

ZONAL LABELING OF VERTEX COMB PRODUCT OF GRAPHS

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Abstract. *Suppose there is a connected plane graph G with a set of vertex $V(G)$ and a set of edges $E(G)$ or $G = (V, E)$. A zonal labeling of graph G is vertex labeling with the two nonzero elements of ring \mathbb{Z}_3 to vertex in graph G such that the sum of the label of the vertices on the boundary of every region of G is the zero elements in \mathbb{Z}_3 . This labeling is zonal and graph G is zonal graph. This paper will discuss zonal labeling on a graph comb product with a graph zonal denoted G . The result states that $F_y \triangleright_o G$, $T \triangleright_o G$, $U \triangleright_o G$ a is graph zonal and $W_z \triangleright_o G$ is not a zonal graph.*

Keywords: Zonal Graph, Zonal Labeling, Vertex Comb Product, Tree, Unicyclic, Fan, Wheel

1. Introduction

A graph G is the set of ordered pairs (V, E) where V is the set of vertex denoted by $V(G)$ and E is the set of edges denoted by $E(G)$ in the graph G . If a graph is drawn on a plane, such that no edges intersect each other, then the graph is called a planar graph [1]. Zonal labeling was first introduced by Chartrand, Cooroo, and Zhang [2]. Expansion of this theory was continued on [3-5]. A zonal labeling of a graph G is vertex labeling with the two nonzero elements in ring \mathbb{Z}_3 to vertex in graph G , denoted by $l(G)$ such that the sum of the label of the vertices on the boundary of every region of G is the zero elements of \mathbb{Z}_3 or written by $l(G) = \sum_{v \in V(G)} l(v) = 0$.

Labeling l is called zonal labeling, and graph G is called zonal graph [2]. Suppose G and G_2 are connected graphs containing x as a vertex in G_2 . The result of the comb product between G and G_2 , denoted by $G \triangleright_o G_2$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of G_2 and identifying the i -th copy of G_2 at the vertex x with the i -th vertex of G [6]. The comb product results in an interior region bounded by $V(G)$ and $V(G_2)$ and also an exterior bounded by $V(G \triangleright_o G_2)$. In this paper, we want to examine the zonal labeling of the comb product between fan, wheel, tree, and unicyclic graph with any zonal graph G .

Before proceeding to the formal definitions and theorems, it is essential to provide a foundational understanding of the basic types of graphs that are central to this study. The following definitions describe fundamental graph structures such as path graphs, cycle graphs, and several graph types namely fan graphs, wheel graphs, trees graphs, and unicyclic graphs.

In addition, a number of established theorems related to the zonality characteristics of these graphs are presented to support the theoretical basis of the subsequent results and proofs.

Definition 1.1. [1] Path graph denoted by $P_w, w \geq 2$, is a graph whose vertex set can be labeled (or relabeled) v_1, v_2, \dots, v_w so that its edges are $v_1v_2, v_2v_3, v_3v_4, \dots, v_{w-1}v_w$.

Definition 1.2. [1] Cycle Graph denoted by $C_x, x \geq 3$, is a graph whose vertex set can be labeled (or relabeled) v_1, v_2, \dots, v_x so that its edges are $v_1v_2, v_2v_3, v_3v_4, \dots, v_{x-1}v_x$, and v_xv_1 . a cycle graph can be said to be a path graph augmented with edge v_xv_1 .

Theorem 1.3. [7] Every cycle graph is zonal graph.

Definition 1.4. Fan Graph denoted by $F_y = K_1 + P_y, y \geq 5$ with $|V(G)| = y + 1$ and can be labeled $v_1, v_2, v_3, \dots, v_y, v_{y+1}$ while $|E(G)| = 2y - 1$ with label $v_1v_2, v_1v_3, v_1v_4, \dots, v_1v_n, v_2v_3, v_3v_4, v_4v_5, \dots, v_yv_{y+1}$.

Theorem 1.5. [7] Every graph F_y is zonal graph if and only if $y = 3k + 2$ and $k \in \mathbb{N}$.

Definition 1.6. Wheel Graph denoted by $W_z, z \geq 3$ is a graph obtained by adding one vertex (call it a) so that vertex a is adjacent to all vertices in the cycle graph.

Theorem 1.7. [7] Every graph (W_z) is zonal if and only if $z = 3k$ and $k \in \mathbb{N}$.

Definition 1.8. A tree graph T is a connected graph that has no cycle graph and a tree graph T has $V(G) = m$ and $E(G) = m - 1$. Whereas a graph that has only exactly one cycle is called unicyclic and a unicyclic graph U has $V(G) = E(G)$.

Theorem 1.9. [7] Every non-trivial tree is zonal graph.

Theorem 1.10. [7] A Unicyclic graph U^* is zonal if and only if $U^* \neq C \star K_2$ for any cycle C .

2. Main Results

In this section, we present some of our research on zonal labeling of some graphs, which are graphs resulting from comb product of a graph with any zonal graph. We explore how this operation on graphs affects the zonality of different graphs and provide formal proof to support our claim. The results are articulated through a series of theorems, starting with Theorem 2.1, which examines that the comb product of fan with any zonal graph is a zonal graph. Although fan graph F_y is

zonal for some $y \geq 3$, it says differently when it combines with any zonal graphs using comb product.

Theorem 2.1. *If graph G is any zonal graph with n vertices, then $F_y \triangleright_o G$ is a zonal graph, for $y \geq 3$.*

Proof. The example of $F_y \triangleright_o G$ is given in the Figure 1 where it shows the graph $F_3 \triangleright_o G$. Based on the Figure 1, we can see that this graph has 2 interior regions, one is the interior region of each copies of G and another is the interior region of F_y . Suppose that $V(G) = \{v_1, v_2, \dots, v_n\}$, $o = v_n$, the chosen vertex o is one that bounds the exterior region of the graph G , $V(F_y) = \{u_i | 1 \leq i \leq y + 1\}$, and $V(F_y \triangleright_o G) = \{v_{i,j} | 1 \leq i \leq y + 1, 1 \leq j \leq n - 1\} \cup \{u_i | 1 \leq i \leq y + 1\}$. Let us define the labeling function $l : V(F_y \triangleright_o G) \rightarrow \{1, 2\}$ as follows. First, based on premise, G is a zonal graph, then there exist $f : V(G) \rightarrow \{1, 2\} \subset \mathbb{Z}_3$ that satisfies the zonal labeling criteria. For the vertex $v_{i,j}$, $l(v_{i,j}) = f(v_j)$, $1 \leq j \leq n - 1$ and $l(u_i) = f(v_n)$, $1 \leq i \leq y + 1$. Next, we show that l is zonal labeling for graph $F_y \triangleright_o G$. Lets check the sum of the label vertices of the interior and exterior region of $F_y \triangleright_o G$.

Bowling [7] explains that every vertex in the graph F_y must have the same label, namely b , such that the sum of vertices which bounding every interior region of F_y is $3b = 0 \in \mathbb{Z}_3$. Since $l(u_i) = f(v_n)$, then all the label of vertices of F_y are the same. Therefore, the condition holds. For the interior region of each copies of G , since the vertex label is induced by G , then the sum of the label is equal to zero. Lastly, we assess the exterior region of $F_y \triangleright_o G$.

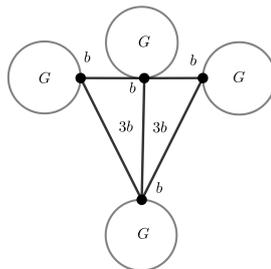


Figure. 1. Graph $F_3 \triangleright_o G$

The exterior region of $F_y \triangleright_o G$ is bounded by the exterior region of each copies of G . Consequently,

$$\begin{aligned}
 \sum_{v \in V(\text{ext}(F_y \triangleright_o G))} l(v) &= (y + 1) \sum_{v \in V(\text{ext}(G))} f(v), \\
 &= (y + 1)3k, \\
 &= 3(ky + k), \\
 &= 3(k(y + 1)), \\
 &= 0 \in \mathbb{Z}_3.
 \end{aligned}$$

Based on the description above, it is obtained that the interior regions of the $F_y \triangleright_o G$ is $0 \in \mathbb{Z}_3$ and the exterior region of $F_y \triangleright_o G$ is $0 \in \mathbb{Z}_3$. Hence, the graph $F_y \triangleright_o G$ is a zonal graph. \square

Theorem 2.1 show that when a zonal graph F_y comb product with any zonal graph, it remains zonal graph. However, there is a zonal graph W_z when comb product with any zonal graph, it becomes a non-zonal graph.

Theorem 2.2. *If graph G is any zonal graph with n vertices, then $W_z \triangleright_o G$ is a non-zonal graph, for $z \geq 3$.*

Proof. Before proving this theorem, let us first examine Figure 2, which shows an example of the graph $W_3 \triangleright_o G$. Based on the figure, we can see that the interior region of graph $W_3 \triangleright_o G$ is similar as that of graph $F_3 \triangleright_o G$ where there are two interior regions, one is the interior region of each copies of G , if exist, and another one is the interior regions of W_z .

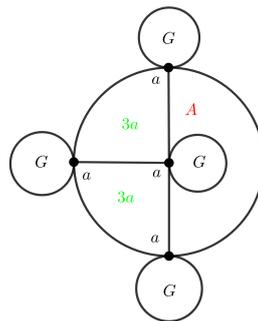


Figure. 2. Graph $W_3 \triangleright_o G$

Assume that f is the vertex labeling of $W_z \triangleright_o G$. In [7], Bowling has explained that all the vertex label of graph W_z are equal. This is because the graph W_z is formed by several C_3 graphs, so every vertex that bounding the interior region and exterior region must be the same. Suppose that a is the vertex label, $a \in \{1, 2\} \subset \mathbb{Z}_3$, then there exist some interior regions in W_z such that the sum of the vertices label is $3a = 0$. Based on premise, G is a zonal graph, then $\sum_{x \in V(\text{ext}(G))} f(x) = 0$. However, there is one interior region in W_z , named A , such that it is bounded by the vertices of the interior region of W_z and the vertices of the exterior region of G . Consequently,

$$\sum_{x \in A} f(x) = \sum_{x \in V(\text{int}W_z)} f(x) + \sum_{x \in V(\text{ext}(G))} f(x) - a = 3a + 3k - a = 2a + 3k \neq 0.$$

Since the sum of the vertices label at one of the interior region of $W_n \triangleright_o G \neq 0$, it is not a zonal graph. \square

Theorem 2.3. *Suppose G is any zonal graph with n vertices and T is any non-trivial tree graph with m vertices, then the graph $T \triangleright_o G$ is a zonal graph.*

Proof. We refer to Figure 3, which illustrates the graph $T \triangleright_o G$. If G is a tree graph, then $T \triangleright_o G$ is also a tree and has shown that it is a zonal graph [7]. If G is not a tree, we can identify that this graph's interior region lies within graph G . Let $V(G) = \{v_1, v_2, \dots, v_n\}$, $o = v_n$, the chosen vertex o is one that bounds the exterior region of the graph G and $V(T) = \{u_i | 1 \leq i \leq m, \}$, then $V(T \triangleright_o G) = \{v_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n - 1\} \cup \{u_i | 1 \leq i \leq m\}$. Next, we will define the labeling function $l : V(T \triangleright_o G) \rightarrow \{1, 2\}$ as follows. First, by assumption, G is a zonal graph, then there exist a function $f : V(G) \rightarrow \{1, 2\} \subset \mathbb{Z}_3$ that satisfies the zonal labeling criteria. For the vertex $v_{i,j}$, we set $l(v_{i,j}) = f(v_j)$, $1 \leq j \leq n - 1$ and $l(u_i) = f(v_n)$ for $1 \leq i \leq m$. Thus, we verify l is zonal labeling for graph $T \triangleright_o G$ by calculating the sum label of the vertices in the interior regions and exterior region of the graph.

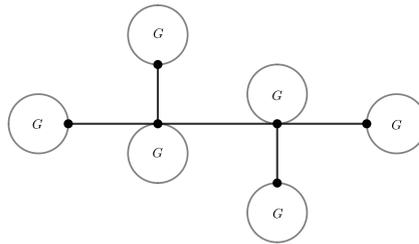


Figure. 3. Graph $T \triangleright_o G$

Since $T \triangleright_o G$ has only one interior region, which is the interior regions of G , and G is zonal, we have $\sum_{x \in V(\text{int}(G))} l(x) = 0$. For the exterior region of $T \triangleright_o G$, we have:

$$\begin{aligned} \sum_{v \in V(\text{ext}(T \triangleright_o G))} l(v) &= (m) \sum_{v \in V(\text{ext}(G))} f(v), \\ &= (m)3k, \\ &= 3(km), \\ &= 0 \in \mathbb{Z}_3. \end{aligned}$$

Therefore, it is proven that the graph $T \triangleright_o G$ is a zonal graph. □

According to [7] a unicyclic graph with the form of $U^* = C \star K_2$, for every cycle graph C is not a zonal graph. But, the comb product of any unicyclic graph U and any zonal graph G is a zonal graph.

Theorem 2.4. *If G is a zonal graph with n vertices and U is a unicyclic with m vertices, then $U \triangleright_o G$ is a zonal graph.*

Proof. Before proceeding with the proof, let us observe the Figure 4, which depicts the graph $U \triangleright_o G$. Based on Figure 4, we can see that the graph $U \triangleright_o G$ has one interior region within graph U and another is the interior of each copies of G , if any. Let $V(U) = \{u_i | 1 \leq i \leq m\}$, and $V(G) = \{v_1, v_2, \dots, v_n\}$, $o = v_n$, the chosen vertex o is one that bounds the exterior region of the graph G then $V(U \triangleright_o G) = \{v_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i | 1 \leq i \leq m\}$. We define the labeling function $l : V(U \triangleright_o G) \rightarrow \{1, 2\}$ as follows. For v any vertex in each copies of G , we induced the labeling function of G , named f , to label each of these copies, since G is a zonal graph. Next, we put specific attention to the vertex of cycle subgraph of U . We appoint different vertex in each copies of G to merge with vertex of cycle such that the sum of each label of vertex of cycle is 0. It is possible because cycle is a zonal graph and there exist $u, v \in V(G)$, such that $f(u) = 1$ and $f(v) = 2$. If $f(x) = 2$ for all $x \in V(G)$, then it is easy to prove that $g(x) = 3 - f(x)$ is also a zonal labeling of G . Next, we verify that l is zonal labeling for graph $U \triangleright_o G$ by calculating the sum of labels for each vertices in the interior region(s) and exterior region of $U \triangleright_o G$.

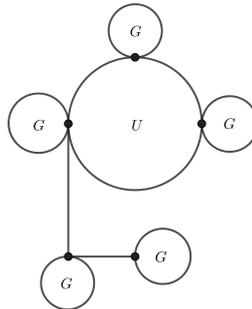


Figure. 4. Graph $U \triangleright_o G$

First, consider the interior region of the cycle. Since cycle is a zonal graph, we have $\sum_{x \in V(\text{int}(G))} l(x) = 0$. For each copy of G , since the vertex labeling is induced by f , we have $\sum_{x \in V(\text{int}(G))} f(x) = 0$. Lastly, for the exterior region of $U \triangleright_o G$, we get:

$$\begin{aligned} \sum_{v \in V(\text{ext}(U \triangleright_o G))} l(v) &= (m) \sum_{v \in V(\text{ext}(G))} f(v), \\ &= (m)3k, \\ &= 3(km), \\ &= 0 \in \mathbb{Z}_3. \end{aligned}$$

Based on the above, we conclude that the graph $U \triangleright_o G$ is a zonal graph. This result also applies to a comb product between cycle graph and any zonal graph. \square

3. Conclusion

This study discusses zonal labeling on graphs that are operated with comb product on zonal graphs. The research shows that certain types of graphs will maintain their zonal properties after this operation, while others will not. Specifically, it was found that Fan, Tree, and Unicyclic graphs operated with the comb product on a zonal graph will remain zonal graphs. Conversely, wheel graph subjected to the same operation will lose their zonal properties. These findings provide deeper insights into how graph operations can affect zonality and pave the way for further research on graph characteristics in the context of zonal labeling.

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Bibliography

- [1] Chartrand, G. , Zhang, P., 2013, *A First Course of Graph Theory*, Courier Corporation
- [2] Chartrand, G., Egan, C., Zhang, P., 2019, *How to Label a Graph*, Springer, New York
- [3] Bowling, A., Xie, W., 2024, Zonal Labeling and Tait Colorings From A New Perspective, *Aequationes Mathematicae* Vol. **6**: 1 – 15
- [4] Bowling, A., Zhang, P., 2022, Absolutely and Conditionally Zonal Graphs, *Electron. Journal of Mathematics* Vol. **4**: 1 – 11
- [5] Chartrand, G., Egan, C., Zhang, P., 2020, Zonal Graph Revisited, *Bulletin of The Institute of Combinatorics and its Applications* Vol. **99**: 132 – 152
- [6] Bollobas, B., 1998, *Modern Graph Theory* (Vol. 184), Springer Science and Business Media.
- [7] Bowling, A., 2023, *Zonality in Graphs*, Western Michigan University, Dissertation, 3934, <https://scholarworks.wmich.edu/dissertations/3934>
- [8] Saputro, S.W., Mardiana, N., Purwasih, I.A., 2017, The metric dimension of comb product graphs, *Matematički Vesnik* Vol. **69**(4): 248 – 258