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THE LOCATING-CHROMATIC NUMBER FOR CERTAIN BARBELL OPERATION ON PIZZA GRAPH AND ITS SUBDIVISION

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Abstract. In graph theory, the locating-chromatic number is a parameter that characterizes the minimum number of colors required to assign to the vertices of a graph such that each vertex can be uniquely identified by its color and the colors of its neighbors. The locating-chromatic number of a graph refers to a concept in graph coloring, which involves assigning colors to the vertices of a graph in such a way that adjacent vertices do not share the same color. It represents the minimum number of colors needed for a proper vertex coloring. This study investigates the locating-chromatic number of certain barbell operation on pizza graphs and its subdivisions.

Keywords : Locating-chromatic number (L-cn), barbell operation, pizza graphs, subdivision

1. Introduction

Chartrand et al. [1] introduced the examination of the partition dimension of connected graphs in their study, aiming to discover an innovative approach to address the challenge of determining the metric dimension in graphs. Metric dimensions play a practical role in guiding a robot represented by a graph during navigation [2], as well as in addressing the challenge of classifying chemical data, which involves determining an effective representation for a set of chemical compounds and ensuring distinct representations for different compounds [3]. The concept of locating-chromatic number is a fundamental aspect of graph theory, introduced by Chartrand et al. [4] in 2002. It involves assigning colors to the vertices of a graph so that the colors of its neighbors can uniquely determine each vertex. This concept has applications in various fields, from robotics to chemical data classification.

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The locating-chromatic number, a concept in graph theory, was first established by Chartrand and his collaborators [4] in 2002. This involves two derived graph concepts: coloring vertices and exploring the graph's partition dimension. The locating-chromatic number for a graph is defined as follows: Let $B = (V, E)$ be a connected graph and e be a proper s-coloring of B with color $\{1, 2, \cdots, s\}$. Let $\Pi = \{T_1, T_2, \cdots, T_s\}$ be a partition of $V(B)$ which is induced by coloring e. The color code $c_\Pi(w)$ of w is the ordered s-tuple $(d(w, T_1), d(w, T_2), \cdots, d(w, T_s))$ where $d(w, T_j) = \min\{d(w, x) : x \in T_j\}$ for any $j \in \{1, 2, \cdots, s\}$. If all distinct vertices of B have distinct color codes, then e is called k-locating coloring of B. The locating-chromatic number, represented by $\chi_L(B)$, is the smallest s such that B has a locating s-coloring. The determination of the locating-chromatic number has also been carried out on the prism graph with a pendant vertex [5], the Buckminsterfullerene graph B_{60} [6], the prism graph $C_{n,n,n}$ [7] and the thorn graph derived from the wheel graph W_3 [8].

Irawan et al. [9] identify a method for calculating the locating-chromatic number of an origami graph and its subdivision, focusing along an outer edge. Asmiati et al. [10] discovered the exploration of the locating-chromatic number in certain operations involving origami graphs. Our exploration centers on a unique class of graphs arising from the amalgamation of barbell operation and pizza graphs, introducing a novel perspective to the existing body of literature. The primary objective of this study is to determine the locating-chromatic number for specific instances of the barbell operation applied to pizza graphs and its subdivisions. The next pizza graph definition is taken from [11]. A pizza graph P_{Z_w} is a graph with $V(P_{Z_w}) = \{a, b_r, c_r : r \in \{1, \cdots, w\}\}$ and $E(P_{Z_w}) = \{ab_r, b_rc_r : r \in \{1, \cdots, w\}\} \cup \{c_rc_{r+1} : r \in \{1, \cdots, w-1\}\} \cup \{c_wc_1\}$.

The subsequent theorems serve as fundamental principles for establishing the lower bound of the locating-chromatic number of a graph. The collection of neighboring vertices of a vertex s in B is represented by $N(s)$.

Theorem 1.1. [4] *Suppose e represents a locating coloring in a connected graph B. If r and s are distinct vertices of B such that $d(r, z) = d(s, z)$ for all $z \in V(B) \setminus \{r, s\}$, then $e(r) \neq e(s)$. In particular, if r and s are non-adjacent vertices such that $N(r) \neq N(s)$, then $e(r) \neq e(s)$.*

Theorem 1.2. [12] *The locating-chromatic number of the pizza graph is 4 for $w = 3$ and w for $w \geq 4$.*

2. Results and Discussion

The following subsection describes the locating-chromatic number for certain barbell operation on pizza graphs and its subdivisions.

2.1. *The locating-chromatic number of certain barbell operation of pizza graphs*

Theorem 2.1. *Let $B_{P_{z_w}}$ be a certain barbell operation of pizza graphs for $w \geq 3$. Then the locating-chromatic number of $B_{P_{z_w}}$ is 5 for $w = 3$ and $w + 1$ for $w \geq 4$.*

Proof. A certain barbell operation of pizza graphs $B_{P_{z_w}}$ is a graph with:

$$\begin{aligned} V(B_{P_{z_w}}) &= \{a_1, a_2, b_r, b_{w+r}, c_r, c_{w+r} : r \in \{1, \cdots, w\}\}, \\ E(B_{P_{z_w}}) &= \{a_1 b_r, a_2 b_{w+r}, b_r c_r, b_{w+r} c_{w+r} : r \in \{1, \cdots, w-1\}\} \\ &\quad \cup \{c_{w+r} c_{w+r+1} : r \in \{1, \cdots, w-1\}\} \cup \{c_w c_1, c_{2w} c_{w+1}\} \cup \{c_w c_{w+1}\}. \end{aligned}$$

Let us distinguish between two cases to prove the locating-chromatic number of certain barbell operation on pizza graphs.

Case 1 For $w = 3$.
First, we aim to identify the lower bound of $\chi_L(B_{P_{z_3}})$. Considering that the certain barbell operation of pizza graphs $B_{P_{z_3}}$ contains the pizza graph P_{z_3}, as per Theorem 1.2 we have the locating-chromatic number of the pizza graph is 4. Next, we will show that 4 colors are not enough. we apply coloring t, utilizing 4 colors in the following way.

$$T_1 = \{b_3, b_6, c_1, c_4\}; T_2 = \{b_1, b_4, c_2, c_5\}; T_3 = \{b_2, b_5, c_3, c_6\}; T_4 = \{a_2, a_1\}.$$

There must be at least two identical color codes, namely $t_\Pi(a_1) = t_\Pi(a_2)$. It can be easily shown that for any barbell operation on the pizza graph $B_{P_{z_3}}$, there must be two vertices with the same color code, which leads to a contradiction. Thus, we have that $\chi_L(B_{P_{z_3}}) \geq 5$.
Following that, we established that the upper bound of $\chi_L(B_{P_{z_3}})$ is 5. To show that $\chi_L(B_{P_{z_3}}) \leq 5$, we apply coloring t, utilizing 5 colors in the following manner.

$$T_1 = \{b_3, b_6, c_1, c_4\}; T_2 = \{b_1, b_4, c_2, c_5\}; T_3 = \{b_2, b_5, c_3, c_6\}; T_4 = \{a_2\}; T_5 = \{a_1\}.$$

Through the use of coloring t we determine the color codes of $V(B_{P_{z_3}})$ as detailed below.

$$\begin{aligned} &t_\Pi(a_1) = (1,1,1,5,0),\ t_\Pi(a_2) = (1,1,1,0,5),\ t_\Pi(b_1) = (1,0,2,5,1), \\ &t_\Pi(b_2) = (2,1,0,5,1),\ t_\Pi(b_3) = (0,2,1,4,1),\ t_\Pi(b_4) = (1,0,2,1,4), \\ &t_\Pi(b_5) = (2,1,0,1,5),\ t_\Pi(b_6) = (0,2,1,1,5),\ t_\Pi(c_1) = (0,1,1,4,2), \\ &t_\Pi(c_2) = (1,0,1,4,2),\ t_\Pi(c_3) = (1,1,0,3,2),\ t_\Pi(c_4) = (0,1,1,2,3), \\ &t_\Pi(c_5) = (1,0,1,2,4),\ t_\Pi(c_6) = (1,1,0,2,4). \end{aligned}$$

Very clearly, the color codes of all vertices in $B_{P_{z_3}}$ are different, then t is a locating coloring. So $\chi_L(B_{P_{z_3}}) \leq 5$.

Case 2 For $w \geq 4$.
First, we will determine the lower bound of $\chi_L(B_{P_{z_w}})$ for $w \geq 4$. Considering that the specific barbell operation of pizza graphs $B_{P_{z_w}}$ contains two isomorphic copies of the pizza graph P_{z_w}, as per Theorem 1.2 we have $\chi_L(B_{P_{z_w}}) \geq w$, for $w \geq 4$.
Next, suppose that t is a locating coloring using w colors, $t : V(B_{P_{z_w}}) \rightarrow \{1, 2, \cdots, w\}$. To determine the lower bound, we will show that w colors

are insufficient. If:

$$t(a_1) = t(a_2) = w,$$
$$t(b_r) = t(b_{w+r}) = \begin{cases} r+1, & \text{for } r \in \{1, \cdots, w-2\}, \\ 1, & \text{for } r \in \{w-1, \cdots, w\}, \end{cases}$$
$$t(c_r) = t(c_{w+r}) = r, \text{ for } r \in \{1, \cdots, r\},$$

then there must be at least two identical color codes, namely $t_\Pi(a_1) = t_\Pi(a_2)$. It can be easily shown that for any barbell operation on the pizza graph $B_{P_{z_w}}$, there must be two vertices with the same color code, which leads to a contradiction. Thus, we have that $\chi_L(B_{P_{z_w}}) \geq w+1$.

Subsequently, to show that $w+1$ serves as an upper bound for the locating-chromatic number in specific barbell operations of the pizza graph $B_{P_{z_w}}$, it is enough to establish the existence of a locating coloring $t : V(B_{P_{z_w}}) \to \{1, 2, \cdots, w+1\}$. For $w \geq 4$, we formulate the coloring t as follows.

$$t(a_1) = w+1,$$
$$t(a_2) = w,$$
$$t(b_r) = t(b_{w+r}) = \begin{cases} r+1, & \text{for } r \in \{1, \cdots, w-2\}, \\ 1, & \text{for } r \in \{w-1, \cdots, w\}, \end{cases}$$
$$t(c_r) = t(c_{w+r}) = r, \text{ for } r \in \{1, \cdots, w\}.$$

Through the use of coloring t we determine the color codes of $V(B_{P_{z_w}})$ as follows.

$$t_\Pi(a_1) = \begin{cases} 0, & \text{for } (w+1)^{th} \text{ component}, \\ 2, & \text{for } w^{th} \text{ component}, \\ 1, & \text{otherwise}, \end{cases}$$

$$t_\Pi(a_2) = \begin{cases} 0, & \text{for } w^{th} \text{ component}, \\ 5, & \text{for } (w+1)^{th} \text{ component}, \\ 1, & \text{otherwise}. \end{cases}$$

$$t_\Pi(b_r) = \begin{cases} 0, & \text{for } (r+1)^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ & \text{for } 1^{st} \text{ component}, \ r \in \{w-1, \cdots, w\}, \\ 1, & \text{for } r^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ & \text{for } (w+1)^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ & \text{for } w^{th} \text{ component}, \ r = w, \text{ and } 1^{st} \text{ component}, \ r = 1, \\ 2, & \text{for } w^{th} \text{ component}, \ r = 1, \\ & \text{for } w^{th} \text{ component}, \ r = w-1, \\ & \text{for } (r+2)^{th} \text{ component}, \ r \in \{1, \cdots, w-3\}, \\ & \text{for } 1^{st} \text{ component}, \ r \in \{2, \cdots, w-2\}, \\ 3, & \text{otherwise}. \end{cases}$$

$$t_\Pi(b_{w+r}) = \begin{cases} 0, & \text{for } (r+1)^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ & \text{for } 1^{st} \text{ component}, \ r \in \{w-1, \cdots, w\}, \\ 1, & \text{for } r^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ & \text{for } w^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ 4, & \text{for } (w+1)^{th} \text{ component}, \ r = 1, \\ 5, & \text{for } (w+1)^{th} \text{ component}, \ r = 2, \\ & \text{for } (w+1)^{th} \text{ component}, \ r = w, \\ 6, & \text{for } (w+1)^{th} \text{ component}, r \in \{3, \cdots, w-1\}, \\ 2, & \text{otherwise.} \end{cases}$$

$$t_\Pi(c_r) = \begin{cases} 0, & \text{for } r^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ 1, & \text{for } (r+1)^{th} \text{ component}, \ r \in \{1, \cdots, w-1\}, \\ & \text{for } (r-1)^{th} \text{ component}, \ r \in \{2, \cdots, w\}, \\ & \text{for } 1^{st} \text{ component}, \ r \in \{w-1, \cdots, w\}, \\ 2 & \text{for } (w+1)^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ & \text{for } (r+2)^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ & \text{for } (r-2)^{th} \text{ component}, \ r \in \{w-1, \cdots, w\}, \ w \geq 5, \\ & \text{for } 2^{nd} \text{ component}, \ r = w, \\ 4, & \text{for } w^{th} \text{ component}, \ r \in \{5, \cdots, w-4\}, \ w \geq 9, \\ r, & \text{for } w^{th} \text{ component}, \ r \in \{1, 2, 3\}, \\ w-r, & \text{for } w^{th} \text{ component}, \ r \in \{w-3, \cdots, w-1\}, \\ 3, & \text{otherwise.} \end{cases}$$

$$t_\Pi(c_{w+r}) = \begin{cases} 0, & \text{for } r^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ 1, & \text{for } (r+1)^{th} \text{ component}, \ r \in \{1, \cdots, w-1\}, \\ & \text{for } (r-1)^{th} \text{ component}, \ r \in \{2, \cdots, w\}, \\ & \text{for } w^{th} \text{ component}, \ r = 1, \\ 2, & \text{for } (r+2)^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ & \text{for } (r-2)^{th} \text{ component}, \ r \in \{w-1, \cdots, w\}, \ w \geq 5, \\ & \text{for } (w-1)^{th} \text{ component}, \ r = 1, \\ & \text{for } w^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ 4, & \text{for } (w+1)^{th} \text{ component}, \ r = 2, \\ & \text{for } (w+1)^{th} \text{ component}, \ r = w, \\ 5, & \text{for } (w+1)^{th} \text{ component}, \ r = 3, \ w \geq 5, \\ & \text{for } (w+1)^{th} \text{ component}, \ r = w-1, \ w \geq 5, \\ 6, & \text{for } (w+1)^{th} \text{ component}, \ r = 3, \ w \geq 6, \\ & \text{for } (w+1)^{th} \text{ component}, \ r = w-1, \ w \geq 6, \\ 7, & \text{for } (w+1)^{th} \text{ component}, \ r = w-1, \ w \geq 8, \\ & \text{for } w^{th} \text{ component}, \ r \in \{w-3, \cdots, w-1\}, \\ 3, & \text{otherwise.} \end{cases}$$

It is clear that the color codes of all vertices in $V(B_{P_{z_w}})$ are different. Then the coloring t is a locating coloring. Thus, $\chi_L(B_{P_{z_w}}) \leq w + 1$. □

2.2. *The locating chromatic number of subdivision of certain barbell operation of pizza graph*

Theorem 2.2. *Let $B^*_{P_{z_w}}$ be a subdivision of certain barbell operation of pizza graphs for $w \geq 3$. Then the locating chromatic number of $B^*_{P_{z_w}}$ is 5 for $w = 3$ and $w + 1$ for $w \geq 4$.*

Proof. A subdivision of certain barbell operation of pizza graphs $B^*_{P_{z_w}}$ is a graph with:

$$\begin{aligned} V(B^*_{P_{z_w}}) &= \{a_1, a_2, b_r, b_{w+r}, c_r, c_{w+r} : r \in \{1, \cdots, w\}\} \cup \{s_r : r \in \{1, \cdots, n\}\}, \\ E(B^*_{P_{z_w}}) &= \{a_1 b_r, a_2 b_{w+r}, b_r c_r, b_{w+r} b_{w+r+1} r \in \{1, \cdots, w-1\}\} \\ &\quad \cup \{c_{w+r} c_{w+r+1} : r \in \{1, \cdots, w-1\}\} \cup \{c_m c_1, c_{2m} c_{w+1}\} \cup \{c_w c_{w+1}\} \\ &\quad \cup \{c_m s_1, s_r c_{w+1}\} \cup \{s_r s_{r+1} : r \in \{1, \cdots, w\}\}. \end{aligned}$$

First, we will determine the lower bound of $\chi_L(B^*_{P_{z_w}})$. Since the subdivision of certain barbell operations of the pizza graph, contains certain barbell operation of the pizza graph, then according to Theorem 2.1, it is evident that $\chi_L(B^*_{P_{z_w}}) \geq \chi_L(P_{z_w}) + 1$.

Let us distinguish between two cases to establish the upper bound for the locating chromatic number of certain barbell operation subdivisions on pizza graphs.

Case 1 For $w = 3$.
We will established that the upper bound of $\chi_L(B^*_{P_{z_3}})$ is 5. To show that $\chi_L(B^*_{P_{z_3}}) \leq 5$, we apply the coloring t, utilizing 5 colors in the following manner.

$$\begin{aligned} T_1 &= \{b_3, b_6, c_1, c_4\}, \\ T_2 &= \{b_1, b_4, c_2, c_5\} \cup \{s_r \mid \text{ for odd } r, r \geq 1\}, \\ T_3 &= \{b_2, b_5, c_3, c_6\} \cup \{s_r \mid \text{ for even } r, r \geq 2\}, \\ T_4 &= \{a_2\}, \\ T_5 &= \{a_1\}. \end{aligned}$$

By using the coloring t, we obtain the color codes of $V(B^*_{P_{z_3}})$ as follows.

$t_\Pi(a_1) = (1, 1, 1, k + 4, 0)$, $t_\Pi(a_2) = (1, 1, 1, 0, k + 5)$, $t_\Pi(b_1) = (1, 0, 2, k + 6, 1)$,
$t_\Pi(b_2) = (2, 1, 0, k + 6, 1)$, $t_\Pi(b_3) = (0, 2, 1, k + 4, 1)$, $t_\Pi(b_4) = (1, 0, 2, 1, k + 4)$,
$t_\Pi(b_5) = (2, 1, 0, 1, k + 6)$, $t_\Pi(b_6) = (0, 2, 1, 1, k + 6)$, $t_\Pi(c_1) = (0, 1, 1, k + 4, 2)$,
$t_\Pi(c_2) = (1, 0, 1, k + 4, 2)$, $t_\Pi(c_3) = (1, 1, 0, k + 3, 2)$, $t_\Pi(c_4) = (0, 1, 1, 2, k + 3)$,
$t_\Pi(c_5) = (1, 0, 1, 2, k + 4)$, $t_\Pi(c_5) = (1, 0, 1, 2, k + 3)$ for $k = 1$,
$t_\Pi(c_6) = (1, 1, 0, 2, k + 4)$,

$$t_{\Pi}(s_r) = \begin{cases} (r+1,0,1,k-r+3,r+2), & \text{for odd } r,\ r \leq \left\lfloor \frac{k}{2} \right\rfloor,\ k \geq 2, \\ (r+1,1,0,k-r+3,r+2), & \text{for even } r,\ r \leq \left\lfloor \frac{k}{2} \right\rfloor,\ k \geq 2, \\ (k-r+1,0,1,k-r+3,r+2), & \text{for odd } r,\ r > \left\lfloor \frac{k}{2} \right\rfloor,\ k \geq 2, \\ (k-r+1,1,0,k-r+3,r+2), & \text{for even } r,\ r > \left\lfloor \frac{k}{2} \right\rfloor,\ k \geq 2. \end{cases}$$

As the color codes for all vertices $B^*_{P_{z_3}}$ are different, then t is a locating-chromatic coloring. Thus $\chi_L(B^*_{P_{z_3}}) \leq 5$.

Case 2 For $w \geq 4$.

To show that $w+1$ serves as an upper bound for the locating chromatic number of certain barbell operation of the pizza graph ($B^*_{P_{z_w}}$), it is enough to establish the existence of a locating coloring $t : V(B^*_{P_{z_w}}) \to \{1, 2, \cdots, w+1\}$. For $w \geq 4$, we formulate the function t. Subsequently, we ascertain the upper bound of $\chi_L(B^*_{P_{z_w}})$ in the following manner.

$$t(a_1) = w+1,\ t(a_2) = w,$$

$$t(b_r) = t(b_{w+r}) = \begin{cases} r+1, & \text{for } r \in \{1, \cdots, w-2\}, \\ 1, & \text{for } r \in \{w-1, \cdots, w\}, \end{cases}$$

$$t(c_r) = t(c_{w+r}) = r,\ \text{for } r \in \{1, \cdots, w\},$$

$$t(s_r) = \begin{cases} w-1, & \text{for odd } r,\ k \geq 1, \\ w, & \text{for even } r,\ k \geq 2. \end{cases}$$

Through the use of coloring t we determine the color codes of $V(B^*_{P_{z_w}})$ as detailed below.

$$t_{\Pi}(a_1) = \begin{cases} 0, & \text{for } (w+1)^{th} \text{ component}, \\ 2, & \text{for } w^{th} \text{ component}, \\ 1, & \text{otherwise}. \end{cases}$$

$$t_{\Pi}(a_2) = \begin{cases} 0, & \text{for } w^{th} \text{ component}, \\ k+5, & \text{for } (w+1)^{th} \text{ component}, \\ 1, & \text{otherwise}. \end{cases}$$

$$t_{\Pi}(b_r) = \begin{cases} 0, & \text{for } (r+1)^{th} \text{ component},\ r \in \{1, \cdots, w-2\}, \\ & \text{for } 1^{st} \text{ component},\ r \in \{w-1, \cdots, w\}, \\ 1, & \text{for } r^{th} \text{ component},\ r \in \{1, \cdots, w\}, \\ & \text{for } (w+1)^{th} \text{ component},\ r \in \{1, \cdots, w\} \\ & \text{for } w^{th} \text{ component},\ r = w, \\ & \text{for } 1^{st} \text{ component},,\ r = 1, \\ 2, & \text{for } w^{th} \text{ component},\ r = 1, \\ & \text{for } w^{th} \text{ component},\ r = w-1, \\ & \text{for } (r+2)^{th} \text{ component},\ r \in \{1, \cdots, w-3\}, \\ & \text{for } 1^{st} \text{ component},\ r \in \{2, \cdots, w-2\}, \\ 3, & \text{otherwise}. \end{cases}$$

$$t_\Pi(b_{w+r}) = \begin{cases} 0, & \text{for } (r+1)^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ & \text{for } 1^{st} \text{ component}, \ r \in \{w-1, \cdots, w\}, \\ 1, & \text{for } r^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ & \text{for } w^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ k+4, & \text{for } (w+1)^{th} \text{ component}, \ r = 1, \\ k+5, & \text{for } (w+1)^{th} \text{ component}, \ r = 2, \\ & \text{for } (w+1)^{th} \text{ component}, \ r = w, \\ k+6, & \text{for } (w+1)^{th} \text{ component}, \ r \in \{3, \cdots, w-1\}, \\ 2, & \text{otherwise.} \end{cases}$$

$$t_\Pi(c_r) = \begin{cases} 0, & \text{for } r^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ 1, & \text{for } (r+1)^{th} \text{ component}, \ r \in \{1, \cdots, w-1\}, \\ & \text{for } (r-1)^{th} \text{ component}, \ r \in \{2, \cdots, w\} \\ & \text{for } 1^{st} \text{ component}, \ r \in \{w-1, \cdots, w\}, \\ 2 & \text{for } (w+1)^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ & \text{for } (r+2)^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ & \text{for } (r-2)^{th} \text{ component}, \ r \in \{w-1, \cdots, w\}, w \geq 5, \\ & \text{for } 2^{nd} \text{ component}, \ r = w, \\ 4, & \text{for } w^{th} \text{ component}, \ r \in \{5, \cdots, w-4\}, \ w \geq 9, \\ r, & \text{for } w^{th} \text{ component}, \ r \in \{1, 2, 3\}, \\ w-r, & \text{for } w^{th} \text{ component}, \ r \in \{w-3, \cdots, w-1\}, \\ 3, & \text{otherwise.} \end{cases}$$

$$t_\Pi(c_{w+r}) = \begin{cases} 0, & \text{for } r^{th} \text{ component}, \ r \in \{1, \cdots, w\}, \\ 1, & \text{for } (r+1)^{th} \text{ component}, \ r \in \{1, \cdots, w-1\}, \\ & \text{for } (r-1)^{th} \text{ component}, \ r \in \{2, \cdots, w\}, \\ & \text{for } w^{th} \text{ component}, \ r = 1, \\ & \text{for } (w-1)^{th} \text{ component}, \ r = 1, k = 1, \\ 2, & \text{for } (r+2)^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ & \text{for } (r-2)^{th} \text{ component}, \ r \in \{w-1, \cdots, w\}, \ w \geq 5, \\ & \text{for } (w-1)^{th} \text{ component}, \ r = 1, \\ & \text{for } w^{th} \text{ component}, \ r \in \{1, \cdots, w-2\}, \\ & \text{for } 2^{nd} \text{ component}, \ r = w, \\ k+3, & \text{for } (w+1)^{th} \text{ component}, \ r = 1, \\ k+4, & \text{for } (w+1)^{th} \text{ component}, \ r = 2, \\ & \text{for } (w+1)^{th} \text{ component}, \ r = w, \\ k+5, & \text{for } (w+1)^{th} \text{ component}, \ r = 3, \ w \geq 5, \\ & \text{for } (w+1)^{th} \text{ component}, \ r = w-1, \ w \geq 5, \\ k+6, & \text{for } (w+1)^{th} \text{ component}, \ r = 4, \ w \geq 6, \\ & \text{for } (w+1)^{th} \text{ component}, \ r = w-2, \ w \geq 6, \\ k+7, & \text{for } (w+1)^{th} \text{ component}, \ r \in \{5, \cdots, w-3\}, \ w \geq 8, \\ 3, & \text{otherwise.} \end{cases}$$

For $r \leq \left\lfloor \frac{k}{2} \right\rfloor, k \geq 1$, we have:

$$t_{\Pi}(s_r) = \begin{cases} 0, & \text{for } (w-1)^{th} \text{ component, odd } r, \\ & \text{for } w^{th} \text{ component, even } r, \\ 1, & \text{for } (w-1)^{th} \text{ component, even } r, \\ & \text{for } w^{th} \text{ component, odd } r, \\ & \text{for } 1^{st} \text{ component, } k=1, \\ 3, & \text{for } 3^{th} \text{ component, } k=1,\ w \geq 5, \\ 1+r, & \text{for } 1^{st} \text{ and } (w-1)^{th} \text{ component,} \\ 2+r, & \text{for } (w+1)^{th}, 2^{nd} \text{ and } (w-2)^{th} \text{ component, } w \geq 5, \\ 3+r, & \text{otherwise.} \end{cases}$$

For $r > \left\lfloor \frac{k}{2} \right\rfloor, k \geq 2$, we have:

$$t_{\Pi}(s_r) = \begin{cases} 0, & \text{for } (w-1)^{th} \text{ component, odd } r, \\ & \text{for } w^{th} \text{ component, odd } r, \\ 1, & \text{for } (w-1)^{th} \text{ component, even } r, \\ & \text{for } w^{th} \text{ component, even } r, \\ 2+r, & \text{for } (w+1)^{th} \text{ component,} \\ 1+k-r, & \text{for } 1^{st} \text{ component,} \\ 2+k-r, & \text{for } 2^{nd} \text{ component,} \\ 3+k-r, & \text{for } 3^{th} \text{ and } (w-1)^{th} \text{ component, } w \geq 5, \\ 4+k-r, & \text{otherwise.} \end{cases}$$

Very clearly, each vertex in $V(B^*_{P_{z_w}})$ possesses a unique color code, which implies that the coloring t qualifies as a locating coloring. Therefore, $\chi_L(B^*_{P_{z_w}}) \leq w+1$. The proof is complete. □

Figure 1 provides a representative example of $\chi_L(B_{P_{z_w}}) = 5$ and Figure 2 provides a representative example of $\chi_L(B^*_{P_{z_w}}) = 5$.

3. Conclusion

The conclusion reached from this discussion is $\chi_L(B_{P_{z_w}}) = \chi_L(B^*_{P_{z_w}}) = \chi_L(P_{z_w}) + 1$. This research analyzes how the locating chromatic number changes as pizza graphs are modified with certain barbell operation and its associated subdivisions.

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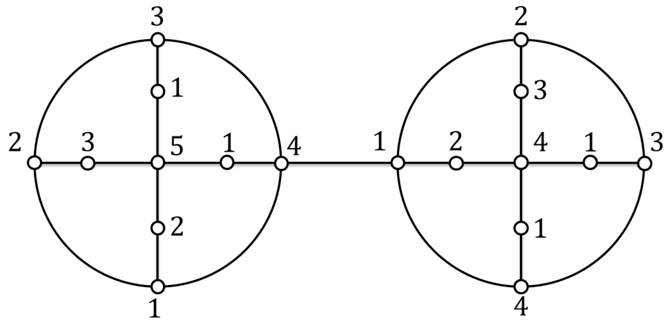

Figure 1. 5-locating coloring of $B_{P_{z_4}}$

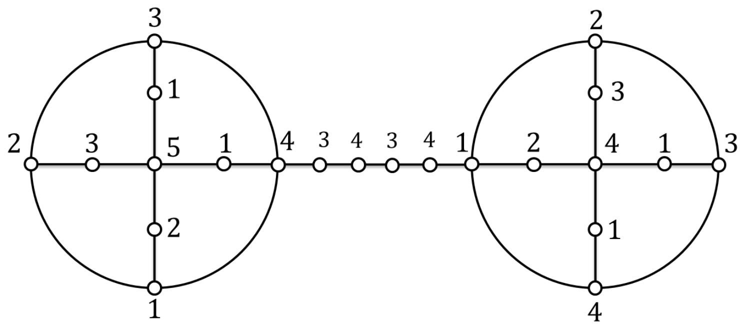

Figure 2. 5-locating coloring of $B^*_{P_{z_4}}$.

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