Jurnal Matematika UNAND
Vol. **15** No. **4** Hal. 460 – 465
Edisi Oktober 2026
ISSN : 2303–291X
e-ISSN : 2721–9410
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https://doi.org/10.25077/jmua.15.4.460-465.2026

THE FIRST FUNDAMENTAL GROUP OF THE PERMUTATION GROUP

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Received February 10, 2025 Received in revised form January 21, 2026
Accepted October 16, 2026 Available online October 22, 2026

Abstract. This study aims to determine the elements of the first fundamental group based on identity graphs derived from permutation groups, particularly those representing symmetric groups. Identity graphs are constructed by treating vertices as group elements and connecting them with edges when their product results in the identity element. The research focuses on identifying the components of the first fundamental group within these graph structures. The elements of the fundamental group are interpreted as sets of equivalent closed paths that start and finish at the same vertex. The findings indicate that the number of these equivalence classes in a fundamental group constructed from an identity graph reflects specific graph structures, offering new perspectives on the connection between permutation groups and graph theory.

Keywords: Permutation Group, The First Fundamental Group, Identity Graph.

1. Introduction

The study of graph theory related to algebraic structures has become an intriguing topic, attracting the attention of many researchers. This can be seen in various studies conducted by several researchers.

The study in [1] examines graphs derived from non-abelian groups, revealing that if two graphs are isomorphic, then the orders of the groups associated with these graphs are identical. Another result from [1] indicates that any group must be nilpotent if its graph is isomorphic to the graph of a non-abelian nilpotent group. Further studies related to [1] are found in [2,3], which discuss the characteristics of non-abelian groups associated with non-abelian AC-groups. It is shown that if

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a non-abelian AC-group is nilpotent and its graph is isomorphic to the graph of another group, then the non-abelian group is also nilpotent, and the order of the center of the AC-group exceeds that of the non-abelian group. Another study in [4] explores the graph characteristics of semidihedral groups, while [5] examines graphs on rings instead of groups.

The study in [6] investigates graphs derived from groups involving central elements, which are elements that commute with all other elements in the group. Based on the graph's structure, it is determined whether the graph is biregular or regular. Conversely, [7] examines graphs that exclude central elements. The study in [7] demonstrates that the non-centralizer graph associated with two isoclinic groups has centers of equal order and is isomorphic.

Other studies address the properties of cyclic graphs of a group in [8], non-cyclic graphs in [9], and non-coprime graphs of several groups, as discussed in [10,11,12].

This article will use an alternative definition of a graph derived from a group, as elaborated in Section 2. Previously, [13] applied this definition to the Kronecker quaternion group. However, this study focuses on permutation or symmetric groups, which are expected to yield different properties. Furthermore, the definitions and theorems related to the first fundamental group, together with the main results of this article, will be presented in Section 3.

2. Identity Graph of Permutation Group

In this section, we discuss the identity graph of the permutation group, starting with a definition of the permutation group.

Definition 2.1. *Let X be a set in $\mathbb{R}^n$. The permutation group of X in $\mathbb{R}^n$ is the set of all isometries of $\mathbb{R}^n$ that carry X onto itself. The group operation is function composition.*

For the following, the permutation group will be denoted by S_n. If $A = \{1, 2, \cdots, n\}$, then the permutation group of A is $S_n = \{\psi : A \to A \mid \psi \text{ is bijective}\}$. Elements in S_n are often represented in two-lined notation, such as:

$$\psi = \begin{pmatrix} a(1) & a(2) & \cdots & a(n) \\ b(1) & b(2) & \cdots & b(n) \end{pmatrix}$$

where $b(i) = \psi(a(i))$ for $i = 1, 2, \cdots,$ n

The inverse of ψ is $\psi^{-1} = \begin{pmatrix} b(1) & b(2) & \cdots & b(n) \\ a(1) & a(2) & \cdots & a(n) \end{pmatrix}$, such that $\psi \cdot \psi^{-1} = \psi^{-1} \cdot \psi = e$. The next theorem discusses about order of $P = \{\psi : A \to A \mid \psi = \psi^{-1}, \psi \neq e\}$.

Theorem 2.2. *If $P = \{\psi : A \to A \mid \psi = \psi^{-1}, \psi \neq e\}$, then:*

$$|P| = \sum_{k=1}^{m} \frac{n!}{2^k \cdot k!(n-2k)!}, \quad \textit{where } m = \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. By the definition of P, the elements of P can be expressed as a composition of 2-cycles. Then, the order of P is the sum of different compositions of

2-cycles. Elements of P are compositions of $m = \left\lfloor \frac{n}{2} \right\rfloor$ different 2-cycles. Assume that $\psi = (a_{i_1}\ a_{j_1})\,(a_{i_2}\ a_{j_2}) \cdots (a_{i_m}\ a_{j_m})$, where $\{a_{i_1}, a_{j_1}, a_{i_2}, a_{j_2}, \cdots, a_{i_m}, a_{j_m}\}$ are all distinct. The probability of the first pair of 2-cycles is C_2^n. For the second pair, since there are only $n-2$ remaining elements, the probability of the second pair is C_2^{n-2}. This continues until the m^{th}-pair of 2-cycles. Since $\alpha \cdot \beta = \beta \cdot \alpha$, where α and β are 2-cycles in ψ, then the probability of ψ is:

$$C_2^n + \frac{C_2^n C_2^{n-2}}{2!} + \cdots + \frac{C_2^n C_2^{n-2} \cdots C_2^{n-2(m-1)}}{m!}.$$

Consider that:

$$\begin{aligned}
|P| &= C_2^n + \frac{C_2^n C_2^{n-2}}{2!} + \cdots + \frac{C_2^n C_2^{n-2} \cdots C_2^{n-2(m-1)}}{m!}, \\
&= \sum_{k=1}^{m} \frac{\prod_{j=0}^{k-1} C_2^{n-2j}}{k!}, \\
&= \sum_{k=1}^{m} \frac{\frac{n(n-1)}{2} \cdot \frac{(n-2)(n-3)}{2} \cdots \frac{(n-2k+2)(n-2k+1)}{2}}{k!}, \\
&= \sum_{k=1}^{m} \frac{n(n-1)(n-2)\cdots(n-2k+1)}{2^k \cdot k!}, \\
&= \sum_{k=1}^{m} \frac{n!}{2^k \cdot k!(n-2k)!}.
\end{aligned}$$

□

Next, this paper considers a simple directed graph without loops or multiple edges. For any graph $\mathcal{J}$, let $V(\mathcal{J})$ and $E(\mathcal{J})$ denote its sets of vertices and edges, respectively. For graph-theoretic terminology, we refer to [14]; for group-theoretic terminology, we refer to [15]. The identity graph of a group is defined as follows [16].

Definition 2.3. *[16] Let H be a group and let e be the identity element of H. The graph $\mathcal{J}_H$ associated with H is defined as follows:*

(1) The vertex set $V(\mathcal{J}_H)$ consists of the elements of H.
(2) The edge set $E(\mathcal{J}_H)$ is defined as follows: two distinct elements $h_1, h_2 \in H$ are joined by an edge if and only if $h_1 h_2 = e$.

Additionally, every element $h \in H \setminus \{e\}$ is adjacent to e, where e is the identity element of H. The graph $\mathcal{J}_H$ is called the identity graph of H.

Theorem 2.4. *[16] Given $A = \{1, 2, \cdots, n\}$. If $S_n = \{\psi : A \to A \mid \psi \text{ is bijective}\}$ and $P = \{\psi : A \to A \mid \psi = \psi^{-1}, \psi \neq e\}$, then the number of possible identity graphs τ_{S_n} is $8^{\frac{n!-(|P|+1)}{2}} \cdot 2^{|P|}$.*

Proof. Since S_n is a group, each element of $S_n \setminus (P \cup \{e\})$ has an inverse distinct from itself. Hence each pair $\{\psi, \psi^{-1}\}$ together with e forms a triangle K_3, while each $\psi \in P$ forms a K_2 with e. Therefore, the number of copies of K_3 in $\mathcal{J}_{S_n}$ is

$$\frac{n! - (|P| + 1)}{2},$$

and the number of copies of K_2 is $|P|$.

As a result, since each K_3 has 8 possible directions and each K_2 has 2 possible directions, the total number of possibilities for the identity graph τ_{S_n} is:

$$8^{\frac{n!-(|P|+1)}{2}} \cdot 2^{|P|}. \qquad \square$$

3. The First Fundamental Group of Permutation Groups

This paper primarily addresses the first fundamental group, focusing on related topics such as directed graphs and their equivalence classes of closed paths. Given that $\mathcal{F}'$ is a directed connected graph, let u be the selected vertex, and $[\delta]$ the equivalence class of a closed path δ based at u, satisfying $\iota(\delta) = \tau(\delta) = u$. Here, $\iota(\delta)$ represents the initial vertex, and $\tau(\delta)$ denotes the terminal vertex of δ. The collection $\{[\delta] \mid \iota(\delta) = \tau(\delta) = u\}$ represents the set of equivalence classes of closed paths at u. Theorem 3.1 discusses about the first fundamental group and its binary operations.

Theorem 3.1. *The algebraic structure* $\pi_1(\mathcal{F}', u) = \{[\delta] \mid \iota(\delta) = \tau(\delta) = u\}$, *equipped with the binary operation:*

$$[\delta_1] \cdot [\delta_2] = [\delta_1 \delta_2],$$

forms a group known as the first fundamental group, with base point u. *In this group,* $[1_u]$ *is the identity element, and* $[\delta^{-1}]$ *is the inverse of* $[\delta]$, *such that* $[\delta]^{-1} = [\delta^{-1}]$.

Next, the first fundamental group is explained by selecting an arbitrary vertex. For the case of a directed identity graph, the first fundamental group can be determined through three cases. These are as follows: the group can be based at a vertex in K_2, a vertex in K_3, or the vertex that serves as the identity element of the group.

Theorem 3.2. *Given* $A = \{1, 2, \cdots, n\}$, $S_n = \{\psi : A \to A \mid \psi \text{ is bijective } \}$ *and* $P = \{\psi : A \to A \mid \psi = \psi^{-1}, \psi \neq e \}$. *Given that* $\mathcal{J}_{S_n}$ *is a directed identity graph of the permutation group, then:*

$$|\pi_1(\mathcal{J}_{S_n}, u)| = \begin{cases} 1 & \text{if } u \text{ is in } K_2, \\ 3 & \text{if } u \text{ is in } K_3, \\ n! - |P| & \text{if } u = e. \end{cases}$$

Proof. We consider three cases based on the vertex u's position in the graph

(Case 1) u is a vertex in K_2 and $u \neq e$ where e is the identity element of S_n. In this case, the set of closed paths at u is given by:

$$\pi_1(\mathcal{J}_{S_n}, u) = \{[\delta] \mid \iota(\delta) = \tau(\delta) = u\} = \{[1_u]\},$$

where $[1_u]$ is the equivalence class of the trivial path at u. Thus, $|\pi_1(\mathcal{J}_{S_n}, u)| = 1$.

(Case 2) u is a vertex in K_3 and $u \neq e$ with e is the identity of S_n. Then u belongs to exactly one triangle, namely $\{e, u, u^{-1}\}$. Let δ_u be the closed path at u along this triangle. The set of closed paths at u is:

$$\pi_1(\mathcal{J}_{S_n}, u) = \{[1_u], [\delta_u], [\delta_u^{-1}]\}.$$

Hence, $|\pi_1(\mathcal{J}_{S_n}, u)| = 3$.

(Case 3) u is the identity element e of the group. The vertex e belongs to all $\frac{n!-(|P|+1)}{2}$ triangles, and each triangle gives two closed paths $[\delta_i]$ and $[\delta_i^{-1}]$. Together with the trivial path $[1_e]$, we have:

$$|\pi_1(\mathcal{J}_{S_n}, e)| = 2 \cdot \frac{n! - (|P| + 1)}{2} + 1 = n! - |P|.$$

In this case, all closed paths include the trivial path, other paths related to the structure of K_3, and the inverses of those paths, which adhere to the properties of the fundamental group.

In all cases, the result follows as stated in the theorem. □

4. Conclusion

This study determines the number of identity graphs and the fundamental group of permutation groups. The identity graph for a permutation group S_n is constructed based on elements that satisfy specific identity relations. The total number of identity graphs for S_n is given by:

$$8^{\frac{n!-(|P|+1)}{2}} \cdot 2^{|P|}$$

where $P = \{\psi : A \to A \mid \psi = \psi^{-1}, \psi \neq e \}$ and size of P is given by:

$$|P| = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n!}{2^k \cdot k!(n-2k)!}$$

The fundamental group of the identity graph depends on the choice of base point, with possible orders of 1, 3, or $n! - |P|$.

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