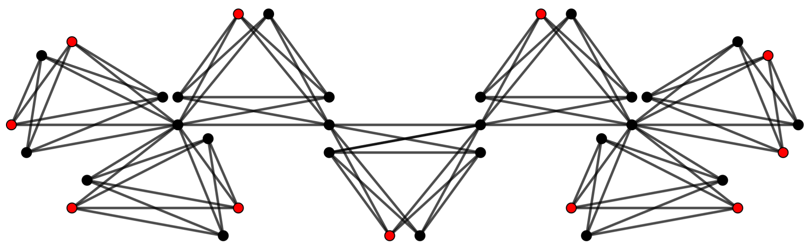

Figure. 1. An edge comb product graph with metric dimension of 11.

Now, we will consider the edge comb product of arbitrary graph with Petersen graph $P(5, 2)$.

Theorem 3.2. *Let T be a tree graph with size $h \geq 2$ and p be the number of pendants in T. For any edge $e \in E(P(5,2))$, we have:*

$$dim(T \rhd_e P(5,2)) = 2h.$$

Proof. Let $e \in E(T)$ be an edge and H^e be a subgraph of $G \rhd_e P(5,2)$ that is isomorphic to $P(5,2)$ and attached to an edge $e \in E(T)$. To prove that $dim(G \rhd_e P(5,2)) \geq 2h$, we will show that every subgraph H^e contains at least two resolving vertices. We can split the problem into two cases.

Case 1. Let e be a pendant edge. Let $v^e_{1,2}$ be a vertex in H^e which is adjacent to vertices outside H^e. It is clear that H^e contains a resolving vertex. Assume that there exists $v^e_{i,j} \in V(H^e)$ as the only resolving vertex in H^e. This implies that $\{v^e_{1,2}, v^e_{i,j}\}$ is a resolving set for $P(5,2)$. This contradicts Theorem 2.1 which states $dim(P(5,2)) = 3$.

Case 2. Let e be a non-pendant edge. Let $v^e_{1,2}$ and $v^e_{2,2}$ be vertices in H^e which are adjacent to vertices outside H^e. Assume that there exists $v^e_{i,j} \in V(H^e)$ as the only resolving vertex in H^e. Let W be the resolving set of $G \rhd_e P(5,2)$. There exists vertices x and y in resolving set W outside of H^e such that $d(x, v^e_{1,2}) = a$ and $d(y, v^e_{2,2}) = b$. By considering only $W' = \{x, y\}$, we have the representation of vertices in H^e below.

$$\begin{aligned}
r(v^e_{1,1} \mid W') &= (a+1, b+2), & r(v^e_{1,2} \mid W') &= (a, b+1),\\
r(v^e_{2,1} \mid W') &= (a+2, b+1), & r(v^e_{2,2} \mid W') &= (a+1, b),\\
r(v^e_{3,1} \mid W') &= (a+2, b+2), & r(v^e_{3,2} \mid W') &= (a+2, b+1),\\
r(v^e_{4,1} \mid W') &= (a+2, b+2), & r(v^e_{4,2} \mid W') &= (a+2, b+2),\\
r(v^e_{5,1} \mid W') &= (a+2, b+2), & r(v^e_{5,2} \mid W') &= (a+1, b+2).
\end{aligned}$$

It could be seen that $v^e_{3,1}, v^e_{4,1}, v^e_{5,1}$ and $v^e_{4,2}$ have the same representation with respect to W'. Since $diam(P(5,2)) = 2$, then

$$\{d(v^e_{i,j}, v^e_{3,1}), d(v^e_{i,j}, v^e_{4,1}), d(v^e_{i,j}, v^e_{5,1}), d(v^e_{i,j}, v^e_{4,2})\} \subseteq \{0,1,2\}$$

This implies that there exists two vertices in $\{v^e_{3,1}, v^e_{4,1}, v^e_{5,1}, v^e_{4,2}\}$ which have the same representation with respect to W. This is a contradiction to the fact that W is a resolving set.

By two preceding cases, it implies that every subgraph H^e contains at least two resolving vertices. Consequently, $dim(G \rhd_e P(5,2)) \geq 2h$.

Next, we will show $dim(G \rhd_e P(5,2)) \leq 2h$. Let W be a resolving set of $G \rhd_e P(5,2)$. Let e and e' be edges in $E(T)$ which are not adjacent. Since for every edge $e \in E(T)$, H^e contains two resolving vertices, then there exists $x \in W \cap H^e$ such that for every $v^e \in V(H^e)$ and $v^{e'} \in V(H^{e'})$ it holds

$$d(x, v^e) \in \{0,1,2\} \qquad d(x, v^{e'}) \in \{2,3,4,5\}$$

Likewise, there exists $y \in W \cap H^{e'}$ such that for every $v^e \in V(H^e)$ and $v^{e'} \in V(H^{e'})$ it holds

$$d(y, v^{e'}) \in \{0,1,2\} \qquad d(y, v^e) \in \{2,3,4,5\}$$

Let v^e be a vertex with $d(x, v^e) = d(y, x^e) = 2$. If y is a cut-vertex, then $d(x, v^e) < d(x, v^{e'})$. Else, if y is not a cut-vertex, then v^e must be cut-vertex. This implies $d(x, v^e) < d(x, v^{e'})$. In any case, there will be only one vertex in $V(H^e \cup H^{e'})$ which has $d(x, v^e) = d(y, x^e) = 2$. Consequently, for any resolving set W, every representation of v^e and $v^{e'}$ will always be distinct.

Therefore, for a given set W, determining vertices in $H^e \cup H^{e'}$ in any two adjacent edges $e, e' \in E(T)$ have distinct representation with respect to W is sufficient to check whether W is a resolving set. Let e and e' be edges in $E(T)$ which are adjacent. We can consider a representation of each $H^e \cup H^{e'}$ with respect to a subset of W. This part can be split into three cases.

Case 1. Let e and e' be pendant edges. Without loss of generality, let $v^e_{1,2} = v^{e'}_{1,2}$. By choosing $W' = \{v^e_{2,1}, v^e_{3,1}, v^{e'}_{2,1}, v^{e'}_{3,1}\}$, we have

$$r(v^e_{1,1} \mid W') = (2, 1, 3, 3), \quad r(v^{e'}_{1,1} \mid W') = (3, 3, 2, 1),$$
$$r(v^e_{2,1} \mid W') = (0, 2, 4, 4), \quad r(v^{e'}_{2,1} \mid W') = (4, 4, 0, 2),$$
$$r(v^e_{3,1} \mid W') = (2, 0, 4, 4), \quad r(v^{e'}_{3,1} \mid W') = (4, 4, 2, 0),$$
$$r(v^e_{4,1} \mid W') = (1, 2, 4, 4), \quad r(v^{e'}_{4,1} \mid W') = (4, 4, 1, 2),$$
$$r(v^e_{5,1} \mid W') = (1, 1, 4, 4), \quad r(v^{e'}_{5,1} \mid W') = (4, 4, 1, 1),$$
$$r(v^e_{1,2} \mid W') = (2, 2, 2, 2), \quad r(v^{e'}_{2,2} \mid W') = (3, 3, 1, 2),$$
$$r(v^e_{2,2} \mid W') = (1, 2, 3, 3), \quad r(v^{e'}_{3,2} \mid W') = (4, 4, 2, 1),$$
$$r(v^e_{3,2} \mid W') = (2, 1, 4, 4), \quad r(v^{e'}_{4,2} \mid W') = (4, 4, 2, 2),$$
$$r(v^e_{4,2} \mid W') = (2, 2, 4, 4), \quad r(v^{e'}_{5,2} \mid W') = (3, 3, 2, 2).$$
$$r(v^e_{5,2} \mid W') = (2, 2, 3, 3),$$

Case 2. Let either one of e or e' be a pendant edge. Without loss of generality, let e be a pendant edge and let $v^e_{1,2} = v^{e'}_{1,2}$. There exists a vertex x in resolving set W outside of $H^e \cup H^{e'}$ such that all shortest paths of x and any vertices in $H^e \cup H^{e'}$ are containing $v^{e'}_{2,2}$. Let $a = d(x, v^{e'}_{2,2})$. Set $W' = \{v^e_{2,1}, v^e_{3,1}, v^{e'}_{1,1}, v^{e'}_{2,1}, x\}$. This implies

$$r(v^e_{1,1} \mid W') = (2, 1, 2, 3, a + 2), \quad r(v^{e'}_{1,1} \mid W') = (3, 3, 0, 2, a + 2),$$
$$r(v^e_{2,1} \mid W') = (0, 2, 3, 4, a + 3), \quad r(v^{e'}_{2,1} \mid W') = (4, 4, 2, 0, a + 1),$$
$$r(v^e_{3,1} \mid W') = (2, 0, 3, 4, a + 3), \quad r(v^{e'}_{3,1} \mid W') = (4, 4, 1, 2, a + 2),$$
$$r(v^e_{4,1} \mid W') = (1, 2, 3, 4, a + 3), \quad r(v^{e'}_{4,1} \mid W') = (4, 4, 1, 1, a + 2),$$
$$r(v^e_{5,1} \mid W') = (1, 1, 3, 4, a + 3), \quad r(v^{e'}_{5,1} \mid W') = (4, 4, 2, 1, a + 2),$$
$$r(v^e_{1,2} \mid W') = (2, 2, 1, 2, a + 1), \quad r(v^{e'}_{2,2} \mid W') = (3, 3, 2, 1, a),$$
$$r(v^e_{2,2} \mid W') = (1, 2, 2, 3, a + 2), \quad r(v^{e'}_{3,2} \mid W') = (4, 4, 2, 2, a + 1),$$
$$r(v^e_{3,2} \mid W') = (2, 1, 3, 4, a + 3), \quad r(v^{e'}_{4,2} \mid W') = (4, 4, 2, 2, a + 2),$$
$$r(v^e_{4,2} \mid W') = (2, 2, 3, 4, a + 3), \quad r(v^{e'}_{5,2} \mid W') = (3, 3, 2, 2, a + 2).$$
$$r(v^e_{5,2} \mid W') = (2, 2, 2, 3, a + 2),$$

Figure 2 illustrates the distinct representations of $H^e \cup H^{e'}$ for this case.

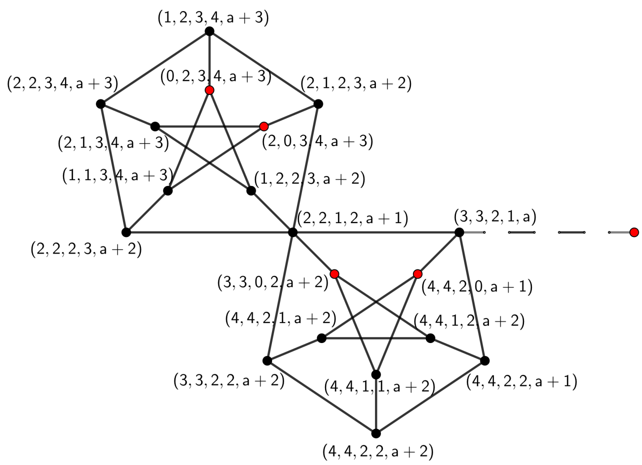

Figure. 2. Representations of two Petersen graphs $P(5,2)$ in a $T \rhd_e P(5,2)$.

Case 3. Let neither e nor e' be a pendant edge. Let $v^e_{1,2} = v^{e'}_{1,2}$. There exists a vertex x in resolving set W outside of $H^e \cup H^{e'}$ such that all shortest paths of x and any vertices in $H^e \cup H^{e'}$ are containing $v^{e'}_{2,2}$. Similarly, there exists a vertex y in resolving set W outside of $H^e \cup H^{e'}$ such that all shortest paths of x and any vertices in $H^e \cup H^{e'}$ are containing $v^e_{1,2}$. Let $a = d(x, v^{e'}_{2,2})$ and $b = d(y, v^e_{1,2})$. By picking $W' = \{v^e_{1,1}, v^e_{2,1}, v^{e'}_{1,1}, v^{e'}_{2,1}, x, y\}$, it follows that

$$\begin{aligned}
r(v^e_{1,1} \mid W') &= (0,2,3,4,a+3,b+1), & r(v^{e'}_{1,1} \mid W') &= (3,2,0,2,a+2,b+2),\\
r(v^e_{2,1} \mid W') &= (2,0,2,3,a+2,b+2), & r(v^{e'}_{2,1} \mid W') &= (4,3,2,0,a+1,b+3),\\
r(v^e_{3,1} \mid W') &= (1,2,3,4,a+3,b+2), & r(v^{e'}_{3,1} \mid W') &= (4,3,1,2,a+2,b+3),\\
r(v^e_{4,1} \mid W') &= (1,1,3,4,a+3,b+2), & r(v^{e'}_{4,1} \mid W') &= (4,3,1,1,a+2,b+3),\\
r(v^e_{5,1} \mid W') &= (2,1,3,4,a+3,b+2), & r(v^{e'}_{5,1} \mid W') &= (4,3,2,1,a+2,b+3),\\
r(v^e_{1,2} \mid W') &= (1,2,2,3,a+2,b), & r(v^{e'}_{2,2} \mid W') &= (3,2,2,1,a,b+2),\\
r(v^e_{2,2} \mid W') &= (2,1,1,2,a+1,b+1), & r(v^{e'}_{3,2} \mid W') &= (4,3,2,2,a+1,b+3),\\
r(v^e_{3,2} \mid W') &= (2,2,2,3,a+2,b+2), & r(v^{e'}_{4,2} \mid W') &= (4,3,2,2,a+2,b+3),\\
r(v^e_{4,2} \mid W') &= (2,2,3,4,a+3,b+2), & r(v^{e'}_{5,2} \mid W') &= (3,2,2,2,a+2,b+2).\\
r(v^e_{5,2} \mid W') &= (2,2,3,4,a+3,b+1), & &
\end{aligned}$$

Let W be the union of W' from all $H^e \cup H^{e'}$ for every adjacent edges e and e'. Since every vertices will have distinct representation, then W is a resolving set with $|W| = 2h$. Hence, $dim(G \rhd_e P(5,2)) = 2h$. □

We present an example of a resolving set in an edge comb product of a tree and $P(5,2)$ in Figure 3.

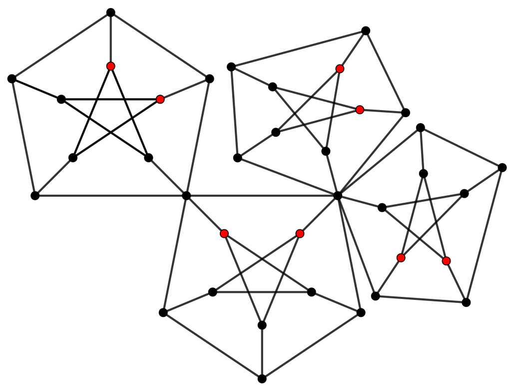

Figure. 3. An edge comb product graph with metric dimension 8.

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