

HAMILTONIAN CYCLES IN WIJAYA KUSUMA FLOWER GRAPH

DINDA KHAIRANI NASUTION, SUSILAWATI*

*Department of Mathematics,
Faculty of Mathematics and Natural Sciences, University of Riau
email : susilawati.math@lecturer.unri.ac.id*

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Abstract. *In 1856, William Rowan Hamilton introduced the Icosian game. From this game, the concept of a Hamiltonian graph is defined. Hamiltonian graph is a graph that contains the Hamiltonian cycle, which is a cycle that passes through each vertex exactly once. We constructed a new class of graph which is inspired by the Wijaya Kusuma flower. In this article, we study the Hamiltonian properties of the Wijaya Kusuma flower graph. Based on the proof, it is concluded that the Wijaya Kusuma flower graph is a Hamiltonian graph.*

Keywords: Hamiltonian cycle, Hamiltonian graph, Wijaya Kusuma flower graph

1. Introduction

In daily life, the application of graph modeling is very useful in various aspects such as determining transportation routes, launching production, optimizing on-line flexible manufacturing systems [1], optimization of transportation routes [2], physical mapping of DNA [3], and calculating the perception limit of point patterns [4]. Graphs are used to solve problems by representing objects as vertices and identifying relationships between them through edges.

In graph theory, many interesting connectedness properties are to be studied, such as the Hamiltonian property. The Hamiltonian cycle problem has many practical applications in daily life. Some examples of Hamiltonian in daily life are the Traveling Salesman Problem (TSP) [5], controlling the stitching trace in the world of computing [?], Knight's tours in chess [7], estimating the state of the electrical energy distribution system [8], and solving mazes [9]. According to [10], a Hamiltonian cycle is a cycle that passes through each vertex in the graph exactly once. A graph is called a Hamiltonian graph if it contains a Hamiltonian cycle.

*Corresponding author

Research on Hamiltonian graphs has received more attention from researchers. In [11], the existence of Hamiltonian and Hypohamiltonian properties on the generalized Petersen graph $GP_{(n,1)}$ and $GP_{(n,2)}$ are discussed. Furthermore, [12] discusses the existence of Hamiltonian and Hypohamiltonian properties on generalized Petersen graph $GP_{(n,6)}$. Next, [13] discussed the Hamiltonian path on the grid graph, and [14] discussed the Hamiltonian cycle on the triangular grid graph. From [13] and [14], it was further developed into [?], which discusses the Hamiltonian cycle on the supergrid graphs. From [?], it was further developed into [15], which discusses the Hamiltonian cycle on the linear-convex supergrid graphs. In [13], it is shown that the results of the Hamiltonian cycle and path problems on the grid graphs are NP-complete. The author refers to [16,17,18] for research on Hamiltonian properties in other graphs.

However, the results obtained from these studies are still about the existence of Hamiltonian cycles on existing graph classes such as in [?,10,11,12,13,14,15,16,17,18] and more. In this article, the author constructs a new graph, namely the Wijaya Kusuma flower graph. This graph is inspired by the Wijaya Kusuma flower, which is also known as the queen of the night flower. Furthermore, the author will determine the existence of the Hamilton cycle in the Wijaya Kusuma flower graph. If the Wijaya Kusuma flower graph contains a Hamilton cycle, then this graph is also a Hamiltonian graph.

2. Preliminaries

2.1. Terminology of graph

Graph theory is a branch of mathematics widely used to describe the structure of objects using vertices and edges. According to [18], a graph G is a pair of sets denoted by $G = (V, E)$, where V is a set of vertices and E is a set of edges connecting several vertices in the graph. The set V must not be empty, while E may be an empty set.

According to [10,19], a graph with only one vertex without any edges is called a trivial graph. Letters, natural numbers, or a combination of both can denote the vertices in the graph. An edge is a fundamental component that connects two vertices (u, v) , denoted by uv or vu .

According to [10,20], the order of graph G is the number of vertices in a graph G , denoted by $p(G)$. Size of graph G is the number of edges in a graph G , denoted by $q(G)$.

According to [10,19], in an undirected graph G , if vertices u and v share a common edge $e = uv$, then u and v are said to be adjacent (*directly connected*). Adjacent denoted by $u \sim v$. For any edge $e = uv$, if e is the edge connecting vertices u and v , then e said to be incident (*directly related*) to vertex u and vertex v . In undirected graph, the number of edges incident with u called the degree of a vertex u , denoted by $deg(u)$.

According to [10,19,20], a path of length n from a starting vertex v_1 to a vertex v_n in a graph G is an alternating sequence of vertices and edges of the form $P = \{v_1, e_1, v_2, e_2, v_3, e_3, \dots, v_{n-1}, e_n, v_n\}$ such that $e_1 = v_1v_2, e_2 = v_2v_3, \dots, e_n = v_{n-1}v_n$

are the edges of the graph G . The vertices and edges passing through a path may be repeated. A simple path is a path with each edge distinct and the vertices are passed through only once. A closed path is a path that starts and ends at the same vertex, and an open path is a path that does not start and end at the same vertex. A cycle is a path that starts and ends at the same vertex (closed path).

2.2. Class of graph

Graphs have many classes that are classes based on edge type, edge weight, cycles, vertex degree, connectivity, and more. According to [10,19,20], graphs are divided into undirected and directed graphs based on their direction orientation of the edges. A graph whose edges do not have a direction orientation is called an undirected graph. In an undirected graph, the order of vertex pairs connected by edges is not considered; therefore, $uv = vu$. Figure 1(a) shows an example of an undirected graph. A graph in which each edge has a specific orientation is called a digraph or directed graph. An edge uv is called an arc. In a directed graph, $uv \neq vu$, where u is the origin vertex, and v is the terminal vertex for the arc uv . Figure 1(b) shows a directed graph.

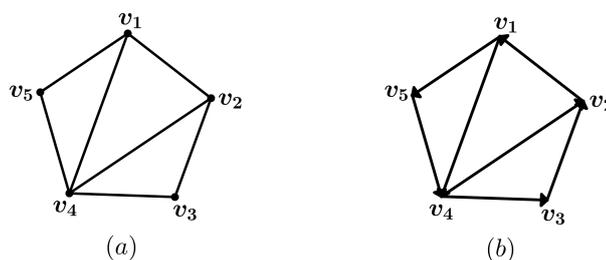


Figure. 1: (a) Undirected graph, and (b) directed graph.

In [9], connected and disconnected graphs are introduced. A connected graph is a graph where for each pair of vertices u and v in the set $V(G)$, there is a path from u to v , and if not, then G is called a disconnected graph. Figure 2(a) and 2(b) show a connected graph and a disconnected graph.

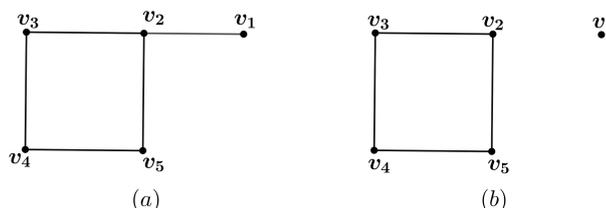


Figure. 2: (a) Connected graph, and (b) disconnected graph.

According to [21], a graph is called a cycle graph if the graph has $n \geq 3$ and is

a simple graph where every vertex has degree two. A cycle graph with n vertices denoted by C_n , where $p(C_n) = n$ and $q(C_n) = n$. Figure 3 shows a cycle graph.

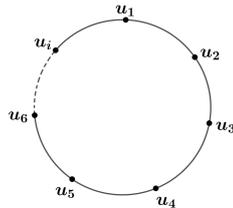


Figure. 3: Cycle graph.

According to [21], a wheel graph is obtained by adding a new vertex to C_n with $n \geq 3$ such that every vertex in C_n is adjacent to the new vertex. The wheel graph has $p(W_n) = n + 1$ and $q(W_n) = 2n$. Figure 4 shows a wheel graph.

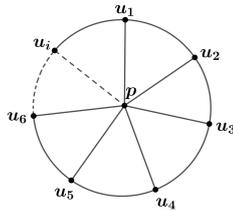


Figure. 4: Wheel graph.

2.3. Properties of Hamiltonian

Based on the properties of the Hamiltonian, Hamiltonian is divided into two categories: Hamiltonian path and Hamiltonian cycle. Hamiltonian path is a path that passes through every vertex of a graph exactly once, and Hamiltonian cycle is a cycle that passes through every vertex exactly once, except the starting vertex ($v_1 = v_n$). A graph that contains a Hamiltonian path is called a semi-Hamiltonian graph. Following is the definition of a Hamiltonian graph.

Definition 2.1. [10] *A graph that contains a Hamiltonian cycle is called a Hamiltonian graph.*

The following lemmas guarantee that the cycle and the wheel graphs are two Hamiltonian graphs.

Lemma 2.2. [10] *Let C_n be a cycle graph with $n \geq 3$, then C_n contains Hamiltonian cycles.*

Proof. We see the graph's structure in Figure 3 to prove this lemma. Consider the cycle graph with $n \geq 3$. First, we construct a Hamiltonian cycle by choosing a vertex v_1 as the starting vertex and passing through all vertices in C_n before returning to v_1 . The Hamiltonian cycle for C_n is as follows.

$$H = \{u_1, u_2, u_3, \dots, u_n, u_1\}.$$

This process yields a Hamiltonian cycle that passes through every vertex in C_n exactly once. It can be concluded that C_n contains a Hamiltonian cycle. \square

Lemma 2.3. [10] *Let W_n be a wheel graph with $n \geq 3$, then W_n contains Hamiltonian cycles.*

Proof. We can see the graph's structure W_n to prove this lemma. Consider the following Figure 4; we construct a Hamiltonian cycle by choosing vertex p as the starting vertex and passing through all vertices in C_n before returning to p . This process yields a Hamiltonian cycle as follows.

$$H = \{p, v_1, v_2, \dots, v_n, p\}. \quad \square$$

2.4. Wijaya Kusuma flower graph

First, we construct a new class of graph called the Wijaya Kusuma flower graph. The Wijaya Kusuma flower graph is inspired by the Wijaya Kusuma flower, also known as the queen of the night, because this flower only blooms for a few hours at night. Figures 5 (a) and (b) show the Wijaya Kusuma flower and the structure of the Wijaya Kusuma flower.



Figure. 5: (a) Wijaya Kusuma flower (b) Structure of the Wijaya Kusuma flower

The Wijaya Kusuma flower consists of four components: carpel, stamen, petal, and sepal. Constructing a graph from the real flower begins by representing each component as a set of vertices and then connecting these components with edges.

The vertex p represents the carpel, the vertices r_i represent the stamen, the vertices $s_i, s_{(2j,1)}, s_{(2j,2)}$ for odd i represent the petals, and the vertices s_i for even i represent the sepals. Figure 6 shows the Wijaya Kusuma flower graph with $n \geq 2$, and the following is the definition of the Wijaya Kusuma flower graph.

Definition 2.4. *The Wijaya Kusuma flower graph, denoted by WK_n , represents the structure of the Wijaya Kusuma flower. The construction begins with a wheel graph, with p as the center vertex and r_i as the vertices on the outer cycle of the wheel. For each odd index i , the vertices r_i are connected to corresponding vertices s_i , which have degree three. For each even index i , the vertices r_i are connected to two vertices $s_{(2j,1)}$ and $s_{(2j,2)}$. Furthermore, the vertices $s_i, s_{(2j,1)}$, and $s_{(2j,2)}$ are connected to form an outer cycle, where the even-indexed vertices s_i lies between the corresponding vertices $s_{(2j,1)}$ and $s_{(2j,2)}$. For $n \geq 2$, the vertex and edge sets of Wijaya Kusuma flower graph are as follows.*

$$V(WK_n) = \{p, r_i, s_i, s_{(2j,1)}, s_{(2j,2)} \mid i = 1, 2, \dots, 2n, j = 1, 2, \dots, n\},$$

$$E(WK_n) = \{pr_i, r_i r_{i+1}, r_{2j-1} s_{2j-1}, r_{2j} s_{(2j,1)}, r_{2j} s_{(2j,2)}, s_{(2j,1)} s_{2j}, s_{(2j,2)} s_{2j}, s_{(2j,2)} s_{2j+1}, s_{(2j,1)} s_{2j-1} \mid i = 1, 2, \dots, 2n, j = 1, 2, \dots, n; i + 1 \bmod 2n, 2j + 1 \bmod 2n\}.$$

The Wijaya Kusuma flower graph is given in Figure 6.

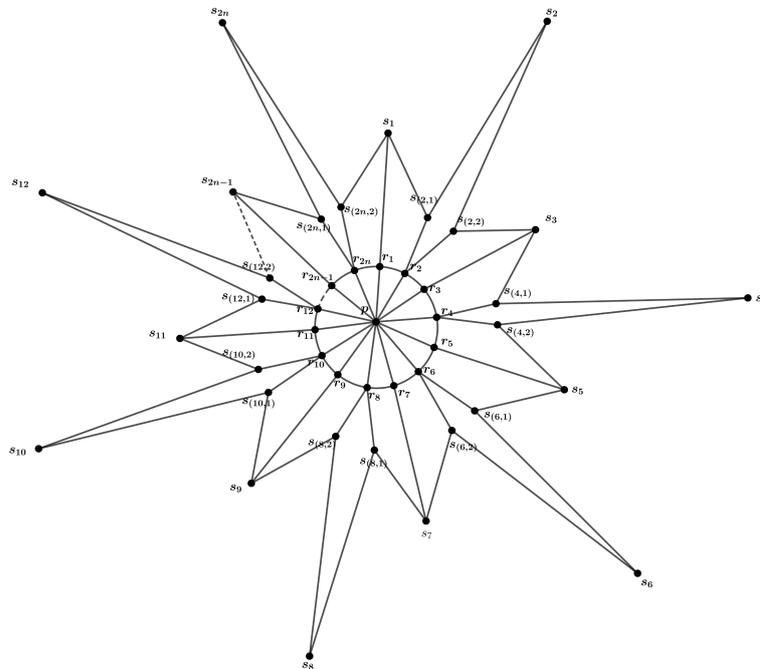


Figure. 6: Wijaya kusuma flower graph.

3. Results and Discussion

In this section, a Hamiltonian cycle in the Wijaya Kusuma flower graph is proven by identifying at least one such cycle. The following lemma guarantees that the graph WK_n , for $n \geq 2$, contains at least one Hamiltonian cycle.

Lemma 3.1. *Let $G = WK_n$ be a Wijaya Kusuma flower graph with $n \geq 2$, then WK_n contains Hamiltonian cycles.*

Proof. The proof of this lemma utilizes the partition method. The idea of the proof using the partition method is to identify two disjoint Hamiltonian cycles in the graph WK_n , denoted by H_1 and H_2 , respectively. Then, two distinct edges are determined, namely $e_1 = \{r_{2i}r_{2i+1} \mid i \text{ even}\}$ and $e_2 = \{s_{(2j,2)}s_i \mid j = 1, 2, \dots, n; i \text{ odd}\}$. The edges e_1 and e_2 are called parallel edges if $r_{2i} \sim s_{(2i,2)}$ and $r_{2i+1} \sim s_{2i+1}$, denoted by $e_1 \approx e_2$. If there exist edges e_1 in H_1 and e_2 in H_2 such that $e_1 \approx e_2$, then the Hamiltonian cycles H_1 and H_2 can be combined into one Hamiltonian cycle in the graph WK_n , denoted by H . If there is more than one pair of edges satisfying $e_1 \approx e_2$, then choose one pair of edges that meets these conditions.

The proof begins by identifying H_1 and H_2 through a partitioning process of the graph WK_n into two disjoint subgraphs, as H_1 and H_2 are two disjoint Hamiltonian cycles. Figure 7 shows the result of the partitioning process on the graph WK_n , where the disjoint subgraphs are indicated. The orange dashed lines indicate the partitioning of the graph.

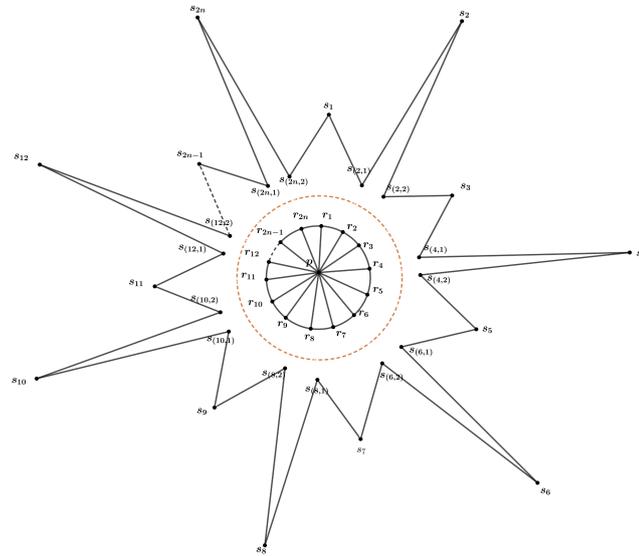


Figure. 7: WK_n partition result.

In Figure 7, the partitioning of the graph WK_n results in two disjoint subgraphs, that is, C_n and W_n . To find the Hamiltonian cycle of each subgraph, use Lemma

2.2 and Lemma 2.3. H_1 is obtained from Lemma 2.3, while H_2 is obtained from Lemma 2.2. In Figure 8, H_1 is shown in blue, and H_2 is shown in green.

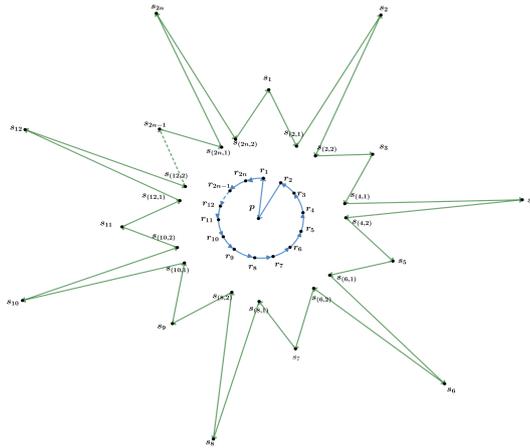


Figure. 8: Hamiltonian cycles for each subgraph.

Next, we obtain $e_1 = \{r_2r_3, r_4r_5, r_6r_7, \dots, r_{2n}r_1\}$ in H_1 , and $e_2 = \{s_{(2,2)}s_3, s_{(4,2)}s_5, s_{(6,2)}s_7, \dots, s_{(2n,2)}s_1\}$ in H_2 . In this proof, the selected edge pair is $e_1 = r_{2n}r_1$ and $e_2 = s_{(2n,2)}s_1$. Furthermore, the vertices $r_{2n} \sim s_{(2n,2)}$ and $r_1 \sim s_1$ are connected using arcs, such that $e_1 \approx e_2$. These arcs must be oriented in opposite directions, following the orientation of the two Hamiltonian cycles in each subgraph. In Figure 9, the added arcs are shown in red. This process results in a cycle formed by combining H_1 and H_2 , but this cycle is not yet a Hamiltonian cycle.

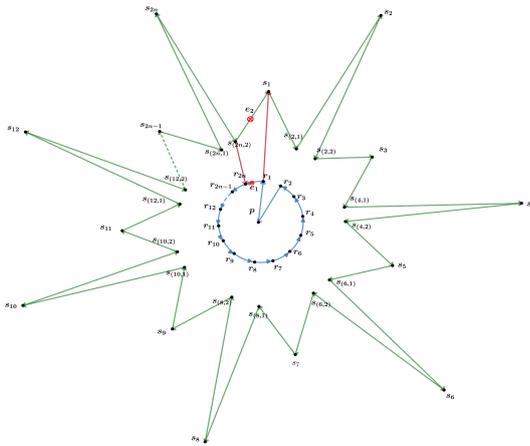


Figure. 9: The edges $e_1 \approx e_2$ in the partitioned graph WK_n .

To obtain a Hamiltonian cycle on WK_n , H_1 and H_2 must be one cycle. Remove the edges e_1 and e_2 to connect H_1 and H_2 . We obtain a cycle H , which is a cycle that passes through each vertex exactly once. In Figure 9, the removal of edges e_1 and e_2 is represented with a red \otimes .

After merging H_1 and H_2 into H , the Hamiltonian cycle for WK_n is obtained. Figure 10 shows the Hamiltonian cycle in the flower graph with $n \geq 2$. It can be concluded that the WK_n contains Hamiltonian cycles.

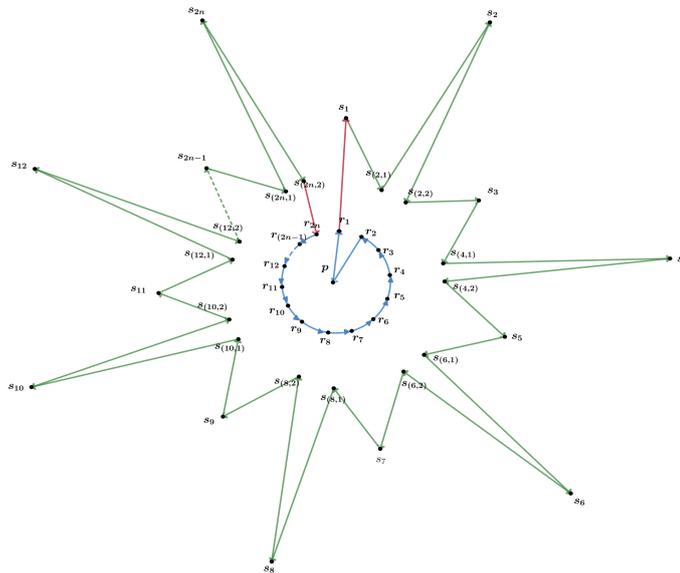


Figure. 10: Hamiltonian cycle in WK_n .

□

The existence of a Hamiltonian cycle in WK_n has been proven; thus, we obtain the following theorem.

Theorem 3.2. *Let $G = WK_n$ be a Wijaya Kusuma flower graph with $n \geq 2$, then WK_n is a Hamiltonian graph.*

Proof. Theorem 3.2 is proven using Definition 2.1 and Lemma 3.1. Definition 2.1 states that a graph is Hamiltonian if it contains a Hamiltonian cycle, and Lemma 3.1 states that the graph WK_n contains a Hamiltonian cycle for every $n \geq 2$. Therefore, it can be concluded that the graph WK_n is a Hamiltonian graph. □

4. Conclusion

In this article, we prove that the Wijaya Kusuma flower graph is a Hamiltonian graph by finding a Hamiltonian cycle. After completing all the steps of the proof, the Hamiltonian cycle for the WK_n is obtained, leading to the conclusion that the WK_n is the Hamiltonian graph.

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