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BOUNDEDNESS OF GENERALIZED FRACTIONAL INTEGRALS ON GENERALIZED WEIGHTED MORREY SPACES OVER METRIC MEASURE SPACES AND APPLICATIONS

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Abstract. In this paper we investigate the boundedness properties of generalized fractional integral on generalized weighted Morrey spaces over metric measure spaces. The measure used in this paper is a doubling measure which satisfies the growth condition. The results show that the generalized fractional integral is bounded from one generalized weighted Morrey spaces to another generalized weighted Morrey space over metric measure spaces either with the same or with the different parameters. Our results extend the known results for fractional integrals on generalized Morrey spaces. We then investigate the regularity of the solution of Dirichlet problem with the data in generalized weighted Morrey spaces by using the boundedness properties of the generalized fractional integral on generalized weighted Morrey space.

Keywords: Sublinear operator, Generalized weighted Morrey space, Metric measure space, Dirichlet problem

1. Introduction

Let $X = (X, \mu, d)$ be a set equipped with a $\sigma-$finite measure μ and a metric d. Here we assume that X is nonempty. The metric d is a nonnegative function d on $X \times X$ that satisfies the following conditions.

(1) For all x, y, z in X we have

$$d(x, y) \le d(x, z) + d(y, z). \tag{1.1}$$

(2) For all $x, y \in X$, it holds that $d(x, y) = 0$ if and only if $x = y$.
(3) $d(x, y) = d(y, x)$ for all $x, y \in X$.

For $a \in X$ and $r > 0$ we denote $B(a, r)$ by a ball centered at a with radius r, that is the set of all $x \in X$ such that $d(x, a) < r$. We assume that the measure μ is doubling measure, i.e., there is a constant $C > 0$ such that:

$$\mu(B(a, 2r)) \le C\mu(B(a, r)), \quad a \in X, r > 0.$$

Moreover, μ satisfies the *growth condition* [1,2], i.e., there are a constant $C > 0$ and an exponent $n > 0$ such that:

$$\mu(B(a,r)) \leq Cr^n, \tag{1.2}$$

for all $a \in X$ and $r > 0$. The assumptions then imply that:

$$C^{-1} \leq \frac{\mu(B(a,t))}{\mu(B(a,r))} \leq C, \quad \frac{1}{2} \leq \frac{t}{r} \leq 2,$$

where C is positive constant that does not depend on $a, t,$ and r. We then call $X = (X, \mu, d) = (X, \mu, d_K)$ as metric measure space. If E is a set in X, then E^c denotes the complement of E on X. Macías and Segovia [3] proved that if (X, d, μ) is given, then we can always find a continuous quasi-distance d' such that there are two positive constants A_1 and A_2 for which:

$$A_1 d'(x,y) \leq d(x,y) \leq A_2 d'(x,y), \quad x, y \in X.$$

Therefore, we always assume that the metric d is continuous. In this paper, we also always assume that $\mathrm{diam}(X) = \infty$. Basics and some results for metric measure spaces may be found in [4,5,6].

With the doubling measures μ satisfying the growth condition, we consider the operators used in this article. First, we define the Hardy-Littlewood maximal operator M as:

$$M(f)(x) = \sup_{t>0} \frac{1}{\mu(B(x,t))} \int_{B(x,t)} |f(y)| d\mu(y).$$

Next, we consider the generalized fractional integral over metric measure spaces. Suppose that ρ is a function from $[0,\infty)$ to $[0,\infty)$. The generalized fractional integral $\mathcal{I}_\rho$ is defined by:

$$\mathcal{I}_\rho(f)(x) = \int_X \frac{\rho(d(x,y))}{d(x,y)^n} f(y) d\mu(y), \quad x \in \mathrm{supp}(\mu),$$

for all suitable function f where n is a constant as in the inequality (1.2). If $\rho(t) = t^\alpha$ where $0 < \alpha < n$, then $\mathcal{I}_\rho$ is just the fractional integral I_α. Moreover, for the function ρ we also define the generalized fractional maximal operator M_ρ as:

$$M_\rho(f)(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)| d\mu(y),$$

for $x \in X$ and suitable function f on X. The relation between $\mathcal{I}_\rho$ and M_ρ is the inequality:

$$M_\rho(f)(x) \leq C\mathcal{I}_\rho(|f|)(x). \tag{1.3}$$

Through this article, the function ρ is doubling, namely there is a constant $C > 0$ such that $1/C \leq \rho(t)/\rho(r) \leq C$ for $12/ \leq t/r \leq 2$. We may see that $\rho(t) = t^\alpha$ is doubling.

Morrey spaces over Euclidean spaces were first introduced by C.B. Morrey in [7] in 1938 to study the local behaviour of solution of elliptic partial differential equations. Morrey spaces were then generalized become generalized Morrey spaces [8,9], weighted Morrey spaces, [10], and generalized weighted Morrey spaces [11]. We shall study the operator $\mathcal{I}_\rho$ on generalized weighted Morrey spaces where Euclidean spaces $\mathbb{R}^n$ is replaced by the metric measure space X.

In the case $X = \mathbb{R}^n$ equipped with the Lebesgue measure and the standard Euclidean distance, $\mathcal{I}_\rho$ is bounded from one generalized Morrey space to another generalized Morrey space with the same parameter p for certain assumptions [12]. By inequality (1.3), it thus implies that M_ρ is bounded on the spaces under the same assumption. Some other researchers studied the boundedness of $\mathcal{I}_\rho$ from one generalized Morrey space with parameter p to another generalized Morrey space with parameter q, see [13,14] for example.

In this article, we study the boundedness property of the generalized fractional integrals on generalized weighted Morrey spaces over metric measure spaces. There are two different versions for boundedness property of the generalized fractional integrals we obtain. The first one states that $\mathcal{I}_\rho$ is bounded from one generalized weighted Morrey space to another generalized weighted Morrey space with the same parameter p. This result extends [12,15].

The second result states that $\mathcal{I}_\rho$ is bounded from one generalized weighted Morrey space to another generalized weghted Morrey space with different parameter. This result extends [13,14,16]. However, the boundedness property we obtain may be viewed as Adams-Gunawan-type inequality. The boundedness properties of $\mathcal{I}_\rho$ that was obtained then imply the boundedness of the generalized fractional maximal operator M_ρ on generalized weighted Morrey spaces over metric measure spaces under the same assumptions. As the application of the boundedness property, we investigate the regularity of the solution of Dirichlet problem with the data in generalized Morrey spaces.

2. A_p weights and generalized weighted Morrey spaces over metric measure spaces

From now, we assume that X is a metric measure space equipped with a metric d and a doubling measure μ satisfying the growth condition. A weight w is a locally $\mu-$integrable function on X taking values in the interval $(0,\infty)$ almost everywhere. If w is a weight on X, then we write $w(E) = \int_E w(y)d\mu(y)$ for all $\mu-$measurable sets E. One of the weight classes used in this paper is $A_p(\mu)$ weight class. The following is the definition of $A_p(\mu)$ weight class [17,18]. We also use the class $A_{p,q}(\mu)$ and $A^\rho_{p,q}(\mu)$.

Definition 2.1. *For $1 < p < \infty$, $A_p(\mu)$ is set of all weights w on X such that there is a constant $C > 0$ for which:*

$$\left(\frac{1}{\mu(B(a,r))}\int_{B(a,r)} w(x)d\mu(x)\right)\left(\frac{1}{\mu(B(a,r))}\int_{B(a,r)} w(x)^{-\frac{1}{p-1}}d\mu(x)\right)^{p-1} \leq C$$

for every ball $B(a,r)$ in X.

If $X = \mathbb{R}^n$, then $A_p(\mu)$ is classical A_p class as in [19,20]. Related to the properties of classical A_p weights as in [19,20,21], we have Theorem 2.2 as follows.

Theorem 2.2. *[22] For each $1 < p < \infty$ and $w \in A_p(\mu)$, there exist two positive constant C such that:*

$$\frac{w(B)}{w(E)} \leq C \left(\frac{\mu(B)}{\mu(E)}\right)^p,$$

for every ball B in X and $\mu-$measurable subset E of B.

We also define the Muckenhoupt class $A_\infty(\mu)$ as in the following definition.

Definition 2.3. *[22] $A_\infty(\mu)$ is the set of all weights w such that there are constants $C > 0$ and $\delta > 0$ for which:*

$$\left(\frac{\mu(B)}{\mu(E)}\right)^\delta \leq C\frac{w(B)}{w(E)},$$

whenever E is a $\mu-$measurable subset of a ball B in X and $\mu(E) > 0$.

By the result in [23], for $1 < p < q < \infty$ it holds that $A_p \subset A_q \subset A_\infty$. Hence, we have the relation:

$$\cup_{1<p<\infty} A_p(\mu) \subseteq A_\infty(\mu). \tag{2.1}$$

Related to the second version of the boundedness properties of generalized fractional integrals, we define $A_{p,q}(\mu)$ as follows.

Definition 2.4. *Let $1 < p < q < \infty$ and p' satisfies $1/p + 1/p' = 1$. We denote by $A_{p,q}(\mu)$ the set of all weights w for which:*

$$\left(\frac{1}{\mu(B(a,r))}\int_{B(a,r)} w(x)^q d\mu(x)\right)^{\frac{1}{q}} \left(\frac{1}{\mu(B(a,r))}\int_{B(a,r)} w(x)^{-p'} d\mu(x)\right)^{\frac{1}{p'}} \leq C,$$

for all $(a,r) \in X \times (0,\infty)$ where $C > 0$ is a constant that does not depend on a and r.

We see that for $1 < p < q < \infty$ and $w \in A_{p,q}(\mu)$, then $w^p \in A_p(\mu)$ and $w^q \in A_q(\mu)$. Therefore, w^p and w^q in $A_\infty(\mu)$ by (2.1) and there is a constant $C > 0$ such that:

$$\frac{1}{C} \leq \frac{w^p(B(a,t))^{\frac{1}{p}}/w^q(B(a,t))^{\frac{1}{q}}}{w^p(B(a,r))^{\frac{1}{p}}/w^q(B(a,r))^{\frac{1}{q}}} \leq C, \quad \frac{1}{2} \leq \frac{t}{r} \leq 2.$$

We generalize the weight class $A_{p,q}(\mu)$ to be the weight class $A^\rho_{p,q}(\mu)$ as in the following definition.

Definition 2.5. *Let $1 < p < q < \infty$ and p' satisfies $1/p + 1/p' = 1$. We denote by $A^\rho_{p,q}(\mu)$ the set of all weights w for which:*

$$\frac{\rho(t)}{\mu(B(a,r))}\left(\int_{B(a,r)} w(x)^q d\mu(x)\right)^{\frac{1}{q}} \left(\int_{B(a,r)} w(x)^{-p'} d\mu(x)\right)^{\frac{1}{p'}} \leq C,$$

for all $(a,r) \in X \times (0,\infty)$ where $C > 0$ is a constant that does not depend on a and r.

For a weight w on X and Ω is a μ-measurable set in X, we define the weighted Lebesgue space $L^{p,w}(\Omega,\mu)$ to be the set of all μ-measurable functions f on Ω for which

$$\|f\|_{L^{p,w}(\Omega,\mu)} := \left(\int_{\Omega} |f(x)|^p w(x) d\mu(x) \right)^{\frac{1}{p}} < \infty.$$

We write $L^{p,w}(\mu) = L^{p,w}(X,\mu)$. Moreover, For $1 < p < \infty$ and a weight w on X, the set $L^{p,w}_{loc}(\mu)$ is the collection of all μ-measurable functions f such that $\mathcal{X}_B \cdot f$ is in $L^{p,w}(\mu)$ for each ball B in X.

We now present the definition of the generalized weighted Morrey spaces over metric measure spaces.

Definition 2.6. *Let $1 < p < \infty, w \in A_p(\mu)$, and ψ be a positive function on $X \times (0,\infty)$. The generalized weighted Morrey space $\mathcal{M}^{p,w}_{\psi}(\mu) = \mathcal{M}^{p,w}_{\psi}(X,d,\mu)$ is the set of all functions $f \in L^{p,w}_{loc}(\mu)$ such that*

$$\|f\|_{\mathcal{M}^{p,w}_{\psi}(\mu)} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,r),\mu)} < \infty.$$

If $X = \mathbb{R}^n$ equipped with a Lebesgue measure and the Euclidean distance, we write $\mathcal{M}^{p,w}_{\psi} = \mathcal{M}^{p,w}_{\psi}(\mu)$. If w is assumed to be constant a.e., then $\mathcal{M}^{p,w}_{\psi}(\mu)$ is the generalized Morrey space $\mathcal{M}^{p}_{\psi}$ on $\mathbb{R}^n$ as in [24]. If $\psi(a,r) = w(B(a,r))^{-\frac{1}{q}}$, then $\mathcal{M}^{p,w}_{\psi}$ is the weighted Morrey spaces [10] where $1 < p < q < \infty$. Moreover, if we set $\psi(a,r) = |B(a,r)|^{-\frac{1}{q}}$ where $1 < p \le q < \infty$ and w is constant a.e., then $\mathcal{M}^{p,w}_{\psi}(\mu)$ is the classical Morrey space as in [7,25,26,27]. With Definitions 2.6, we shall investigate the boundedness of the Hardy-Littlewood maximal operator M and the generalized fractional integral $\mathcal{I}_\rho$ from $\mathcal{M}^{p,w}_{\psi_1}(\mu)$ to $\mathcal{M}^{p,w}_{\psi_2}(\mu)$ where $1 < p < \infty$. Moreover, we also investigate the boundedness of $\mathcal{I}_\rho$ from $\mathcal{M}^{p,w}_{\psi}(\mu)$ to $\mathcal{M}^{q,w}_{\psi^{\frac{p}{q}}}(\mu)$ where $1 < p < q < \infty$.

3. Generalized fractional integrals on generalized weighted Morrey spaces over metric measure spaces

In this section, we prove the boundedness of generalized fractional integral $\mathcal{I}_\rho$ on generalized weighted Morrey spaces over metric measure spaces. The following two theorems are the boundedness properties of the Hardy-Littlewood maximal operator M. The theorems as in [28] tell us that M is bounded from one generalized weighted Morrey space to another generalized weighted Morrey spaces that generalized some results in [29] particularly for M.

Theorem 3.1. *Let $1 < p < \infty$ and $w \in A_p(\mu)$. Suppose that ψ_1 and ψ_2 are two positive function on $X \times (0,\infty)$ for which there is a positive constant $C > 0$ such that:*

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \le C\psi_2(a,r), \quad (a,r) \in X \times (0,\infty). \tag{3.1}$$

Then, M is bounded from $\mathcal{M}^{p,w}_{\psi_1}(\mu)$ to $\mathcal{M}^{p,w}_{\psi_2}(\mu)$.

Theorem 3.2. *Let $1 < p < \infty$ and $w \in A_p(\mu)$. Suppose that ψ_1 and ψ_2 are two positive function on $X \times (0, \infty)$ for which there is a positive constant $C > 0$ such that:*

$$\sup_{r<t<\infty} \psi_1(a,t) \le C\psi_2(a,r), \quad (a,r) \in X \times (0,\infty). \tag{3.2}$$

Then, M is bounded from $\mathcal{M}^{p,w}_{\psi_1}(\mu)$ to $\mathcal{M}^{p,w}_{\psi_2}(\mu)$.

We denote $Z_p(w)$ by the set of all pairs of functions (ψ_1, ψ_2) which satisfy Eq. (3.1) or Eq. (3.2). By the notations, we provide two theorems about the boundedness of generalized fractional integral $\mathcal{I}_\rho$ on generalized weighted Morrey spaces over metric measure spaces. We prove the the theorems by using the boundedness of Hardy-Littlewood maximal operator on the spaces. The following theorem is our first main result about the boundedness of generalized fractional integral $\mathcal{I}_\rho$ on generalized weighted Morrey spaces over metric measure spaces.

Theorem 3.3. *Let $1 < p < \infty$ and $w \in A_p(\mu)$. Assume that ρ is doubling and there exist a positive constant $C > 0$ such that:*

$$\frac{1}{C} \le \frac{\rho(t)\psi_1(a,t)}{\rho(r)\psi_1(a,r)} \le C,$$

for $a \in X$ and $\frac{1}{2} \le \frac{t}{r} \le 2$. If

$$\psi_1(a,r)\int_0^r \rho(t)\frac{dt}{t}dt + \int_r^\infty \rho(t)\psi_1(a,t)\frac{dt}{t} \le C\psi_2(a,r), \quad a \in X, r > 0,$$

and $(\psi_1, \psi_2) \in Z_p(w)$, then, $\mathcal{I}_\rho$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}(\mu)$ to $\mathcal{M}^{p,w}_{\psi_2}(\mu)$.

Proof of Theorem 3.3. For $r > 0$ and $f \in \mathcal{M}^{p,w}_{\psi_1}(\mu)$, we write:

$$\begin{aligned}\mathcal{I}_\rho(f)(x) &= \int_{B(x,r)} \frac{\rho(d(x,y))}{d(x,y)^n} f(y)d\mu(y) + \int_{X\setminus B(x,r)} \frac{\rho(d(x,y))}{d(x,y)^n} f(y)d\mu(y) \\ &:= \mathcal{I}_1(x) + \mathcal{I}_2(x).\end{aligned}$$

Note that, by the assumption, we have that:

$$\int_{2^k r}^{2^{k+1}r} \rho(t)\frac{dt}{t} = \int_1^2 \frac{\rho(2^k rt)}{2^k rt} 2^k dt = \int_{\frac{1}{2}}^1 \rho(2^k rt)\frac{dt}{t} \ge C\rho(2^k r),$$

where C is not dependent on $r > 0$. We shall find the estimate for $\mathcal{I}_1(x)$ and $\mathcal{I}_2(x)$.

First, for $\mathcal{I}_1(x)$, by the last inequalities,

$$\begin{aligned}
|\mathcal{I}_1(x)| &\leq \int_{d(x,y)<r} \frac{\rho(d(x,y))}{d(x,y)^n}|f(y)|d\mu(y), \\
&= \sum_{k=-\infty}^{-1} \int_{2^k r\leq d(x,y)<2^{k+1}r} \frac{\rho(d(x,y))}{d(x,y)^n} f(y)d\mu(y), \\
&\leq C \sum_{k=-\infty}^{-1} \int_{2^k r\leq d(x,y)<2^{k+1}r} \frac{\rho(2^k r)}{d(x,y)^n}, \\
&\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^k r)}{(2^k r)^n} \int_{d(x,y)<2^{k+1}r} |f(y)|d\mu(y), \\
&\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^k r)}{\mu(B(x,2^{k+1}r))} \int_{d(x,y)<2^{k+1}r} |f(y)|d\mu(y), \\
&\leq CMf(x) \sum_{k=-\infty}^{-1} \rho(2^k r), \\
&\leq CMf(x) \sum_{k=-\infty}^{-1} \int_{2^k r}^{2^{k+1}r} \rho(t)\frac{dt}{t} = CMf(x)\int_0^r \rho(t)\frac{dt}{t}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{I}_1\|_{L^{p,w}(B(a,r),\mu)} &\leq C\|Mf\|_{L^{p,w}(B(a,r),\mu)} \int_0^r \rho(t)\frac{dt}{t}, \\
&\leq C\|Mf\|_{L^{p,w}(B(a,r),\mu)} \frac{\psi_2(a,r)}{\psi_1(a,r)},
\end{aligned}$$

and

$$\frac{1}{\psi_2(a,r)}\|\mathcal{I}_1\|_{L^{p,w}(B(a,r),\mu)} \leq C\frac{1}{\psi_1(a,r)}\|Mf\|_{L^{p,w}(B(a,r),\mu)}.$$

Therefore,

$$\frac{1}{\psi_2(a,r)}\frac{1}{w(B(a,r))^{\frac{1}{p}}}\|\mathcal{I}_1\|_{L^{p,w}(B(a,r),\mu)} \leq C\|Mf\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)}.$$

By the assumption, Theorem 3.1 and Theorem 3.2 imply that:

$$\frac{1}{\psi_2(a,r)}\frac{1}{w(B(a,r))^{\frac{1}{p}}}\|\mathcal{I}_1\|_{L^{p,w}(B(a,r),\mu)} \leq C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)}. \tag{3.3}$$

For $\mathcal{I}_2(x)$ wee see that:

$$\begin{aligned}
|\mathcal{I}_2(x)| &\leq \sum_{k=1}^{\infty} \int_{2^k r\leq d(x,y)<2^{k+1}r} \frac{\rho(d(x,y))}{d(x,y)^n}|f(y)|d\mu(y), \\
&\leq C\sum_{k=1}^{\infty} \frac{\rho(2^{k+1}r)}{(2^{k+1}r)^n} \int_{d(x,y)<2^{k+1}r} |f(y)|d\mu(y).
\end{aligned}$$

If $x \in B(a,r)$ and $d(x,y) < 2^{k+1}r$, then $d(y,a) \le d(x,y) + d(x,a) < 2^k r + r < 2^{k+1}r$. Hence, by the assumption, we have that:

$$\begin{aligned} |\mathcal{I}_2(x)| &\le C \sum_{k=1}^{\infty} \frac{\rho(2^{k+1}r)}{(2^{k+1}r)^n} \int_{d(a,y)<2^{k+2}r} |f(y)| d\mu(y), \\ &\le C \sum_{k=1}^{\infty} \frac{\rho(2^{k+2}r)}{(2^{k+2}r)^n} \int_{d(a,y)<2^{k+2}r} |f(y)| d\mu(y), \\ &\le C \sum_{k=1}^{\infty} \frac{\rho(2^{k+2}r)}{\mu(B(a,2^{k+2}r))} \int_{d(a,y)<2^{k+2}r} |f(y)| d\mu(y), \end{aligned}$$

for $x \in B(a,r)$. By Lemma 5 in [28], for all $k \in \mathbb{N}$ it holds that:

$$\frac{1}{\mu(B(a,2^{k+1}r))} \int_{d(x,y)<2^{k+1}r} |f(y)| d\mu(y) \le C \frac{1}{w(B(a,2^{k+1}r))} \|f\|_{L^{p,w}(B(a,2^{k+1}r),\mu)},$$

where $C > 0$ does not depend on a and r. Moreover, by the assumption,

$$\begin{aligned} \int_{2^k r}^{2^{k+1}r} \rho(t)\psi_1(a,Kt)\frac{dt}{t} &= \int_1^2 \frac{\rho(2^k rt)\psi_1(a,2^k rt)}{2^k rt} 2^k r dt, \\ &= \int_{\frac{1}{2}}^1 \rho(2^k rt)\psi_1(a,2^k rt)\frac{dt}{t} \ge C\rho(2^k r)\psi_1(a,2^k r), \end{aligned}$$

where C does not dependent on $a \in X$ and $r > 0$. Thus,

$$\begin{aligned} |\mathcal{I}_2(x)| &= C \sum_{k=1}^{\infty} \rho(2^{k+2}r) \frac{1}{w(B(a,2^{k+2}r))} \|f\|_{L^{p,w}(B(a,2^{k+1}r),\mu)}, \\ &\le C \sum_{k=1}^{\infty} \rho(2^{k+2}r)\psi_1(a,2^{k+2}r) \|f\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)}, \\ &\le C \|f\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)} \sum_{k=1}^{\infty} \int_{2^{k+2}r}^{2^{k+3}r} \rho(t)\psi_1(a,t)\frac{dt}{t}, \\ &\le C \|f\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)} \int_r^{\infty} \rho(t)\psi_1(a,t)\frac{dt}{t}, \end{aligned}$$

and by the assumption,

$$|\mathcal{I}_2(x)| \le C_2 \|f\|_{\mathcal{M}^{p,w}_{\psi}(\mu)} \psi_2(a,r), \quad x \in B(a,r).$$

Therefore,

$$\|\mathcal{I}_2\|_{L^{p,w}(B(a,r),\mu)} \le C_2 \|f\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)} \psi_2(a,r) w(B(a,r))^{\frac{1}{p}}.$$

Hence,

$$\frac{1}{\psi_2(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|\mathcal{I}_2\|_{L^{p,w}(B(a,r),\mu)} \le C_2 \|f\|_{\mathcal{M}^{p,w}_{\psi}(\mu)}. \tag{3.4}$$

By combining Eq. (3.3) and Eq. (3.4), we have that:

$$\frac{1}{\psi_2(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|\mathcal{I}_\rho(f)\|_{L^{p,w}(B(a,r),\mu)} \le C \|f\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)},$$

and $\|\mathcal{I}_\rho(f)\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)} \le C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}(\mu)}$. This proves that $\mathcal{I}_\rho$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}(\mu)$ to $\mathcal{M}^{p,w}_{\psi_2}(\mu)$. □

For M_ρ, we have the following corollary.

Corollary 3.4. *Let $1 < p < \infty$ and $w \in A_p(\mu)$. Assume that ρ is doubling and there exist a positive constant $C > 0$ such that $1/C \le (\rho(t)\psi_1(a,t))/(\rho(r)\psi_1(a,r)) \le C$ for $a \in X$ and $\frac{1}{2} \le \frac{t}{r} \le 2$. If*

$$\psi_1(a,r)\int_0^r \rho(t)\frac{dt}{t}dt + \int_r^\infty \rho(t)\psi_1(a,t)\frac{dt}{t} \le C\psi_2(a,r), \quad a \in X, r > 0,$$

and $(\psi_1,\psi_2) \in Z_p(w)$, then M_ρ is bounded from $\mathcal{M}^{p,w}_{\psi_1}(\mu)$ to $\mathcal{M}^{p,w}_{\psi_2}(\mu)$.

Based on Theorem 3.3, $\mathcal{I}_\rho$ is bounded from $\mathcal{M}^{p,w}_{\psi_1}(\mu)$ to $\mathcal{M}^{p,w}_{\psi_2}(\mu)$ under some assumptions. This theorem generalize the result in [12,15]. However, the result cannot be viewed as a generalization of the known results for I_α as in [9]. Therefore, we also provide the extension for the known results for I_α as in the following theorem.

Theorem 3.5. *Let $1 < p < q < \infty$ and $w^{\frac{1}{p}} \in A^\rho_{p,q}(\mu)$. Suppose that there are positive constants C_1 and C_2 such that ρ is doubling, and $1/C_1 \le \rho(t)\psi(a,t)/(\rho(r)\psi(a,r)) \le C_1$ for $1/2 \le t/r \le 2$ and $a \in X$. If*

$$\psi(a,r)\int_0^r \rho(t)\frac{dt}{t} + \int_r^\infty \frac{w(B(a,t))^{\frac{1}{p}}}{w^{\frac{q}{p}}(B(a,t))^{\frac{1}{q}}}\psi(a,t)\frac{dt}{t} \le C_2\psi(a,r)^{\frac{p}{q}}, \quad a \in X, r > 0,$$

and $\psi \in Z_p(w)$, then $\mathcal{I}_\rho$ is bounded from $\mathcal{M}^{p,w}_{\psi}(\mu)$ to $\mathcal{M}^{q,w}_{\psi^{\frac{p}{q}}}(\mu)$.

Proof. Let $f \in \mathcal{M}^{p,w}_{\psi}(\mu)$ and we may assume that $\|f\|_{\mathcal{M}^{p,w}_{\psi}(\mu)} = 1$. For $a \in X$ and $r > 0$, we write $\mathcal{I}_\rho(f)(x) = \mathcal{I}_1(x) + \mathcal{I}_2(x)$ where $x \in B(a,r)$. We shall estimate $\mathcal{I}_1(x)$ and $\mathcal{I}_2(x)$. For $\mathcal{I}_1(x)$, we have that:

$$\begin{aligned}
|\mathcal{I}_1(x)| &\le \int_{d(x,y)<r} \frac{\rho(d(x,y))}{d(x,y)^n}|f(y)|d\mu(y),\\
&= \sum_{k=-\infty}^{-1}\int_{2^k r\le d(x,y)<2^{k+1}r} \frac{\rho(d(x,y))}{d(x,y)^n}|f(y)|d\mu(y),\\
&\le \sum_{k=-\infty}^{-1}\frac{\rho(2^k r)}{(2^k r)^n}\int_{d(x,y)<2^{k+1}}|f(y)|d\mu(y),\\
&\le C\sum_{k=-\infty}^{-1}\frac{\rho(2^k r)}{\mu(B(x,2^{k+1}r))}\int_{d(x,y)<2^{k+1}r}|f(y)|d\mu(y),\\
&\le C\sum_{k=-\infty}^{-1}\rho(2^k r)M(f)(x) \le CM(f)(x)\sum_{k=-\infty}^{-1}\int_{2^k r}^{2^{k+1}r}\rho(t)\frac{dt}{t},\\
&= CM(f)(x)\int_0^r \rho(t)\frac{dt}{t} \le CM(f)(x)\psi(a,r)^{(p-q)/q}.
\end{aligned}$$

For $\mathcal{I}_2(x)$, by Hölder's inequality, the assumption that $w^{\frac{1}{p}} \in A^\rho_{p,q}(\mu)$, we have that for $x \in B(a,r)$,

$$
\begin{aligned}
&|\mathcal{I}_2(x)| \\
&\leq \sum_{k=0}^{\infty} \int_{2^k r \leq d(x,y) < 2^{k+1} r} \frac{\rho(d(x,y))}{d(x,y)^n} |f(y)| d\mu(y), \\
&\leq \sum_{k=0}^{\infty} \frac{\rho(2^k r)}{(2^k r)^n} \int_{2^k r \leq d(x,y) < 2^{k+1} r} |f(y)| d\mu(y), \\
&\leq \sum_{k=0}^{\infty} \frac{\rho(2^{k+2} r)}{(2^{k+2} r)^n} \int_{d(a,y) < 2^{k+2} r} |f(y)| d\mu(y), \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^{k+2} r)}{\mu(B(a, 2^{k+2} r))} \int_{d(a,y) < 2^{k+2} r} |f(y)| d\mu(y), \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^{k+2} r)}{\mu(B(a, 2^{k+2} r))} \|f\|_{L^{p,w}(B(a,2^{k+2}r)} \left(\int_{B(a,2^{k+2}r)} w(y)^{\frac{-p'}{p}} d\mu(y) \right)^{\frac{1}{p'}}, \\
&\leq C \sum_{k=0}^{\infty} \frac{w(B(a, 2^{k+2} r))^{\frac{1}{p}}}{w^{\frac{q}{p}}(B(a, 2^{k+2} r))^{\frac{1}{q}}} \psi(a, 2^{k+2} r) \|f\|_{\mathcal{M}_{\psi}^{p,w}}, \\
&\leq C \sum_{k=0}^{\infty} \int_{2^{k+2} r}^{2^{k+3} r} \frac{w(B(a, 2^{k+2} r))^{\frac{1}{p}}}{w^{\frac{q}{p}}(B(a, 2^{k+2} r))^{\frac{1}{q}}} \psi(t) \frac{dt}{t}, \\
&\leq C \int_r^{\infty} \frac{w(B(a,t))^{\frac{1}{p}}}{w^{\frac{q}{p}}(B(a,t))^{\frac{1}{q}}} \psi(t) \frac{dt}{t} \leq C \psi(a,r)^{\frac{p}{q}}.
\end{aligned}
$$

Combining the two estimates, we have that:

$$
\mathcal{I}_\rho(f)(x) \leq C \left[M(f)(x) \psi(a,r)^{\frac{p-q}{q}} + \psi(a,r)^{\frac{p}{q}} \right], \quad x \in B(a,r). \tag{3.5}
$$

As in the proof of Theorem 22 in [30], by the last inequality and choosing $M(f)(x)/\psi(a,r) = \|f\|_{\mathcal{M}_{\psi}^{p,w}}$ yields:

$$
|\mathcal{I}_\rho(f)| \leq C M(f)(x)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{\psi}^{p,w}}^{1-\frac{p}{q}}, \quad x \in B(a,r). \tag{3.6}
$$

It then implies that:

$$
|\mathcal{I}_\rho(f)(x)|^q w(x) \leq C M(f)(x)^p w(x) \|f\|_{\mathcal{M}_{\psi}^{p,w}}^{q-p}, \quad x \in B(a,r).
$$

By integrating the both sides over the ball $B(a,r)$, we may obtain:

$$
\begin{aligned}
&\frac{1}{w(B(a,r))} \int_{B(a,r)} |\mathcal{I}_\rho(f)(x)|^q w(x)^q d\mu(x) \\
&\qquad\qquad \leq C \|f\|_{\mathcal{M}_{\psi}^{p,w}}^{q-p} \frac{1}{w(B(a,r))} \int_{B(a,r)} M(f)(x)^p w(x)^p d\mu(x).
\end{aligned}
$$

Therefore, by Theorem 3.1 or Theorem 3.2,

$$\begin{aligned}\frac{1}{\psi(a,r)^p}\frac{1}{w(B(a,r))}&\int_{B(a,r)}|\mathcal{I}_\rho(f)(x)|^q w(x)d\mu(x)\\ &\le C\|f\|_{\mathcal{M}_\psi^{p,w}}^{q-p}\frac{1}{\psi(a,r)^p}\frac{1}{w(B(a,r))}\int_{B(a,r)}M(f)(x)^p w(x)d\mu(x),\\ &\le C\|f\|_{\mathcal{M}_\psi^{p,w}}^{q-p}\|Mf\|_{\mathcal{M}_\psi^{p,w}(\mu)}^p,\\ &\le C\|f\|_{\mathcal{M}_\psi^{p,w}}^{q-p}\|f\|_{\mathcal{M}_\psi^{p,w}(\mu)}^p = C\|f\|_{\mathcal{M}_\psi^{p,w}}^q,\end{aligned}$$

and

$$\frac{1}{\psi(a,r)^p}\frac{1}{w(B(a,r))}\int_{B(a,r)}|\mathcal{I}_\rho(f)(x)|^q w(x)d\mu(x)\le C\|f\|_{\mathcal{M}_\psi^{p,w}}^q,$$

which implies that:

$$\frac{1}{\psi(a,r)^{\frac{p}{q}}}\frac{1}{w(B(a,r))^{\frac{1}{q}}}\left(\int_{B(a,r)}|\mathcal{I}_\rho(f)(x)|^q w(x)d\mu(x)\right)^{\frac{1}{q}}\le C\|f\|_{\mathcal{M}_\psi^{p,w}},$$

for $a\in X$ and $r>0$. By taking the supremum over $a\in X$ and $r>0$, we obtain: $\|\mathcal{I}_\rho(f)\|_{\mathcal{M}^{q,w}_{\psi^{\frac{p}{q}}}(\mu)}\le C\|f\|_{\mathcal{M}_\psi^{p,w}}^q$. This means that $\mathcal{I}_\rho$ is bounded from $\mathcal{M}_\psi^{p,w}(\mu)$ to $\mathcal{M}^{q,w}_{\psi^{\frac{p}{q}}}(\mu)$ and completes the proof of Theorem 3.5. □

Corollary 3.6. *Let $1<p<q<\infty$ and $w^{\frac{1}{p}}\in A^\rho_{p,q}(\mu)$. Suppose that there are positive constants C_1 and C_2 such that ρ is doubling and, $1/C_1\le\rho(t)\psi(a,t)/(\rho(r)\psi(a,r))\le C_1$ for $1/2\le t/r\le 2$ and $a\in X$. If*

$$\psi(a,r)\int_0^r\rho(t)\frac{dt}{t}+\int_r^\infty\frac{w(B(a,t))^{\frac{1}{p}}}{w^{\frac{q}{p}}(B(a,t))^{\frac{1}{q}}}\psi(a,t)\frac{dt}{t}\le C_2\psi(a,r)^{\frac{p}{q}},\quad a\in X, r>0,$$

and $\psi\in Z_p(w)$, then M_ρ are bounded from $\mathcal{M}_\psi^{p,w}(\mu)$ to $\mathcal{M}^{q,w}_{\psi^{\frac{p}{q}}}(\mu)$.

By adapting the definition of weight class we use and the assumption, we obtain the following well-know result about the boundedness of $\mathcal{I}_\rho$ on generalized Morrey spaces over $\mathbb{R}^n$.

Corollary 3.7. *[14] Let $1<p<q<\infty$ and $\phi\in\mathcal{G}_p$. If there is a constant $C>0$ such that:*

$$\phi(r)\int_0^r\rho(t)\frac{dt}{t}+\int_r^\infty\phi(t)\rho(t)\frac{dt}{t}dt\le C\phi(r)^{p/q}$$

for $r>0$, then $\mathcal{I}_\rho$ is bounded from $\mathcal{M}_\phi^{p,1}$ to $\mathcal{M}_\phi^{q,1}$.

Corollary 3.8. *[16] Suppose that there is a constant $C>0$ such that:*

(1) ρ and ϕ are doubling.
(2) For $r>0$,

$$\int_r^\infty\phi(t)^p\frac{dt}{t}\le C\phi(r)^p,$$

and

$$\phi(r)\int_0^r \rho(t)\frac{dt}{t}+\int_r^\infty \rho(t)\phi(t)\frac{dt}{t}\le C\phi(r)^{p/q}.$$

If ϕ is surjective, then $\mathcal{I}_\rho$ is bounded from $\mathcal{M}^{p,1}_{\phi}(\mu)$ to $\mathcal{M}^{p,1}_{\phi^{p/q}}(\mu)$.

Corollary 3.9. *[31] Let $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n-\lambda}, 0<\alpha<n$, and $1<p<\frac{n}{\alpha}$. If $L^{p,\lambda}=\mathcal{M}^{p,1}_{\psi}$ where $\psi(r)=\frac{1}{r^{\lambda/p}}$ for $r>0$ and $L^{q,\lambda}=\mathcal{M}^{q,1}_{\varphi}$ where $\varphi(r)=\frac{1}{r^{\lambda/q}}$, then I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda}$.*

4. Applications to regularity of Dirichlet problem with the data belongs to generalized weighted Morrey spaces

In this section, we apply the boundedness properties of generalized fractional integral I_ρ on generalized weighted Morrey spaces over metric measure spaces. In particular, the properties are applied to the fractional integral $I_\rho=I_\alpha$, namely $\rho(t)=t^\alpha$ where $0<\alpha<n$ and $X=\mathbb{R}^n$ equipped with the Euclidean distance and the Lebesgue measure.

Throughout this section, let Ω be an open, bounded, and connected subset of $\mathbb{R}^n$, with $n\ge 3$. We denote $\mathcal{M}^{p,w}_{\psi}(\Omega)$ by the set is of all functions $f\in L^{p,w}_{\text{loc}}(\Omega)$, that is the functions for which $\|f\|_{L^{p,w}(B(a,r)\cap\Omega)}<\infty$ for $(a,r)\in X\times(0,\infty)$, such that:

$$\|f\|_{\mathcal{M}^{p,w}_{\psi}(\Omega)}=\sup_{a\in\Omega,r>0}\frac{1}{\psi(a,r)}\frac{1}{w(B(a,r))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,r)\cap\Omega)}<\infty.$$

By the definition and the fact that:

$$\|M(f)\|_{\mathcal{M}^{p,w}_{\psi}(\Omega)}\le C\|M(f\cdot\mathcal{X}_\Omega)\|_{\mathcal{M}^{p,w}_{\psi}}\le C\|M(f\cdot\mathcal{X}_\Omega)\|_{\mathcal{M}^{p,w}_{\psi}}\le C\|f\|_{\mathcal{M}^{p,w}_{\psi}(\Omega)},$$

we have the following corollary.

Corollary 4.1. *Let $1<p<q<\infty$, $\alpha/n=1/p-1/q$, and $w\in A_p$. Suppose that there are positive constants C_1 and C_2 such that:*

(1) $1/C_1\le\psi(a,t)/\psi(a,r)\le C_1$ for $\frac{1}{2}\le t/r\le 2$ and $a\in X$.
(2) For $a\in X$ and $r>0$,

$$\psi(a,r)r^\alpha+\int_r^\infty\frac{w(B(a,t))^{\frac{1}{p}}}{w^{\frac{q}{p}}(B(a,t))^{\frac{1}{q}}}\psi(a,t)\frac{dt}{t}\le C_2\psi(a,r)^{\frac{p}{q}}.$$

Suppose that $\psi\in Z_p(w)$. Then, $\mathcal{I}_\alpha$ is bounded from $\mathcal{M}^{p,w}_{\psi}(\Omega)$ to $\mathcal{M}^{q,w}_{\psi^{\frac{p}{q}}}(\Omega)$.

We denote $\mathbb{A}^{p,q}(w)$ by the set of all functions ψ which satisfy the assumptions in Corollary 4.1. We shall use this notation to provide the applications of Corollary 4.1 regarding to the regularity property of particular solutions of certain Dirichlet problems.

For $q=1,2$, we denote $W^{1,q}(\Omega)$ by the Sobolev spaces and under the Sobolev norm, the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$ is denoted by $W^{1,q}_0(\Omega)$. Moreover, we denote $H^{-1}(\Omega)$ by the dual space of $W^{1,2}_0(\Omega)$.

We shall investigate the following Dirichlet problem:

$$Lu = f, \quad u \in W_o^{1,2}(\Omega), \tag{4.1}$$

where L is the divergent elliptic operator. The function f is called the data which is belongs to generalized weighted Morrey spaces. The operator L is defined by:

$$Lu = -\sum_{i,j=1}^{\infty} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right),$$

where $u \in W_0^{1,2}(\Omega)$, $a_{ij} = a_{ji} \in L^\infty(\Omega)$ for $i, j, \in \{1, 2, \cdots, n\}$, and there exists $\Lambda > 0$ for which:

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n,$$

for $x \in \Omega$ a.e. Moreover, we assume that the coefficients of L satisfy the Dini-continuous condition.

Theorem 4.2. *[32] There exists a unique function $K : \Omega \times \Omega \to [0, \infty]$ such that*

$$G(\cdot, y) \in W^{1,2}(\Omega \setminus B(y, r)) \cap W_0^{1,1}(\Omega), \quad (y, r) \in \Omega \times \mathbb{R}^+,$$

and for $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial G(x, y)}{\partial x_i}\frac{\partial(x)}{\partial x_j}dx = \phi(y).$$

Furthermore,

$$G(x, y) \leq C\frac{1}{|x - y|^{n-2}}, \quad x, y \in \Omega, x \neq y,$$

and

$$|\nabla_x G(x, y)| \leq \frac{1}{|x - y|^{n-1}}, \quad x, y \in \Omega, x \neq y.$$

The function G in the theorem is then called the Green function for the operator L and the domain Ω. For $f \in \mathcal{M}_\psi^{p,w}(\Omega) \cap H^{-1}(\Omega)$, we define:

$$u(x) = \int_\Omega G(x, y)f(y)dy, \quad x \in \Omega. \tag{4.2}$$

Lemma 4.3. *[33] The weak derivative of u is given by:*

$$\frac{\partial u(x)}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\int_\Omega G(x, y)f(y)dy\right) = \int_\Omega \frac{\partial G(x, y)}{\partial x_i}f(y)dy.$$

Theorem 4.4. *Suppose that $\psi \in \mathbb{A}^{p_0,q_0}(w)$ where $1/q_0 = 1/p_0 - 2/n$. Then, there exists a positive constant C such that:*

$$\|u\|_{\mathcal{M}^{q_0,w}_{\psi^{\frac{p_0}{q_0}}}(\Omega)} \leq C\|f\|_{\mathcal{M}_\psi^{p_0,w}(\Omega)}.$$

Proof. One may apply Corollary 4.1 and use the definition of u as in Eq. (4.2) to get the desired inequality. □

By the previous theorem and Theorem 4 in [33], we have the following theorem.

Theorem 4.5. *Suppose that* $\psi \in \mathbb{A}^{p_0,q_0}(w)$ *where* $1/q_0 = 1/p_0 - 2/n$. *Then,* u *is the weak solution of Dirichlet problem (4.1) and* $u \in \mathcal{M}^{q_0,w}_{\psi^{\frac{p_0}{q_0}}}(\Omega)$.

Theorem 4.6. *Suppose that* $\psi \in \mathbb{A}^{p_1,q_1}(w)$ *where* $1/q_1 = 1/p_1 - 1/n$. *Then, there exists a constant* C *such that:*

$$\|\nabla u\|_{\mathcal{M}^{q_1,w}_{\psi^{\frac{p_1}{q_1}}}(\Omega)} \leq C\|f\|_{\mathcal{M}^{p_1,w}_{\psi}(\Omega)}.$$

Proof. Note that by Eq. (4.3) and Theorem 4.2, we have that

$$|\nabla u| = \left(\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right)^{\frac{1}{2}} \leq \sqrt{n}\int_{\Omega}|\nabla_x G(x,y)||f(y)|dy \leq \sqrt{n}\int_{\Omega}\frac{1}{|x-y|^{n-1}}|f(y)|dy.$$

Hence, by Corollary 4.1, we obtain that:

$$\|\nabla u\|_{\mathcal{M}^{q_1,w}_{\psi^{\frac{p_1}{q_1}}}(\Omega)} \leq C\|f\|_{\mathcal{M}^{p_1,w}_{\psi}(\Omega)}$$

and prove the theorem. □

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