Jurnal Matematika UNAND
Vol. **15** No. **3** Hal. 346 – 357
Edisi Juli 2026
ISSN : 2303–291X
e-ISSN : 2721–9410

https://doi.org/10.25077/jmua.15.3.346-357.2026

ON CONSTRUCTING *H*-IRREGULARITY LABELING WITH MINIMUM LABEL FOR CERTAIN BALLOON GRAPHS

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Received April 3, 2025, Received in revised form February 5, 2026
Accepted July 10, 2026 Available online July 31, 2026

Abstract. If ϱ and H are simple, connected, undirected graphs, and ϱ can be covered by an H-covering, then for a positive integer p, a total p-labeling φ on ϱ is considered as a total H-irregular p-labeling if, for every subgraph K of ϱ that is isomorphic to H, the weight of K (the sum of the labels of all vertices and edges of K) is a unique number. The smallest integer p for which graph ϱ can be labeled with a total H-irregular p-labeling is called the total H-irregularity strength of graph G. This paper presents the exact total H-irregularity strength values for some particular graphs including balloon graphs, double balloon graphs, and double balloon ladder graphs.

Keywords: balloon graph, double balloon graph, double balloon ladder graph, total $H-$irregularity strength

1. Introduction

Wilson (see [1]) defines a graph ϱ as $(V(\varrho), E(\varrho))$, where $V(\varrho)$ represents a finite set of nonempty vertices and $E(\varrho)$ is a finite set of unordered pairs of distinct elements from $V(\varrho)$, known as edges.

Wallis in [2] describes a labeling on ϱ as a function whose domain is either $V(\varrho)$, $E(\varrho)$, or $V(\varrho) \cup E(\varrho)$, and its codomain is a set of positive or nonnegative integers. When the domain is $V(\varrho)$, it is called vertex labeling; when it is $E(\varrho)$, it is edge labeling; and when it is $V(\varrho) \cup E(\varrho)$, it is total labeling. Various labeling techniques can be employed, one of which is irregular labeling with the domain $V(\varrho) \cup E(\varrho)$ mapping to a set of positive integers. This labeling is discussed in this paper.

Gallian in [3] states that there are about 200 types of labeling techniques, evolving in approximately 3000 papers. These techniques include irregular labeling, introduced by Chartrand et al. (see [4]). In irregular labeling, the weight of a vertex $v \in V(\varrho)$ relative to a total labeling is defined as the sum of the label of

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v and all labels of edges incident to v. From some types of irregular labeling on a graph ϱ, the concepts of total edge irregularity strength (tes) and total vertex irregularity strength (tvs) arise.

Ashraf et al. (see [5]) introduced the strength of irregularity-H, an extension of tes and tvs. A graph ϱ with an H-covering means that for any edge of ϱ, there exists a subgraph of ϱ isomorphic to H that includes the edge. Let $\varphi : V(\varrho) \cup E(\varrho) \to \{1, 2, \cdots, p\}$ be any total p-labeling on ϱ. For any subgraph K of ϱ isomorphic to H, the H-weight of K with respect to φ, denoted by $wt_\varphi(K)$, is defined as $wt_\varphi(K) = \sum_{v\in V(K)} \varphi(v) + \sum_{e\in E(K)} \varphi(e)$. A total p-labeling on ϱ is called H-irregular if $wt_\varphi(K') \neq wt_\varphi(K'')$ for every two distinct subgraphs K' and K'' isomorphic to H. The smallest integer p such that ϱ can be labeled with an H-irregular total p-labeling is called the total H-irregularity strength of ϱ, denoted by $tHs(\varrho, H)$.

Recently, $tHs(\varrho, H)$ has been extensively studied. Agustin et al. in [6] obtained results on $tHs(\varrho, H)$ for shackles and amalgamation graphs. Nisviasari et al. in [7] researched $tHs(\varrho, H)$ for triangular ladder and grid graphs. Hidayatul et al. (see [8]) obtained results on $tHs(\varrho, H)$ for grid, butterfly, hexagonal, and diamond graphs. Suni et al. in [9] investigated diamond ladder, circular triple ladder, and prism graphs. Wahyujati et al. (see [10]) studied the total H-irregularity strength of edge comb product graphs. Shulhany et al. (see [11]) examined the same parameter for some classes of graphs, while Labane et al. (see [12]) have done so for cycles and diamond graphs. In this work, we introduce a new type of graph that has not been previously studied by other researchers, which involves applying specific graph operations. These results not only provide fresh insights but also extend the applicability of $tHs(\varrho, H)$ to a broader spectrum of graph types. The use of graph operations in the investigation further highlights the structural complexity and versatility of the studied graphs. This finding will significantly enrich the theory of $tHs(\varrho, H)$ by deepening our understanding of its behavior in diverse graph classes and paving the way for future studies in this area.

Theorem 1.1. *[5] Let ϱ be a graph admitting an $H-$covering and have w subgraphs isomorphic to H. Then:*

$$tHs(\varrho, H) \geq \left\lceil 1 + \frac{w-1}{|V(H)| + |E(H)|} \right\rceil.$$

2. Result and Discussion

Our research findings will be presented in this section. First, we begin with the following definition.

Definition 2.1. *The balloon graph Bl_η, $\eta \geq 1$, is a graph with:*

$$V(Bl_\eta) = \{x_q, y_q : q = 1, 2, \cdots, \eta\} \cup \{z_q : q = 1, 2, \cdots, \eta+1\}, \text{ and}$$
$$E(Bl_\eta) = \{x_q y_q, y_q z_q, y_q z_{q+1}, z_q z_{q+1} : q = 1, 2, \cdots, \eta\}.$$

Figure 1 provided an example of balloon graph, denoted by Bl_5.

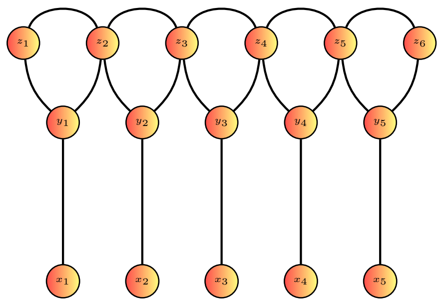

Figure. 1. Balloon Graph Bl_5

Theorem 2.2. *Given $Bl_\eta, \eta \geq 1$ with $Bl_\zeta-$covering, for $1 \leq \zeta \leq \eta$. Then,*

$$tHs(Bl_\eta, Bl_\zeta) = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil.$$

Proof. The balloon Bl_η, $\eta \geq 1$ admits $Bl_\zeta-$covering with exactly $\eta - \zeta + 1$ balloons Bl_ζ, $1 \leq \zeta \leq \eta$ and η is a positive integer. From Theorem 1.1, it holds:

$$tHs(Bl_\eta, Bl_\zeta) \geq \left\lceil 1 + \frac{(\eta - \zeta + 1) - 1}{7\zeta + 1} \right\rceil = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil.$$

Next, put $p = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil$, then $tHs(Bl_\eta, Bl_\zeta) \leq p$ will be proven. On the balloon graph Bl_η with $Bl_\zeta-$covering, $1 \leq \zeta \leq \eta$, we define:

$$\Psi_{\eta,\zeta} : V(Bl_\eta) \cup E(Bl_\eta) \to \{1, 2, \cdots, p\},$$

for $\zeta = 1, 2, \cdots, \eta$. In this manner, for $q = 1, 2, \cdots, \eta$:

$$\Psi_{\eta,\zeta}(x_q) = \left\lceil \frac{q}{7\zeta + 1} \right\rceil, \Psi_{\eta,\zeta}(y_q) = \left\lceil \frac{q + 2\zeta}{7\zeta + 1} \right\rceil, \Psi_{\eta,\zeta}(x_q y_q) = \left\lceil \frac{q + \zeta}{7\zeta + 1} \right\rceil,$$

$$\Psi_{\eta,\zeta}(y_q z_q) = \left\lceil \frac{q + 3\zeta}{7\zeta + 1} \right\rceil, \Psi_{\eta,\zeta}(y_q z_{q+1}) = \left\lceil \frac{q + 4\zeta}{7\zeta + 1} \right\rceil, \Psi_{\eta,\zeta}(z_q z_{q+1}) = \left\lceil \frac{q + 5\zeta}{7\zeta + 1} \right\rceil.$$

For $q = 1, 2, \cdots, \eta + 1$,

$$\Psi_{\eta,\zeta}(z_q) = \left\lceil \frac{q + 6\zeta}{7\zeta + 1} \right\rceil.$$

The largest label is reached when $q = \eta + 1$, that is,

$$\Psi_{\eta,\zeta}(z_q) = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil.$$

For every $r = 1, 2, 3, \cdots, \eta - \zeta + 1$, the vertex set is:

$$V(Bl_\zeta) = \{x_{r+q}, y_{r+q} : q = 0, 1, 2, \cdots, \zeta - 1\} \cup \{z_{r+q} : q = 0, 1, 2, \cdots, \zeta\},$$

and the edge set is:

$$E(Bl_\zeta) = \{x_{r+q} y_{r+q}, y_{r+q} z_{r+q}, y_{r+q} z_{r+q+1}, z_{r+q} z_{r+q+1} : q = 0, 1, 2, \cdots, \zeta - 1\}.$$

Furthermore, for every subgraph Bl_{ζ}^{r} in Bl_{ζ} the weight of $Bl_{\zeta}^{r}, r = 1, 2, \cdots, \eta - \zeta + 1$, with respect to $\Psi_{\eta,\zeta} : V(Bl_{\zeta}) \cup E(Bl_{\zeta}) \to \{1, 2, \cdots, p\}$ is:

$$wt_{\Psi_{\eta,\zeta}}(Bl_{\zeta}^{r}) = \sum_{v \in V(Bl_{\zeta}^{r})} \Psi_{\eta,\zeta}(v) + \sum_{e \in E(Bl_{\zeta}^{r})} \Psi_{\eta,\zeta}(e).$$

Then, for $r = 1, 2, \cdots, \eta - \zeta$,

$$\begin{aligned} wt_{\Psi_{\eta,\zeta}} Bl_{\zeta}^{r+1} - wt_{\Psi_{\eta,\zeta}} Bl_{\zeta}^{r} &= \Psi_{\eta,\zeta}(x_{r+\zeta}) + \Psi_{\eta,\zeta}(y_{r+\zeta}) + \Psi_{\eta,\zeta}(z_{r+\zeta+1}) \\ &+ \Psi_{\eta,\zeta}(x_{r+\zeta}y_{r+\zeta}) + \Psi_{\eta,\zeta}(y_{r+\zeta}z_{r+\zeta}) \\ &+ \Psi_{\eta,\zeta}(y_{r+\zeta}z_{r+\zeta+1}) + \Psi_{\eta,\zeta}(z_{r+\zeta}z_{r+\zeta+1}) - \Psi_{\eta,\zeta}(x_{r}) \\ &- \Psi_{\eta,\zeta}(y_{r}) - \Psi_{\eta,\zeta}(z_{r}) - \Psi_{\eta,\zeta}(x_{r}y_{r}) - \Psi_{\eta,\zeta}(y_{r}z_{r}) \\ &- \Psi_{\eta,\zeta}(y_{r}z_{r+1}) - \Psi_{\eta,\zeta}(z_{r}z_{r+1}) \\ &= \left\lceil \frac{r+\zeta}{7\zeta+1} \right\rceil + \left\lceil \frac{r+3\zeta}{7\zeta+1} \right\rceil + \left\lceil \frac{r+7\zeta+1}{7\zeta+1} \right\rceil + \left\lceil \frac{r+2\zeta}{7\zeta+1} \right\rceil \\ &+ \left\lceil \frac{r+4\zeta}{7\zeta+1} \right\rceil + \left\lceil \frac{r+5\zeta}{7\zeta+1} \right\rceil + \left\lceil \frac{r+6\zeta}{7\zeta+1} \right\rceil - \left\lceil \frac{r}{7\zeta+1} \right\rceil \\ &- \left\lceil \frac{r+2\zeta}{7\zeta+1} \right\rceil - \left\lceil \frac{r+6\zeta}{7\zeta+1} \right\rceil - \left\lceil \frac{r+\zeta}{7\zeta+1} \right\rceil - \left\lceil \frac{r+3\zeta}{7\zeta+1} \right\rceil \\ &- \left\lceil \frac{r+4\zeta}{7\zeta+1} \right\rceil - \left\lceil \frac{r+5\zeta}{7\zeta+1} \right\rceil \\ &= 1. \end{aligned}$$

For every $r = 1, 2, \cdots, \eta - \zeta$ we get:

$$wt_{\Psi_{\eta,\zeta}}(Bl_{\zeta}^{r}) < wt_{\Psi_{\eta,\zeta}}(Bl_{\zeta}^{r+1}).$$

Hence,

$$wt_{\Psi_{\eta,\zeta}}(Bl_{\zeta}^{r+1}) = 1 + wt_{\Psi_{\eta,\zeta}}(Bl_{\zeta}^{r}).$$

Moreover, all $DBL_{\zeta}-$weights are distinct. Therefore, $\Psi_{\eta,\zeta}$ is total $Bl_{\zeta}-$irregular labeling of Bl_{η}. Since $tHs(Bl_{\eta}, Bl_{\zeta}) \geq p$ and $tHs(Bl_{\eta}, Bl_{\zeta}) \leq p$, we have:

$$tHs(Bl_{\eta}, Bl_{\zeta}) = p = \left\lceil \frac{6\zeta + \eta + 1}{7\zeta + 1} \right\rceil. \qquad \square$$

Figure 2 is an example of Bl_{10} with $Bl_{3}-$covering and that $tHs(Bl_{10}, Bl_{3}) = 2$.

Now, by double balloon graph, we mean the graph defined in Definition 2.3 as follows.

Definition 2.3. *A double ballon graph DBl_{η}, $\eta \geq 1$, is a graph with:*

$$V(DBl_{\eta}) = \{x_q, y_q : q = 1, 2, \cdots, \eta\} \cup \{w_q, z_q : q = 1, 2, \cdots, \eta + 1\} \text{ and}$$
$$E(DBl_{\eta}) = \{w_q w_{q+1}, x_q w_q, x_q w_{q+1}, x_q y_q, y_q z_q, y_q z_{q+1}, z_q z_{q+1} : q = 1, 2, \cdots, \eta\}.$$

Figure 3 provided an example of double balloon graph, denoted by DBl_5.

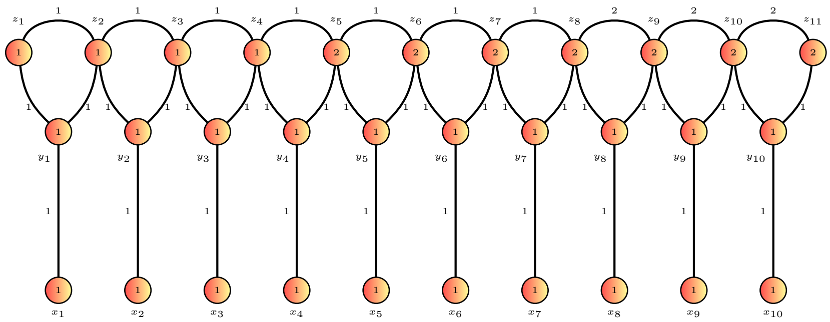

Figure. 2. Balloon graph Bl_{10} with Bl_3-covering

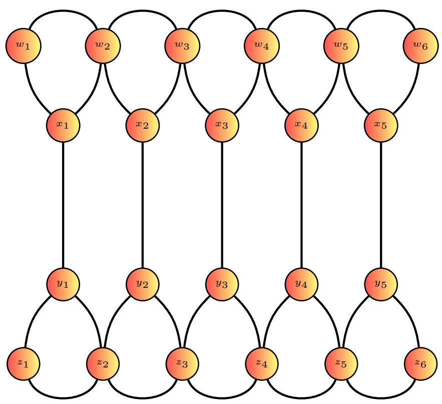

Figure. 3. Double Balloon Graph DBl_5

Theorem 2.4. *Given $DBl_\eta, \eta \geq 1$ with $DBl_\zeta-$covering, for $1 \leq \zeta \leq \eta$. Then,*

$$tHs(DBl_\eta, DBl_\zeta) = \left\lceil \frac{10\zeta + \eta + 2}{11\zeta + 2} \right\rceil.$$

Proof. The double balloon graph, $DBl_\eta, \eta \geq 1$ admits $DBl_\zeta-$covering with exactly $\eta-\zeta+1$ double balloon DBl_ζ, $1 \leq \zeta \leq \eta$ and η is a positive integer. From Theorem 1.1, we obtain:

$$tHs(DBl_\eta, DBl_\zeta) \geq \left\lceil 1 + \frac{(\eta - \zeta + 1) - 1}{11\zeta + 2} \right\rceil = \left\lceil \frac{10\zeta + \eta + 2}{11\zeta + 2} \right\rceil.$$

Next, put $p = \left\lceil \frac{10\zeta + \eta + 2}{11\zeta + 2} \right\rceil$, we prove that $tHs(DBl_\eta, DBl_\zeta) \leq p$. On the double balloon graph DBl_η with $DBl_\zeta-$covering, $1 \leq \zeta \leq \eta$, we construct:

$$\Psi_{\eta,\zeta} : V(DBl_\eta) \cup E(DBl_\eta) \to \{1, 2, \cdots, p\},$$

for $\zeta = 1, 2, \cdots, \eta$, as follows.

For $q = 1, 2, \cdots, \eta$,

$$\begin{aligned}
\Psi_{\eta,\zeta}(x_q) &= \left\lceil \frac{q+7\zeta}{11\zeta+2} \right\rceil, \Psi_{\eta,\zeta}(y_q) = \left\lceil \frac{q+4\zeta}{11\zeta+2} \right\rceil, \Psi_{\eta,\zeta}(x_q y_q) = \left\lceil \frac{q+5\zeta}{11\zeta+2} \right\rceil, \\
\Psi_{\eta,\zeta}(y_q z_q) &= \left\lceil \frac{q+2\zeta}{11\zeta+2} \right\rceil, \Psi_{\eta,\zeta}(y_q z_{q+1}) = \left\lceil \frac{q+3\zeta}{11\zeta+2} \right\rceil, \\
\Psi_{\eta,\zeta}(z_q z_{q+1}) &= \left\lceil \frac{q}{11\zeta+2} \right\rceil, \Psi_{\eta,\zeta}(w_q w_{q+1}) = \left\lceil \frac{q+\zeta}{11\zeta+2} \right\rceil, \\
\Psi_{\eta,\zeta}(x_q w_q) &= \left\lceil \frac{q+6\zeta}{11\zeta+2} \right\rceil, \Psi_{\eta,\zeta}(x_q w_{q+1}) = \left\lceil \frac{q+8\zeta}{11\zeta+2} \right\rceil.
\end{aligned}$$

For $q = 1, 2, \cdots, \eta + 1$,

$$\Psi_{\eta,\zeta}(w_q) = \left\lceil \frac{q+10\zeta+1}{11\zeta+2} \right\rceil, \quad \Psi_{\eta,\zeta}(z_q) = \left\lceil \frac{q+9\zeta}{11\zeta+2} \right\rceil.$$

The largest label is reached when $q = \eta + 1$, that is:

$$\Psi_{\eta,\zeta}(w_q) = \left\lceil \frac{10\zeta+\eta+2}{11\zeta+2} \right\rceil.$$

For every $r = 1, 2, 3, \cdots, \eta - \zeta + 1$, the vertex set and the edge set are:

$$\begin{aligned}
V(DBl_\zeta) &= \{x_{r+q}, y_{r+q} : q = 0, 1, 2, \cdots, \zeta - 1\} \cup \{w_{r+q}, z_{r+q} : q = 0, 1, 2, \cdots, \zeta\}, \\
E(DBl_\zeta) &= \{w_{r+q}w_{r+q+1}, x_{r+q}w_{r+q}, x_{r+q}w_{r+q+1}, x_{r+q}y_{r+q}, y_{r+q}z_{r+q}, \\
&\qquad y_{r+q}z_{r+q+1}, z_{r+q}z_{r+q+1} : q = 0, 1, 2, \cdots, \zeta - 1\}.
\end{aligned}$$

Furthermore, for every subgraph DBl_ζ^r in DBl_ζ, the weight of $DBl_\zeta^r, r = 1, 2, \cdots, \eta - \zeta + 1$, with respect to $\Psi_{\eta,\zeta} : V(DBl_\zeta) \cup E(DBl_\zeta) \to \{1, 2, \cdots, p\}$ is:

$$wt_{\Psi_{\eta,\zeta}}(DBl_\zeta^r) = \sum_{v \in V(DBl_\zeta^r)} \Psi_{\eta,\zeta}(v) + \sum_{e \in E(DBl_\zeta^r)} \Psi_{\eta,\zeta}(e).$$

Then, for $r = 1, 2, \cdots, \eta - \zeta$, we have:

$$\begin{aligned}
wt_{\Psi_{\eta,\zeta}} DBl_\zeta^{r+1} - wt_{\Psi_{\eta,\zeta}} DBl_\zeta^r &= \Psi_{\eta,\zeta}(w_{r+\zeta+1}) + \Psi_{\eta,\zeta}(w_{r+\zeta}w_{r+\zeta+1}) \\
&+ \Psi_{\eta,\zeta}(x_{r+\zeta}w_{r+\zeta}) + \Psi_{\eta,\zeta}(x_{r+\zeta}w_{r+\zeta+1}) \\
&+ \Psi_{\eta,\zeta}(x_{r+\zeta}) + \Psi_{\eta,\zeta}(y_{r+\zeta}) + \Psi_{\eta,\zeta}(z_{r+\zeta+1}) \\
&+ \Psi_{\eta,\zeta}(x_{r+\zeta}y_{r+\zeta}) + \Psi_{\eta,\zeta}(y_{r+\zeta}z_{r+\zeta}) \\
&+ \Psi_{\eta,\zeta}(y_{r+\zeta}z_{r+\zeta+1}) + \Psi_{\eta,\zeta}(z_{r+\zeta}z_{r+\zeta+1})
\end{aligned}$$

$$
\begin{aligned}
&- \Psi_{\eta,\zeta}(x_r) - \Psi_{\eta,\zeta}(y_r) - \Psi_{\eta,\zeta}(z_r) - \Psi_{\eta,\zeta}(w_r)\\
&- \Psi_{\eta,\zeta}(w_r w_{r+1}) - \Psi_{\eta,\zeta}(x_r w_r) - \Psi_{\eta,\zeta}(x_r w_{r+1})\\
&- \Psi_{\eta,\zeta}(x_r y_r) - \Psi_{\eta,\zeta}(y_r z_r) - \Psi_{\eta,\zeta}(y_r z_{r+1})\\
&- \Psi_{\eta,\zeta}(z_r z_{r+1}),\\
= &\left\lceil \frac{r+11\zeta+2}{11\zeta+2} \right\rceil + \left\lceil \frac{r+2\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+7\zeta}{11\zeta+2} \right\rceil\\
&+ \left\lceil \frac{r+9\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+8\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+5\zeta}{11\zeta+2} \right\rceil\\
&+ \left\lceil \frac{r+10\zeta+1}{11\zeta+2} \right\rceil + \left\lceil \frac{r+6\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+3\zeta}{11\zeta+2} \right\rceil\\
&+ \left\lceil \frac{r+4\zeta}{11\zeta+2} \right\rceil + \left\lceil \frac{r+\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+7\zeta}{11\zeta+2} \right\rceil\\
&- \left\lceil \frac{r+4\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+9\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+10\zeta+1}{11\zeta+2} \right\rceil\\
&- \left\lceil \frac{r+\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+6\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+8\zeta}{11\zeta+2} \right\rceil\\
&- \left\lceil \frac{r+5\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+2\zeta}{11\zeta+2} \right\rceil - \left\lceil \frac{r+3\zeta}{11\zeta+2} \right\rceil\\
&- \left\lceil \frac{r}{11\zeta+2} \right\rceil,\\
= &\ 1.
\end{aligned}
$$

For every $r = 1, 2, \cdots, \eta - \zeta$ we get:

$$wt_{\Psi_{\eta,\zeta}}(DBl_\zeta^r) < wt_{\Psi_{\eta,\zeta}}(DBl_\zeta^{r+1}).$$

Hence,

$$wt_{\Psi_{\eta,\zeta}}(DBl_\zeta^{r+1}) = 1 + wt_{\Psi_{\eta,\zeta}}(DBl_\zeta^r).$$

Furthermore, all $DBL_\zeta-$weights are distinct. Thus, $\Psi_{\eta,\zeta}$ is total $DBl_\zeta-$irregular labeling of DBl_η. Since $tHs(DBl_\eta, DBl_\zeta) \geq p$ and $tHs(DBl_\eta, DBl_\zeta) \leq p$, we have:

$$tHs(DBl_\eta, DBl_\zeta) = p = \left\lceil \frac{10\zeta+\eta+2}{11\zeta+2} \right\rceil. \qquad \square$$

Figure 4 shows DBl_{19} with DBl_2-covering, with $tHs(DBl_{19}, DBl_2) = 2$.

Subsequently, Definition 2.5 furnishes the definition of a double balloon ladder graph.

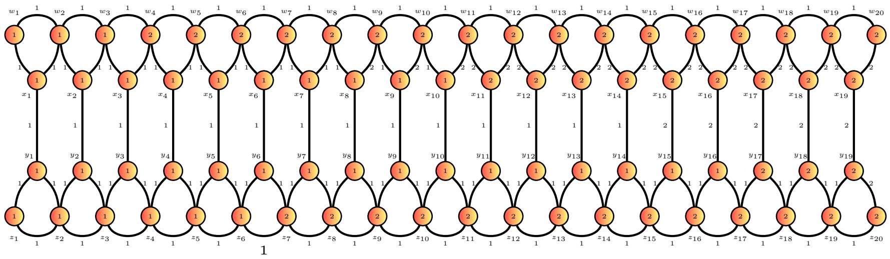

Figure. 4. Double Ballon Graph DBL_{19} with DBL_2-covering

Definition 2.5. *A double balloon ladder graph $DBlL_\eta$, $\eta \geq 1$, is a graph with:*

$$V(DBlL_\eta) = \{x_q, y_q : q = 1, 2, \cdots, \eta\} \cup \{w_q, z_q : q = 1, 2, \cdots, \eta + 1\}, \text{ and}$$
$$E(DBlL_\eta) = \{w_q w_{q+1}, x_q w_q, x_q w_{q+1}, x_q y_q, y_q z_q, y_q z_{q+1}, z_q z_{q+1} : q = 1, 2, \cdots, \eta\}$$
$$\cup \{w_q z_q : q = 1, 2, \cdots, \eta + 1\}.$$

Figure 5 provided an example of double balloon ladder graph, denoted by $DBlL_5$.

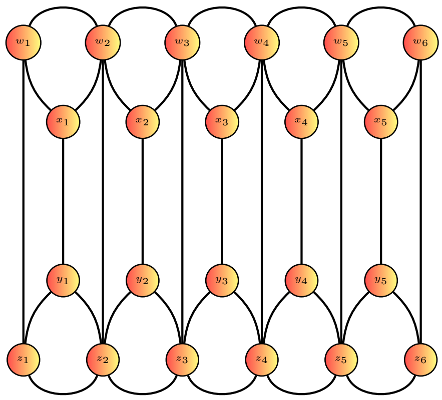

Figure. 5. Double Balloon Ladder Graph $DBlL_5$

Theorem 2.6. *Given $DBlL_\eta, \eta \geq 1$ with $DBl_\zeta-$covering, for $1 \leq \zeta \leq \eta$. Then,*

$$tHs(DBlL_\eta, DBlL_\zeta) = \left\lceil \frac{11\zeta + \eta + 3}{12\zeta + 3} \right\rceil.$$

Proof. The double balloon ladder graph $DBlL_\eta$, $\eta \geq 1$ admits $DBlL_\zeta-$covering with exactly $\eta - \zeta + 1$ double balloon ladder graphs $DBlL_\zeta$, $1 \leq \zeta \leq \eta$ and η is a positive integer. By Theorem 1.1,

$$tHs(DBlL_\eta, DBlL_\zeta) \geq \left\lceil 1 + \frac{(\eta - \zeta + 1) - 1}{12\zeta + 3} \right\rceil = \left\lceil \frac{11\zeta + \eta + 3}{12\zeta + 3} \right\rceil.$$

Next, let $p = \left\lceil \frac{11\zeta+\eta+3}{12\zeta+3} \right\rceil$, we show that $tHs(DBlL_\eta, DBlL_\zeta) \leq p$. On the double balloon ladder graph $DBlL_\eta$ with $DBlL_\zeta-$covering, $1 \leq \zeta \leq \eta$, we define a function:

$$\Psi_{\eta,\zeta} : V(DBlL_\eta) \cup E(DBlL_\eta) \to \{1, 2, \cdots, p\},$$

for $\zeta = 1, 2, \cdots, \eta$. In this way, for $q = 1, 2, \cdots, \eta$:

$$\begin{aligned}
&\Psi_{\eta,\zeta}(x_q) = \left\lceil \frac{q+6\zeta}{12\zeta+3} \right\rceil, \Psi_{\eta,\zeta}(y_q) = \left\lceil \frac{q+3\zeta}{12\zeta+3} \right\rceil, \Psi_{\eta,\zeta}(x_q y_q) = \left\lceil \frac{q+4\zeta}{12\zeta+3} \right\rceil,\\
&\Psi_{\eta,\zeta}(y_q z_q) = \left\lceil \frac{q+\zeta}{11\zeta+3} \right\rceil, \Psi_{\eta,\zeta}(y_q z_{q+1}) = \left\lceil \frac{q+2\zeta}{12\zeta+3} \right\rceil,\\
&\Psi_{\eta,\zeta}(z_q z_{q+1}) = \left\lceil \frac{q}{12\zeta+3} \right\rceil, \Psi_{\eta,\zeta}(w_q w_{q+1}) = \left\lceil \frac{q}{12\zeta+3} \right\rceil,\\
&\Psi_{\eta,\zeta}(x_q w_q) = \left\lceil \frac{q+5\zeta}{12\zeta+3} \right\rceil, \Psi_{\eta,\zeta}(x_q w_{q+1}) = \left\lceil \frac{q+7\zeta}{12\zeta+3} \right\rceil.
\end{aligned}$$

For $q = 1, 2, \cdots, \eta+1$,

$$\begin{aligned}
&\Psi_{\eta,\zeta}(w_q) = \left\lceil \frac{q+11\zeta+2}{12\zeta+3} \right\rceil, \Psi_{\eta,\zeta}(z_q) = \left\lceil \frac{q+9\zeta}{12\zeta+3} \right\rceil,\\
&\Psi_{\eta,\zeta}(w_q z_q) = \left\lceil \frac{q+10\zeta+1}{12\zeta+3} \right\rceil.
\end{aligned}$$

The largest label is obtained when $q = \eta + 1$, that is:

$$\Psi_{\eta,\zeta}(w_q) = \left\lceil \frac{11\zeta+\eta+3}{12\zeta+3} \right\rceil.$$

For every $r = 1, 2, 3, \cdots, \eta - \zeta + 1$, the vertex set is:

$$V(DBlL_\zeta) = \{x_{r+q}, y_{r+q} : q = 0, 1, 2, \cdots, \zeta - 1\} \cup \{w_{r+q}, z_{r+q} : q = 0, 1, 2, \cdots, \zeta\},$$

and the edge set:

$$\begin{aligned}
E(DBlL_\zeta) = \{&w_{r+q}w_{r+q+1}, x_{r+q}w_{r+q}, x_{r+q}w_{r+q+1}, x_{r+q}y_{r+q}, y_{r+q}z_{r+q},\\
&y_{r+q}z_{r+q+1}, z_{r+q}z_{r+q+1}, w_q z_q : q = 0, 1, 2, \cdots, \zeta - 1\}.
\end{aligned}$$

Furthermore, for every subgraph $DBlL_\zeta^r$ in DBl_ζ, we have weight $DBlL_\zeta^r, r = 1, 2, \cdots, \eta - \zeta + 1$, with respect to $\Psi_{\eta,\zeta} : V(DBlL_\zeta) \cup E(DBlL_\zeta) \to \{1, 2, \cdots, p\}$ is

$$wt_{\Psi_{\eta,\zeta}}(DBl_\zeta^r) = \sum_{v \in V(DBlL_\zeta^r)} \Psi_{\eta,\zeta}(v) + \sum_{e \in E(DBlL_\zeta^r)} \Psi_{\eta,\zeta}(e).$$

Then, for $r = 1, 2, \cdots, \eta - \zeta$, we have:

$$\begin{aligned}
wt_{\Psi_{\eta,\zeta}} DBl_\zeta^{r+1} - wt_{\Psi_{\eta,\zeta}} DBl_\zeta^{r} &= \Psi_{\eta,\zeta}(w_{r+\zeta+1}) + \Psi_{\eta,\zeta}(w_{r+\zeta} w_{r+\zeta+1}) \\
&+ \Psi_{\eta,\zeta}(x_{r+\zeta} w_{r+\zeta}) + \Psi_{\eta,\zeta}(x_{r+\zeta} w_{r+\zeta+1}) \\
&+ \Psi_{\eta,\zeta}(x_{r+\zeta}) + \Psi_{\eta,\zeta}(y_{r+\zeta}) + \Psi_{\eta,\zeta}(z_{r+\zeta+1}) \\
&+ \Psi_{\eta,\zeta}(x_{r+\zeta} y_{r+\zeta}) + \Psi_{\eta,\zeta}(y_{r+\zeta} z_{r+\zeta}) \\
&+ \Psi_{\eta,\zeta}(y_{r+\zeta} z_{r+\zeta+1}) + \Psi_{\eta,\zeta}(z_{r+\zeta} z_{r+\zeta+1}) \\
&+ \Psi_{\eta,\zeta}(w_{r+\zeta+1} z_{r+\zeta+1}) - \Psi_{\eta,\zeta}(x_r) - \Psi_{\eta,\zeta}(y_r) \\
&- \Psi_{\eta,\zeta}(z_r) - \Psi_{\eta,\zeta}(w_r) - \Psi_{\eta,\zeta}(w_r w_{r+1}) \\
&- \Psi_{\eta,\zeta}(x_r w_r) - \Psi_{\eta,\zeta}(x_r w_{r+1}) - \Psi_{\eta,\zeta}(x_r y_r) \\
&- \Psi_{\eta,\zeta}(y_r z_r) - \Psi_{\eta,\zeta}(y_r z_{r+1}) - \Psi_{\eta,\zeta}(z_r z_{r+1}) \\
&- \Psi_{\eta,\zeta}(w_r z_r), \\
&= \left\lceil \frac{r+12\zeta+3}{12\zeta+3} \right\rceil + \left\lceil \frac{r+9\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+6\zeta}{12\zeta+3} \right\rceil \\
&+ \left\lceil \frac{r+8\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+7\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+4\zeta}{12\zeta+3} \right\rceil \\
&+ \left\lceil \frac{r+10\zeta+1}{12\zeta+3} \right\rceil + \left\lceil \frac{r+5\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+2\zeta}{12\zeta+3} \right\rceil \\
&+ \left\lceil \frac{r+3\zeta}{12\zeta+3} \right\rceil + \left\lceil \frac{r+\zeta}{13\zeta+2} \right\rceil + \left\lceil \frac{r+11\zeta+2}{12\zeta+3} \right\rceil \\
&- \left\lceil \frac{r+6\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+3\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+9\zeta}{12\zeta+3} \right\rceil \\
&- \left\lceil \frac{r+11\zeta+2}{12\zeta+3} \right\rceil - \left\lceil \frac{r+8\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+5\zeta}{12\zeta+3} \right\rceil \\
&- \left\lceil \frac{r+7\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+4\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r+\zeta}{12\zeta+3} \right\rceil \\
&- \left\lceil \frac{r+2\zeta}{12\zeta+3} \right\rceil - \left\lceil \frac{r}{12\zeta+3} \right\rceil - \left\lceil \frac{r+10\zeta+1}{12\zeta+3} \right\rceil, \\
&= 1.
\end{aligned}$$

For every $r = 1, 2, \cdots, \eta - \zeta$ we get

$$wt_{\Psi_{\eta,\zeta}}(DBlL_\zeta^r) < wt_{\Psi_{\eta,\zeta}}(DBlL_\zeta^{r+1}).$$

Hence,

$$wt_{\Psi_{\eta,\zeta}}(DBlL_\zeta^{r+1}) = 1 + wt_{\Psi_{\eta,\zeta}}(DBlL_\zeta^r).$$

Furthermore, all $DBL_\zeta-$weights are distinct. Thus, $\Psi_{\eta,\zeta}$ is total $DBlL_\zeta-$irregular labeling of $DBlL_\eta$. Since $tHs(DBlL_\eta, DBlL_\zeta) \geq p$ and $tHs(DBlL_\eta, DBlL_\zeta) \leq p$

we have:

$$tHs(DBlL_\eta, DBlL_\zeta) = p = \left\lceil \frac{11\zeta + \eta + 3}{12\zeta + 3} \right\rceil. \qquad \square$$

Figure 6 is an evidence of $DBlL_{28}$ with $DBlL_2-$covering, with $tHs(DBlL_{28}, DBlL_2) = 2$.

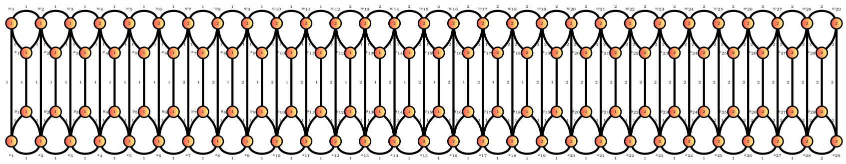

Figure. 6. Total $DBlL-$Irregularity Labeling of $DBlL_{28}$

3. Conclusion

This research focuses on the total $H-$irregularity strength of balloon graph Bl_η with $Bl_\zeta-$covering, double balloon graph DBl_η, and double balloon ladder graph $DBlL_\eta$ with $DBl_\zeta-$covering. We found that the lower bound given in Theorem 1.1 is sharp for all these graphs.

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