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ON THE ENERGY OF p-SEMISHADOW AND p-NEW DUPLICATE GRAPH

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Abstract. *Graph is mathematical structures used to represent relationships between objects as a vertex and edge (relationship between vertices). The energy of the graph is the absolute sum of the eigenvalues of its adjacency matrix, the graph representation matrix with entries 1 and 0. This article introduces a new graph operation, p-new duplicate, and determines energy of p-semishadow and p-new duplicate.*

Keywords: Energy, p-semishadow graph, p-new duplicate graph

1. Introduction

A graph is a set of pairs (V, E) where $V(G)$ is the set of vertices on the graph denoted as $V(G) = \{v_1, v_2, \cdots, v_n\}$, and $E(G)$ is the set of edges on the graph G that contains an unordered pair of two vertices on the graph G, notated by $v_iv_j \in E(G)$ with $1 \leq i, j \leq n$. The graph that is used in this study is a simple graph without a loop and double edge so that the set of edges is $\{v_iv_j\}$ with $i \neq j$ and $1 \leq i, j \leq n$. The number of vertices on the graph G is called order, $|V(G)| = n$, and the number of edges on the graph G is called size, $|E(G)| = m$. Two vertices v_i and v_j are adjacent if $v_iv_j \in E(G)$. The neighbors of a vertex v are the vertices adjacent to it, and their set is called the open neighborhood of v denoted by $N(v)$. Degree vertex v_i of graph G is denoted by $d_G(v_i)$, which is many edges that are incident on the vertex v_i. The adjacency matrix of graphs G, notated by $A(G)$, is a symmetrical matrix with a size the same as its graph order, and the entry of that matrix is 1 for $v_iv_j \in E(G)$ and 0 for others. Energy of the graph G, notated as $\varepsilon(G)$, is the absolute sum of the eigenvalues of the adjacency matrix. The spectrum of graph G, notated as $spec(G)$, is the set of eigenvalues and their multiples, e.g., the spectrum on graph (G) with λ_i, for eigenvalues and m_i for multiples of the

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eigenvalue λ_i, that is notated as:

$$spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

A graph is one of the tools of applied mathematics in various fields, such as in the chemical field, where a molecule is represented as a graph, its atomic collection is assumed to be a set of vertices, and the relationships between its atoms are assumed to be edges. The energy from the graph was first introduced by Gutman in [1], inspired by the Huckel Molecular Orbital theory used to explain the behavior of electrons in molecules [2]. Various applications of graph energy in different fields are presented [3] at chemistry, [4] at ecology, and [5] at military.

Until now, energy has been widely studied and developed; various previous researchers obtained energy, and their development of the graph with operation. Vaidya and Popat examined the energy of the operation graph in [6], namely the splitting graph and shadow graph that were introduced by Sampathkumar [7]. The two operations were generalized by Abdel-Aal in [8] and mentioned as p-splitting graph and q-shadow graph, for $p = 1, 2, \cdots$ and $q = 2, 3, \cdots$. Then in another article, Vaidya and Popat examine energy from p-splitting and q-shadow graphs [9]. Both operations are of great interest to other researchers because the characteristics of the p-splitting graph and q-shadow graph can be related to the characteristics of the graphs G. Billal and Munir [10] in obtaining energy ABC (atomic bound connectivity) and spectral radii ABC (greatest eigenvalues of the matrix ABC) of the p-splitting graph and q-shadow graph. Zhang and others in [11] examined the maximum and minimum spectral radii of the p-splitting graph and q-shadow graph. Furthermore, Patel et al. in [12] introduced the p-semishadow graph and obtained the energy degree sum from the p-splitting graph and p-semishadow graph.

The duplicate graph was first introduced by [13] as a unitary operation on the graph. [14] defined the generalized duplicate graph as a tensor operation of the graph G with K_2 (the complete graph with two vertices), and they obtained its energy. In this study, a new operation is introduced, namely a p-new duplicate graph D_p, which is inspired by the p-splitting graph, q-shadow graph, and q-semishadow graph.

In this article we obtain the properties of the p-semishadow graph, namely order, size, degree of vertices, and energy. In addition, we introduce a new operation, specifically the p-new duplicate graph, and obtain its properties, specifically order, size, degree of vertices, and energy.

2. Preliminaries

In this section, several definitions and propositions are presented, which will be utilized in the main results section.

Definition 2.1. [*12*] *The p-semishadow graph is obtained by copying each vertex p times for $p \geq 1$ be an integers, so that the vertex set is given by $\{i = 1, 2, \cdots, n \mid v_i^1, v_i^2, \cdots, v_i^p\}$. Each vertex v_i^p is adjacent to neighbors of vertex $v_i \in G$, and vertex v_i^p is adjacent to v_j^p if v_i and v_j are adjacent in G, with $1 \leq i, j \leq n$. The p-semishadow graph of the graph G is notated with $Ssh_p(G)$.*

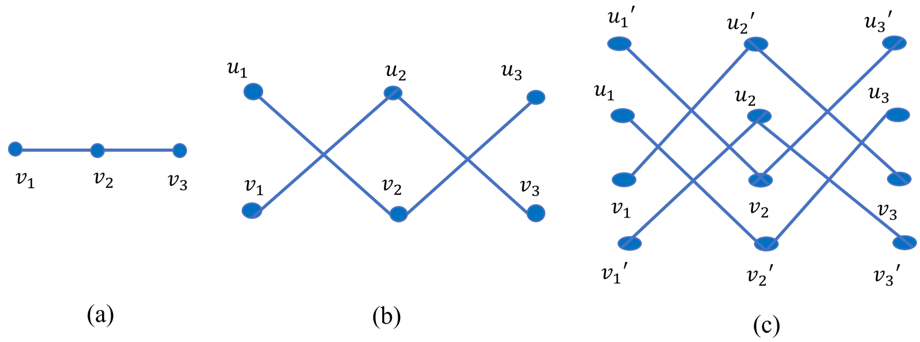

Figure. 1. (a) Graph P_3, (b) Graph $D(P_3)$, (c) Graph $D^2(P_3)$

Definition 2.2. *[13] Let G be a graph with n vertices, and V' be a set such that $|V| = |V'|$ and $f : V \to V'$ be a bijective map, for $v_i \in V$, where $v_i \mapsto v_i'$. Suppose a duplicate graph of G denoted by $D(G)$ that is a graph with a set of vertices $V(D(G)) = V \cup V'$, whose edges are given as follows. In the graph G, v_iv_j is an edge if and only if $v_i'v_j$ and v_iv_j' are edges in $D(G)$.*

Sampathkumar [13] also defines the duplicate graph in general terms as $D^p(G)$, obtained recursively by $D^{p-1}\ (D(G))$. Later, Patil and Raja at [14] provided an alternative and simpler definition of the duplicate graph using a tensor product, namely $D^p(G) = G \otimes pK_2$, and easily computed its energy. By Sampathkumar definition, graph $D^2(P_3)$ is obtained recursively as $D^2(P_3) = D\,(D(P_3))$, whereas according to Patil and Raja, it is simply obtained as $P_3 \otimes 2K_3$. The graph P_3, $D(P_3)$, and $D^2(P_3)$ are shown in Figure 1. After obtaining the graph $D^2(P_3)$, it can be observed that this resulting graph is isomorphic to $4P_3$.

Next, we introduce the definition of the Kronecker product of matrices. This property is particularly useful for determining the eigenvalues of duplicate graphs defined via the Kronecker product, as it simplifies the spectral analysis significantly.

Definition 2.3. *[15] Kronecker products of two matrices are defined as follows. Let A and B be matrices of order $m \times m$ and $n \times n$. The Kronecker product $A \otimes B$ is given as:*

$$A \otimes B := [a_ijB] = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix}.$$

Proposition 2.4. *[15] Let $A_{m\times m}$ and $B_{n\times n}$ be two square matrices, the eigenvalues of A and B are consecutive by λ_A and λ_B, then the eigenvectors corresponding to λ_A and λ_B are consecutive by $\overrightarrow{x}_{m\times 1}$ and $\overrightarrow{y}_{n\times 1}$. The eigenvalues of $A \otimes B$ are $\lambda_A \cdot \lambda_B$ that corresponds to the eigenvector $\overrightarrow{x} \otimes \overrightarrow{y}$ with ordo $mn \times 1$.*

We also provide the definition of a block matrix and its determinant to facilitate the computation of eigenvalues.

Proposition 2.5. *[16] Suppose A is a matrix partitioned as:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

such as A_{11} and A_{22} are square matrices and invertible. Matrix determinant of A obtained by

$$\begin{aligned} |A| &= |A_{11}|.|(A_{22} - A_{21}(A_{11})^{-1}A_{12})|, \text{ or} \\ &= |A_{22}|.|(A_{11} - A_{12}(A_{22})^{-1}A_{21})|. \end{aligned}$$

3. Main Result

This section is divided into two subsections. The first subsection discusses the characteristics and energy of the p-semishadow graph, while the second subsection focuses on the characteristics and energy of the p-new duplicate graph.

3.1. *Graph p-Semishadow*

The characteristics of p-semishadow graph $Ssh_p(G)$, such as order, size, and degree of each vertex, are explained at Lemma 3.1.

Lemma 3.1. *Let G be a graph with $|V(G)| = n$, $|E(G)| = m$, and let $(Ssh_p(G))$ be a p-semishadow graph of G for $p = 1, 2, \cdots$. Then:*

a. $|V(Ssh_p(G))| = (p+1) \cdot |V(G)|$.
b. $|E(Ssh_p(G))| = (3p+1) \cdot |E(G)|$.
c. $d_{Ssh_p(G)}(v_i) = (p+1) \cdot d_G(v_i)$.
d. $d_{Ssh_p(G)}(v_i^p) = 2 \cdot d_G(v_i)$.

Theorem 3.2 provides a detailed formulation and analysis of the energy of graph $Ssh_p(G)$.

Theorem 3.2. *Let G be a simple graph with ordo n, then the energy of $Ssh_p(G)$ is:*

$$\varepsilon(Ssh_p(G)) = \varepsilon(G).((p-1) + 2\sqrt{p}).$$

Proof. Let G be a graph on n vertices. The adjacency matrix of G is a n-square matrix with $a = 1$ for $v_iv_j \in E$, $1 \le i, j \le n$ and 0 for others.

$$A(G) = \begin{bmatrix} 0 & a & \cdots & a \\ a & 0 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & 0 \end{bmatrix}. \tag{3.1}$$

Graph p-semishadow of G, for $p = 1, 2, \cdots$, has order

$$|V(Ssh_p(G))| = (p+1) \cdot |V(G)| = (p+1) \cdot n = pn + n,$$

based on Lemma 3.1, so that the adjacency matrix has an order, $A(Ssh_p(G))_{pn+n}$. Let $\{v_i^1, v_i^2, \cdots, v_i^p\}$, for $i = 1, 2, \cdots, n$ be the vertices in $Ssh_p(G)$. Then the vertices

$\{v_i^k\}$ are adjacent to each vertex v_i such that $N(v_i^1) = N(v_i^2) = \cdots = N(v_i^p) = N(v_i)$. Therefore, matrix $A(Ssh_p(G))$ can be simplified to Kronecker product of the adjacency matrix G, $A(G)$ with the following matrix.

$$A(Ssh_p(G)) = \begin{bmatrix} A(G) & A(G) & A(G) & \cdots & A(G) \\ A(G) & A(G) & 0 & \cdots & 0 \\ A(G) & 0 & A(G) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & 0 & 0 & \cdots & A(G) \end{bmatrix}_{p-1} = A(G) \otimes \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Thus, the matrix $A(Ssh_p(G))$ can be represented as the Kronecker product of two matrices, namely $A(G)$ and B, which is written in the notation $A(Ssh_p(G)) = A(G) \otimes B$. At this stage, to determine the eigenvalues of the matrix $A(Ssh_p(G))$, Proposition 2.4 is applied by assuming that μ is an eigenvalue of the matrix $A(G)$ and λ is an eigenvalue of the matrix B. Therefore, to get the eigenvalues it is divided into several steps. The first step is to determine the eigenvalues of the matrix B. The eigenvalues of B satisfy this equation $|B - \lambda I| = 0$:

$$\begin{vmatrix} 1-\lambda & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 0 & \cdots & 0 \\ 1 & 0 & 1-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1-\lambda \end{vmatrix} = 0.$$

Then matrix B is partitioned as follows.

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where:

$$B_{11} = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}, B_{12} = \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix}_{2\times(p-1)},$$
$$B_{21} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix}_{(p-1)\times 2}, B_{22} = \begin{bmatrix} 1-\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1-\lambda \end{bmatrix}_{(p-1)\times(p-1)}.$$

According to Proposition 2.5, the eigenvalues are obtained by:

$$|B - \lambda I| = |B_{22}|.|(B_{11} - B_{12}(B_{22})^{-1}B_{21})|,$$

with $|(B_{11} - B_{12}(B_{22})^{-1}B_{21})|$ acquired.

$$
\begin{aligned}
|(B_{11} - B_{12}(B_{22})^{-1}B_{21})| &= \left| B_{11} - \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix} \cdot \left(\frac{1}{1-\lambda}\right) \begin{bmatrix} 1-\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1-\lambda \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix} \right|, \\
&= \left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - \left(\frac{1}{1-\lambda}\right) \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix} \right|, \\
&= \left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - \left(\frac{1}{1-\lambda}\right) \begin{bmatrix} p-1 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix} \right|, \\
&= \begin{bmatrix} \dfrac{(1-\lambda)^2 - (p-1)}{1-\lambda} & 1 \\ 1 & 1-\lambda \end{bmatrix}.
\end{aligned}
$$

The eigenvalues of B satisfy the following equation:

$$
\begin{aligned}
& |B_{22}|.|(B_{11} - B_{12}(B_{22})^{-1}B_{21})| = 0, \\
\Longleftrightarrow & \begin{bmatrix} 1-\lambda & 0 & \cdots & 0 \\ 0 & 1-\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\lambda \end{bmatrix} \cdot \begin{bmatrix} \dfrac{(1-\lambda)^2 - (p-1)}{1-\lambda} & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0, \\
\Longleftrightarrow & (1-\lambda)^{p-1}.(\lambda^2 - 2\lambda - p + 1) = 0.
\end{aligned}
$$

The characteristic polynomial of B is:

$$
p(B) = (1-\lambda)^{p-1}(\lambda^2 - 2\lambda - p + 1) = 0.
$$

Next, the second step is to express the spectrum of $Ssh_p(G)$ as follows.

$$
spec(Ssh_p(G)) = \mu_i \begin{pmatrix} 1+\sqrt{p} & 1 & 1-\sqrt{p} \\ 1 & p-1 & 1 \end{pmatrix},
$$

where μ_i are the eigenvalues of the $A(G)$ for $i = 1, 2, \cdots, n$. Next, the third step is to obtain the energy of graph $Ssh_p(G)$:

$$
\begin{aligned}
\varepsilon(Ssh_p(G)) &= \sum_{i=1}^{n} |\mu_i| \cdot |(p-1)| + |1-\sqrt{p}| + |1+\sqrt{p}| \\
&= \varepsilon(G) \cdot ((p-1) + 1 + \sqrt{p} - 1 + \sqrt{p}) \\
&= \varepsilon(G) \cdot ((p-1) + 2\sqrt{p}).
\end{aligned}
$$

□

In Example 3.3, we give the application of Lemma 3.1 and Theorem 3.2.

Example 3.3. Let P_3 and $Ssh_2(P_3)$ be shown in the Figure 2. Then:

a. The order of $Ssh_2(P_3)$ is $|V(Ssh_2(P_3))| = (p+1).|V(P_3)| = 9$.
b. The size of $Ssh_2(P_3)$ is $|E(Ssh_2(P_3))| = (3p+1).|E(P_3)| = 14$.
c. The degree of every vertex $v_i \in Ssh_2(G)$, for $1 \leq i \leq 3$ are shown in Table 1.

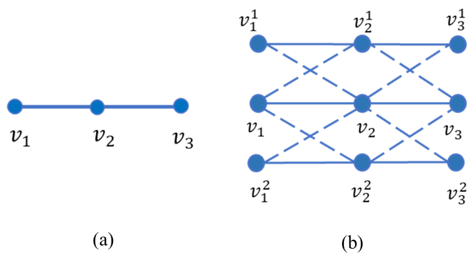

Figure. 2. (a) Graph P_3 (b)Graph $Ssh_2(P_3)$

Table 1. Degree of vertex $v_i \in Ssh_2(P_3)$

$d_{P_3}(v_i)$	$d_{Ssh_2(P_3)}(v_i)$
$v_1 = 1$	$v_1 = 3 \cdot 1 = 3$
$v_2 = 2$	$v_2 = 3 \cdot 2 = 6$
$v_3 = 1$	$v_3 = 3 \cdot 1 = 3$

Table 2. Degree of vertex $v_i^k \in Ssh_2(P_3)$

$d_{P_3}(v_i)$	$d_{Ssh_2(P_3)}(v_i^k)$
$v_1 = 1$	$v_1^k = 2 \cdot 1 = 2$
$v_2 = 2$	$v_2^k = 2 \cdot 2 = 4$
$v_3 = 1$	$v_3^k = 2 \cdot 1 = 2$

d. Degree of each vertex $v_i^k \in Ssh_2(G)$, for $1 \leq i \leq 3$ and $1 \leq k \leq 2$ are shown at Table 2.

It is obvious that the eigenvalues and energy of P_3 are as follows.

$$spec(P_3) = \begin{pmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}.$$

The energy of $Ssh_p(G)$, with $\varepsilon(P_3) = 2\sqrt{2}$ is:

$$\begin{aligned} \varepsilon(Ssh_2(P_3)) &= \varepsilon(P_3).((2-1) + 2\sqrt{2}), \\ &= 2\sqrt{2} \cdot (1 + 2\sqrt{2}), \\ &= 10.8284. \end{aligned}$$

3.2. *Graph p-New Duplicate*

We introduce new operation that is expanded from $D^p(G)$ by Sampathkumar in [12], namely p-new duplicate graph, which inspired by p-splitting, p-shadow and p-semishadow graph ([8], [12]).

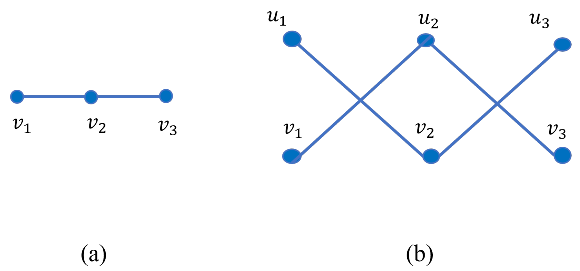

Figure. 3. (a) Graph P_3, (b) Graph $ND_2(P_3)$

Definition 3.4. *Let G be a graph with n vertices, and V^p, for $p = 1, 2, \cdots,$ be a set such that $|V| = |V^p|$ and $f : V \to V^p$ be a bijective map, for $v_i \in V$, where $v_i \mapsto v_i^p$. The p-new duplicate graph of G, denoted by $ND_p(G)$, has vertex set $V(ND_p(G))$ and edge set $E(ND_p(G))$ as follows.*

$$V(ND_p(G)) = V \cup V^1 \cup V^2 \cup \cdots \cup V^p,$$
$$E(ND_p(G)) = \left\{v_i v_j^p | v_i \in V, v_j^p \in V^p, v_i v_j^p \in E(G)\right\}.$$

Several characteristics of the graph $ND_p(G)$, such as order, size, and the degree of each vertex, are given at Lemma 3.5.

Lemma 3.5. *Suppose that G is a graph with $|V(G)| = n$, $|E(G)| = m$, and p-new duplicate graph of G is $(ND_p(G))$ for $p = 1, 2, \cdots$. Then:*

a. $|V(ND_p(G))| = (p + 1).|V(G)|$.
b. $|E(ND_p(G))| = 2p.|E(G)|$.
c. $d_{ND_p(G)}(v_i) = p.d_G(v_i)$.
d. $d_{ND_p(G)}(v_i^p) = d_G(v_i)$.

Subsequently, the energy of the graph $ND_p(G)$ is determined in Theorem 3.6.

Theorem 3.6. *Let G be a graph, then the energy of $ND_p(G)$ is:*

$$\varepsilon(ND_p(G)) = \varepsilon(G) \cdot 2\sqrt{p}.$$

Proof. Let G be graph with n vertices. The Adjacency matrix of G is a n-square matrix with $a = 1$ for $v_i v_j \in E$, $1 \le i, j \le n$ and 0 for others. The Adjacency matrix of G shown in Equation 3.1, Graph p-new duplicate of G, for $p = 1, 2, \cdots$, based on Lemma 3.2, has an order $|V(ND_p(G))| = (p+1).|V(G)| = (p+1).n = pn+n$. Therefore, the adjacency matrix has an order $A(ND_p(G))_{pn+n}$. Let $\left\{v_i^1, v_i^2, \cdots, v_i^p\right\}$, for $i = 1, 2, \cdots, n$ be the vertices in $Ssh_p(G)$. Then the vertices $\left\{v_i^k\right\}$ adjacent to each vertex v_i such that $N(v_i^1) = N(v_i^2) = \cdots = N(v_i^p) = N(v_i)$. Next, the matrix $A(ND_p(G))$ can be simplified to Kronecker product of the adjacency matrix G,

$A(G)$ with the following matrix.

$$A(ND_p(G)) = \begin{bmatrix} 0 & A(G) & A(G) & \cdots & A(G) \\ A(G) & 0 & 0 & \cdots & 0 \\ A(G) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & 0 & 0 & \cdots & 0 \end{bmatrix}_{p+1} = A(G) \otimes \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, the matrix $A(ND_p(G))$ can be represented as the Kronecker product of two matrices, namely $A(G)$ and C, which is written in the notation $A(ND_p(G)) = A(G) \otimes C$. At this stage, to determine the eigenvalues of the matrix $A(ND_p(G))$, Proposition 2.4 is applied by assuming that μ is an eigenvalue of the matrix $A(G)$ and λ is an eigenvalue of the matrix C. Therefore, there are several steps to determine the eigenvalues.

The first step is to determine the eigenvalues of the matrix C. The eigenvalues of C satisfy the equation $|C - \lambda I| = 0$.

$$\begin{vmatrix} -\lambda & 1 & 1 & \cdots & 1 \\ 1 & -\lambda & 0 & \cdots & 0 \\ 1 & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -\lambda \end{vmatrix} = 0.$$

The matrix can be represented as a partitioned matrix:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

with:

$$C_{11} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}, C_{12} = \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix}_{2\times(p-1)},$$
$$C_{21} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix}_{(p-1)\times 2}, C_{22} = \begin{bmatrix} 1-\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1-\lambda \end{bmatrix}_{(p-1)\times(p-1)}.$$

The second step, based on Proposition 2.5, the eigenvalues are obtained by:

$$|C - \lambda I| = |C_{22}|.|(C_{11} - C_{12}(C_{22})^{-1}C_{21})|,$$

with $|(C_{11} - C_{12}(C_{22})^{-1}C_{21})|$ acquired.

$$|(C_{11} - C_{12}(C_{22})^{-1}C_{21})| = \left| C_{22} - \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix} \cdot \left(\frac{1}{-\lambda}\right) \begin{bmatrix} 1-\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1-\lambda \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix} \right|,$$

$$= \left| \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} - \left(\frac{1}{-\lambda}\right) \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix} \right|,$$

$$= \left| \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} - \left(\frac{1}{-\lambda}\right) \begin{bmatrix} p-1 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix} \right|,$$

$$= \begin{bmatrix} -\lambda + \dfrac{(p-1)}{\lambda} & 1 \\ 1 & -\lambda \end{bmatrix}.$$

The eigenvalues of C satisfy the following equation.

$$|C - \lambda I| = |C_{22}|.|(C_{11} - C_{12}(C_{22})^{-1}C_{21})| = 0,$$

$$\Longleftrightarrow \begin{bmatrix} -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda \end{bmatrix} \cdot \begin{bmatrix} -\lambda + \frac{(p-1)}{\lambda} & 1 \\ 1 & -\lambda \end{bmatrix} = 0,$$

$$\Longleftrightarrow p(C) = (-\lambda)^{p-1}(\lambda^2 - p) = 0.$$

The characteristic polynomial of C is $p(C) = (-\lambda)^{p-1}(\lambda^2 - p) = 0$. Next, the spectrum of $ND_p(G)$ is:

$$spec(ND_p(G)) = \mu_i \begin{pmatrix} \sqrt{p} & 1 & -\sqrt{p} \\ 1 & p-1 & 1 \end{pmatrix},$$

where μ_i is the eigenvalues of the matrix $A(G)$ for $i = 1, 2, \cdots, n$. Moreover, the third step is to obtain the energy of graph $ND_p(G)$ as follows.

$$\begin{aligned} \varepsilon(ND_p(G)) &= \sum_{i=1}^{n} |\mu_i|.| - \sqrt{p}| + |\sqrt{p}|, \\ &= \varepsilon(G).\left(\sqrt{p} + \sqrt{p}\right), \\ &= \varepsilon(G).\left(2\sqrt{p}\right). \end{aligned}$$

□

In Example 3.7, we give the application of Lemma 3.5 and Theorem 3.6.

Example 3.7. Let P_3 be a path graph with three vertices and two edges. The 2-semishadow graph of P_3, denoted $ND_2(P_3)$, is shown in Figure 3. It will be shown that:

a. The order of $ND_2(P_3)$ is $|V(ND_2(P_3))| = 9$.
b. The size of $ND_2(P_3)$ is $|E(ND_2(P_3))| = 8$.
c. The degree of each vertex $v_i \in ND_2(G)$, for $1 \leq i \leq 3$ are shown in Table 3.

d. The degree of each vertex $v_i^k \in ND_2(G)$, for $1 \leq i \leq 3$ and $1 \leq k \leq 3$ are shown in Table 4.

It obvious that the energy of P_3 is as follows.

$$spec(P_3) = \begin{pmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}.$$

The energy of $ND_2(P_3)$, with $\varepsilon(P_3) = 2 : \sqrt{2}$ is

$$\begin{aligned} \varepsilon(D_2(P_3)) &= \varepsilon(P_3).(2\sqrt{2}), \\ &= 2\sqrt{2} \cdot (2\sqrt{2}), \\ &= 8. \end{aligned}$$

Table 3. Degree of vertex $v_i \in ND_2(P_3)$

$d_{P_3}(v_i)$	$d_{ND_2(P_3)}(v_i)$
$v_1 = 1$	$v_1 = 2 \cdot 1 = 2$
$v_2 = 2$	$v_2 = 2 \cdot 2 = 4$
$v_3 = 1$	$v_3 = 2 \cdot 1 = 2$

Table 4. Degree of vertex $v_i^k \in ND_2(P_3)$

$d_{P_3}(v_i)$	$d_{ND_2(P_3)}(v_i^k)$
$v_1 = 1$	$v_1^k = 1$
$v_2 = 2$	$v_2^k = 2$
$v_3 = 1$	$v_3^k = 1$

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