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ON THE PARTITION DIMENSION OF A SUBDIVISION OF COMPLETE GRAPH AND COMPLETE BIPARTITE GRAPH

ISMAIL MULIA HASIBUAN[a]*, WAHYUNI ABIDIN[b]

[a] *Universitas Islam Negeri Sultan Syarif Kasim Riau,*
[b] *Universitas Islam Negeri Alauddin Makassar,*
email : imhasibuan@uin-suska.ac.id, wahyuniabidin@uin-alauddin.ac.id.

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Abstract. *Let $G(V, E)$ be a connected graph. The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is the length of the shortest path from u to v in G. The distance from a vertex $v \in V(G)$ to a set $S \subset V(G)$ is defined as $\min\{d(v, x)|x \in S\}$. The partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$ is called a resolving partition of G if the vectors $(d(v, S_1), d(v, S_2), \cdots, d(v, S_k))$ for all $v \in V(G)$ are distinct. The partition dimension of G, denoted by* pd(G)*, is the smallest k such that G has a resolving k-partition. Let $A = \{e_1, e_2, \cdots, e_t\} \subseteq E(G)$, for some t. The subdivision of a graph G on the edge set A, denoted by $S(G(A; n_1, n_2, \cdots, n_t))$, is a graph obtained from the graph G by replacing edge e_1 with a path of length $n_1 + 2$, edge e_2 with a path of length $n_2 + 2$, up to edge e_t with a path of length $n_t + 2$, respectively. In this paper we determine the partition dimension of $S(K_n(A; r_1, r_2, \cdots, r_t))$ for $n \geq 3$ and $t \leq 3$. We also derive the partition dimension of $S(K_{m,n}(A; r_1, r_2, \cdots, r_t))$ for $m \geq n \geq 2$ and $t \leq 3$.*

Keywords: Resolving partition, partition dimension, subdivision, complete graph, complete bipartite graph.

1. Introduction

The concept of metric dimension of a graph was first introduced by Slater [1] as *a graph locating set*, and by Harary and Melter [2] independently as *a graph resolving set*. All connected graphs of order n which have metric dimension 1, $n-1$, or $n-2$ have been characterized by Chartrand et al. [3]. Meanwhile, some authors investigated the metric dimension of certain graphs obtained by a graph operation [4,5]. Then, Chartrand et al. [6] introduced a variant of this concept called *a resolving partition* of a graph. In this matter, the study focuses on finding the minimal partition of the vertex set of graph G such that the representations of all vertices in G with respect to such a partition are distinct. The representation of a vertex in G, in this case, is determined by its distances to all the partition classes.

*Corresponding author

Let $G = (V, E)$ be a connected graph and $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a partition of $V(G)$. The *representation* of vertex $v \in G$ with respect to Π is $(d(v, S_1), d(v, S_2), \cdots, d(v, S_k))$. The partition Π is called a *resolving partition* of G if the representations of all vertices are different. The *partition dimension* of G, denoted by *pd(G)*, is the smallest number k such that G has a resolving partition with k partition classes. The study on the partition dimension of graphs has received much attention. As the first results, Chartrand et al. [7] determined the partition dimension of some classes of trees, namely double-stars and caterpillars. Furthermore, Chartrand et. al. [6] characterized all graphs of order n and having partition dimension either 2, n or $n-1$. Other known results on the partition dimension of graphs can be also found in [8,9,10,11,12].

Let $G = (V, E)$ be a connected graph and $A = \{e_1, e_2, \cdots, e_k\} \subseteq E(G)$. The *subdivision* of a graph G on the edge set A, denoted by $S(G(A; n_1, n_2, \cdots, n_k))$, is a graph obtained from the graph G by replacing edge e_1 with a path of length $n_1 + 2$, edge e_2 with a path of length $n_2 + 2$, up to edge e_k with a path of length $n_k + 2$, respectively. The internal vertices of the path replacing edge e_i, for each i, are called the *subdivision vertices*. In particular, if $A = \{e\}$ and $n_1 = k \geq 1$, then the subdivision of graph G on A is simply denoted by $S(G(e; k))$. In [6], Chartrand et al. determined the partition dimension of the complete graph K_n and the complete bipartite graph $K_{m,n}$. In [13], Amrullah et al. determine the partition dimension of subdivision a star graph. Then in [14] Amrullah et al. determine the partition dimension of subdivision of a complete graph where every edges of complete graph are subdivided by one vertex. In this paper, we will determine the partition dimension of $S(G(A, r_1, r_2, \cdots, r_t))$ if G is either a complete graph or a complete bipartite graph and $t \leq 3$. Let us begin to state the useful result shown by Chartrand et al. [6] as follows.

Lemma 1.1. *Let Π be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$, then u and v are not in the same set in Π.*

2. Result and Discussion

Let $G_1 \cong S(K_n(e; r))$ with $V(G_1) = \{v_1, v_2, \cdots, v_n, x_1, x_2, \cdots, x_r\}$ and $e = v_1v_2$. Note that $x_1, x_2, \cdots, x_r$ are the subdivision vertices in G_1. Then, we obtain the following theorem.

Theorem 2.1. *Let $G_1 \cong S(K_n(e; r))$,$n \geq 3$, $r \geq 1$ and for any edge e. Then:*

$$pd(G_1) = \begin{cases} 3 & , n = 3, \\ n-1 & , n \geq 4. \end{cases}$$

Proof. If $n = 3$ then $G_1 = C_{r+3}$; and so $pd(G_1) = 3$. For $n \geq 4$, let us define a partition $\Pi = \{S_1, S_2, \cdots, S_{n-1}\}$ of $V(G_1)$ with $S_1 = \{v_1, v_n, x_1, x_2, \cdots, x_r\}$, and $S_i = \{v_i\}$ for $i = 2, 3, \cdots, n-1$.

For $i < j$, we have that $d(x_i, S_2) \neq d(x_j, S_2)$ or $d(x_i, S_3) \neq d(x_j, S_3)$. So x_i and x_j can be distinguished. Since $d(x_i, S_3) \geq 2$, $d(v_1, S_3) = d(v_n, S_3) = 1$, $d(v_1, S_2) \geq 2$

and $d(v_n, S_2) = 1$, then $r(x_i|\Pi), r(v_1|\Pi)$ and $r(v_n|\Pi)$ are distinct for all i. Therefore, the representations of all vertices in G_1 are distinct. Therefore $pd(G_1) \leq n - 1$.

Furthermore, by Lemma 1.1, each pair of v_i and v_j, $3 \leq i < j \leq n$ must be in different partition classes. If v_1 and v_i are in the same partition class for some $i \in [3, n]$ then v_2 must be in different partition classes with any v_j, for $j \in [3, n]$. Therefore $pd(G_1) \geq n - 1$. This concludes our proof. □

Now, for $n \geq 3$ and $r_1, r_2 \geq 1$, define $G_2 = S(K_n(A; r_1, r_2))$ with $A = \{e_1, e_2\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_n, x_1, x_2, \cdots, x_{r_1}, y_1, y_2, \cdots, y_{r_2}\}$. Note that the vertices $x_1, x_2, \cdots, x_{r_1}$, $y_1, y_2, \cdots, y_{r_2}$ are the subdivision vertices in G_2. We obtain the following theorem.

Theorem 2.2. *If edges e_1 and e_2 form a path P_3 then:*

$$pd(G_2) = \begin{cases} 3 & , 3 \leq n \leq 4, \\ n-2 & , n \geq 5. \end{cases}$$

Proof. Again, if $n = 3$ then we have $G_2 = C_{r_1+r_2+3}$; and then $pd(G_2) = 3$. Now, let $e_1 = v_1v_2$ and $e_2 = v_2v_3$. For $n = 4$, define a partition $\Pi = \{S_1, S_2, S_3\}$ of $V(G_2)$ with $S_1 = \{v_1, v_4, x_1, x_2, \cdots, x_{r_1}\}$, $S_2 = \{v_2\}$, and $S_3 = \{v_3, y_1, y_2, \cdots, y_{r_2}\}$. It is clear that for $i < j$, either $d(x_i, S_2) \neq d(x_j, S_2)$ or $d(x_i, S_3) \neq d(x_j, S_3)$ holds; and so x_i and x_j can be distinguished. Since $d(x_i, S_3) \geq 2$, $d(v_1, S_3) = d(v_4, S_3) = 1$, $d(v_1, S_2) \geq 2$ and $d(v_4, S_2) = 1$ then the vertices x_i, v_1 and v_4 can be distinguished for all i. The set S_2 is a singleton partition class, so v_2 has a unique representation. By a similar argument, the representations of v_3 and y_i for all i are also distinct. Therefore, $pd(G_2) \leq 3$. By Lemma 1.1 we conclude that $pd(G_2) = 3$ if $n = 4$.

Now consider if $n \geq 5$. Define a partition $\Pi = \{S_1, S_2, \cdots, S_{n-2}\}$ of $V(G_2)$ with $S_1 = \{v_1, v_n, x_1, x_2, \cdots, x_{r_1}\}$, $S_2 = \{v_2\}$, $S_3 = \{v_3, v_4, y_1, y_2, \cdots, y_{r_2}\}$, and $S_i = \{v_{i+1}\}$, for $i = 4, 5, \cdots, n-2$. We shall show that Π is a resolving partition. To do so we must show that the representations of all vertices in S_1 and S_3 are distinct. It is clearly that for $i < j$, either $d(x_i, S_2) \neq d(x_j, S_2)$ or $d(x_i, S_3) \neq d(x_j, S_3)$ holds; and so x_i and x_j can be distinguished. Since $d(x_i, S_3) \geq 2$, $d(v_1, S_3) = d(v_n, S_3) = 1$, $d(v_1, S_2) \geq 2$ and $d(v_n, S_2) = 1$ then the vertices x_i, v_1 and v_n can be distinguished for all i. By a similar argument, the representations of v_3, v_4 and y_i for all i are also distinct. Therefore, we have $pd(G_2) \leq n - 2$.

By Lemma 1.1, each pair of v_i and v_j for $4 \leq i < j \leq n$ must be in different partition classes. One of these vertices v_1, v_2 and v_3 must be in a different partition class with all vertices v_i for $i \in (4, n)$. Therefore, we require at least $n - 2$ partition classes. So, we obtain $pd(G_2) \geq n - 2$. This means that $pd(G_2) = n - 2$. □

Let $G_3 \simeq S(K_n(A; r_1, r_2, r_3))$, with $n \geq 4$, $A = \{e_1, e_2, e_3\}$, and $r_1, r_2, r_3 \geq 1$. Then, we have the following theorem.

Theorem 2.3. *If the edges e_1, e_2 and e_3 forms a path P_4 in K_n then:*

$$pd(G_3) = \begin{cases} 3 & , n = 4, 5, \\ n-2 & , n \geq 6. \end{cases}$$

Proof. For the cases of $n = 4$ or $n = 5$, it is easy to prove. Now consider $n \geq 6$ and let $\Pi = \{S_1, S_2, \cdots, S_{n-2}\}$ be a partition of $V(G_3)$ with $S_1 = \{v_1, v_n, x_1, x_2, \cdots, x_{r_1}\}$, $S_2 = \{v_2\}$, $S_3 = \{v_3, y_1, y_2, \cdots, y_{r_2}\}$, $S_4 = \{v_4, v_5, z_1, z_2, \cdots, z_{r_3}\}$, and $S_i = \{v_{i+1}\}$, for $i = 5, 7, \cdots, n-2$.

We shall show that Π is a resolving partition. To do so we must show that the representations of all vertices in S_1, S_3, and S_4 are distinct. It is clearly that for $i < j$, either $d(x_i, S_2) \neq d(x_j, S_2)$ or $d(x_i, S_3) \neq d(x_j, S_3)$ holds; and so x_i and x_j can be distinguished. Since $d(x_i, S_3) \geq 2$, $d(v_1, S_3) = d(v_n, S_3) = 1$, $d(v_1, S_2) \geq 2$ and $d(v_n, S_2) = 1$ then the vertices x_i, v_1 and v_n can be distinguished for all i. By a similar argument, the representations of v_3, and y_i for all i are also distinct. Then, by a similar argument, the representations of v_4, v_5, and z_i for all i are also distinct. Therefore, we have $pd(G_3) \leq n-2$.

By Lemma 1.1, all vertices v_i for $5 \leq i \leq n$ must be in different partition classes. Now, if v_1 is contained in the same partition class with vertex v_i for some $i \in (5, n)$, then v_2 must be in different partition classes with all vertices v_j, for all $j \in [5, n]$. Furthermore, if v_4 is contained in the same partition class with vertex v_i for some $i \in (5, n)$ then v_3 must be in different partition classes v_2 and v_j for all $j \in [5, n]$. Therefore, we require at least $n-2$ partition classes, and so $pd(G_3) \geq n-2$. This concludes our proof. □

Now, we consider the subdivision of a complete bipartite graph. For $m \geq n \geq 2$, $r \geq 1$, define $G_4 = S(K_{m,n}(r))$ with $V(G_4) = \{a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n, x_1, x_2, \cdots, x_r\}$. The vertices $x_1, x_2, \cdots, x_r$, are the subdivision vertices in G_4. We obtain the following theorem.

Theorem 2.4. *Let $G_4 = S(K_{m,n}(r))$, $m \geq n \geq 2$, $r \geq 1$. Then,*

$$pd(G_4) = \begin{cases} 3 & ,(m,n) = (2,2), r \geq 1; \text{ or } (m,n) = (3,3), r \geq 2, \\ 4 & ,(m,n) = (3,3), r = 1, \\ m & ,n \leq m \leq n+1, \\ m-1 & ,m \geq n+2. \end{cases}$$

Proof. Let a_1b_1 be the subdivision edge in G_4. Then, we divide the proof into three cases.

Case 1. $((m,n) = (2,2), r \geq 1)$ or $((m,n) = (3,3), r \geq 2)$.

If $(m,n) = (2,2)$, then $G_4 = C_{r+4}$, for $r \geq 1$. So, $pd(G_4) = 3$. For $(m,n) = (3,3), r \geq 2$, define a partition $\Pi = \{S_1, S_2, \cdots, S_3\}$ of $V(G_4)$ with $S_1 = \{a_1, a_3, b_3, x_1, x_2, \cdots, x_r\}$, $S_2 = \{a_2, b_2\}$, and $S_3 = \{b_1\}$. Now, we must show that the representations of all vertices in S_1, and S_2 are distinct. It is clearly that for $i < j$, either $d(x_i, S_2) \neq d(x_j, S_2)$ or $d(x_i, S_3) \neq d(x_j, S_3)$ holds; and so x_i and x_j can be distinguished. Since $d(x_i, S_2) \geq 2$, $d(a_1, S_2) = d(a_3, S_2) = 1$, $d(a_1, S_3) \geq 2$ and $d(a_3, S_1) = 1$, then the vertices x_i, a_1, a_3, and b_3 can be distinguished for all i. Therefore, $pd(G_4) \leq 3$. On the other hand, by Lemma 1.1 we have that $pd(G_4) \geq 3$. Hence, $pd(S(K_{m,n}(r))) = 3$ if $(m,n) = (2,2)$ or $((m,n) = (3,3)$ and $r \geq 2)$.

Case 2. $(m,n) = (3,3), r = 1$.

Now, define a partition $\Pi = \{S_1, S_2, S_3, S_4\}$ of $V(G_4)$ with $S_1 = \{a_1, a_3, b_3\}$, $S_2 = \{b_2\}$, $S_3 = \{a_2, b_1\}$ and $S_4 = \{x_1\}$.

Now, we must show that the representations of all vertices in S_1, and S_3 are distinct. It is clearly that either $d(a_1, S_4) = 1$ and $d(a_3, S_4) = 2$, $d(a_1, S_4) = 1$ and $d(b_3, S_4) = 2$; and $d(a_3, S_2) = 1$ and $d(b_3, S_2) = 2$ holds; and so a_1, a_3, and b_3 can be distinguished. Since $d(a_2, S_4) = 2$ and $d(b_1, S_4) = 1$, then the vertices a_2 and b_1 can be distinguished. Therefore, $pd(G_4) \leq 4$.

Since G_4 is not a path then $pd(G_4) \geq 3$. For a contradiction, assume $pd(G_4) = 3$. By Lemma 1.1, vertices a_2, a_3 must be in different partition classes. Similarly, it holds for b_2 and b_3. If all vertices $a_1, a_2, a_3, b_1, b_2, b_3$ are only in two different classes then vertex x_1 must be in the third partition class. But, then there are two vertices having the same representation, a contradiction. Therefore, either all a_is or all b_is are in three different classes. Without loss of generality, let $a_1 \in S_1, a_2 \in S_2$, $a_3 \in S_3$, and $S_1 \cup S_2 \cup S_3 = V(G_4)$. To avoid having the same representation, vertex b_1 must be in the same class with either b_2 or b_3. This implies that x_1 and b_1 are in the same partition class. However, now either (x_1 and one of a_is) or (b_1 and one of a_is) have the same representation, a contradiction. Therefore, $pd(G_4) \geq 4$. Hence, $pd(S(K_{m,n}(r))) = 4$ if $(m, n) = (3, 3)$ and $r = 1$.

Case 3. $n \leq m \leq n + 1$.
If $m = n$ then define a partition $\Pi = \{S_1, S_2, \cdots, S_{m-1}, S_m\}$ of $V(G_4)$ with $S_1 = \{a_1, a_m, b_n, x_1, x_2, \cdots, x_r\}$, $S_i = \{a_i, b_i\}$ for $2 \leq i \leq m - 2$, $S_{m-1} = \{b_{n-1}\}$ and $S_m = \{a_{m-1}, b_1\}$. Since, any two vertices can be distinguished by either S_{m-1} or S_m, then Π is a resolving partition of G_4. So, $pd(G_4) \leq m$. Furthermore, by Lemma 1.1, all vertices a_is for $2 \leq i \leq m$ must be in different partition classes. Now, if a_1 is in the same partition class with other a_i for some $i \in [2, m]$ then all b_js for $1 \leq j \leq n$ must be in different partition classes. Therefore, $pd(G_4) \geq m$. This implies that $pd(G_4) = m$.

Now, let $m = n + 1$. Define a partition $\Pi = \{S_1, S_2, \cdots, S_{m-1}, S_m\}$ of $V(G_4)$ with $S_1 = \{a_1, b_1, x_1, x_2, \cdots, x_r\}$, $S_i = \{a_i, b_i\}$ for $2 \leq i \leq m - 1$, and $S_m = \{a_m\}$. Now, we must show that the representations of all vertices in S_1, and S_i for $2 \leq i \leq m - 1$ are distinct. It is clearly that for $i < j$, either $d(x_i, S_2) \neq d(x_j, S_2)$ or $d(x_i, S_m) \neq d(x_j, S_m)$ holds; and so x_i and x_j can be distinguished. Since $d(x_i, S_2) \geq 2$, $d(a_1, S_2) = d(b_1, S_2) = 1$, $d(a_1, S_m) = 2$ and $d(b_1, S_m) = 1$, then the vertices x_i, a_1, and b_1 can be distinguished for all i. Then, $d(a_i, S_m) = 2$ and $d(b_i, S_m) = 1$ for some i. Therefore, $pd(G_4) \leq m$. Furthermore, by Lemma 1.1, all vertices a_is for $2 \leq i \leq m$ must be in different partition classes. Now, if a_1 is in the same partition class with other a_i for some $i \in [2, m]$ then all b_js for $1 \leq j \leq n$ must be in different partition classes. If there is no new partition class other than the ones having vertices a_is then the representations of vertices a_i and b_j for some i and j are same. Therefore, $pd(G_4) \geq m$. This implies that $pd(G_4) = m$.

Case 4. $m \geq n + 2$.
Define a partition $\Pi = \{S_1, S_2, \cdots, S_{m-1}\}$ of $V(G_4)$ with $S_i = \{a_{i+1}, b_i\}$ for $1 \leq i \leq n$, $S_i = \{a_{i+1}\}$ for $n + 1 \leq i \leq m - 2$, and $S_{m-1} = \{a_1, a_m, x_1, x_2, \cdots, x_r\}$. Now, we must show that the representations of all vertices in S_i for $1 \leq i \leq n$ and S_{m-1} are distinct. It is clear that $d(a_{i+1}, S_{m-1}) = 2$ and $d(b_i, S_{m-1}) = 2$ for some i $(1 \leq i \leq n)$. Then, for $i < j$, either $d(x_i, S_1) \neq d(x_j, S_1)$ or $d(x_i, S_2) \neq d(x_j, S_2)$ holds; and so x_i and x_j can be distinguished. Since $d(x_i, S_2) \geq 2$ and $d(a_1, S_2) =$

$d(a_m, S_2) = 1$, $d(a_1, S_1) \geq 2$ and $d(a_m, S_1) = 1$, then the vertices x_i, a_1, and a_m can be distinguished for all i. So, $pd(G_4) \leq m - 1$. By Lemma 1.1, all vertices a_is for all $i \in [2, m]$ must be in different partition classes. Therefore $pd(G_4) \geq m - 1$. This concludes our proof. □

Now, we consider the subdivision of a complete bipartite graph. For $m \geq n \geq 4; r_1, r_2 \geq 1$, define $G_5 = S(K_{m,n}(\{e_1, e_2\}; r_1, r_2))$ with $V(G_5) = \{a_1, a_2, \cdots, a_m, \; b_1, b_2, \cdots, b_n, \; x_1, x_2, \cdots, x_{r_1}, y_1, y_2, \cdots, y_{r_2}\}$. The vertices $x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_{r_2}$ are the subdivision vertices in G_5. We obtain the following theorem.

Theorem 2.5. *If edges e_1 and e_2 form a path P_3 starting from a vertex in the partite set of order n in $K_{m,n}$ then:*

$$pd(S(G_5)) = \begin{cases} m & , m = n, n+1, \\ m-1 & , m \geq n+2. \end{cases}$$

Proof. Let $e_1 = a_1 b_1$ and $e_2 = a_1 b_2$. Now, consider the following two cases.
Case 1. $n \leq m \leq n + 1$.
If $m = n$ then define a partition $\Pi = \{S_1, S_2, \cdots, S_m\}$ of $V(G_5)$, with $S_1 = \{a_1, a_m, b_n, x_1, x_2, \cdots, x_{r_1}\}$, $S_2 = \{a_2, b_2\}$, $S_3 = \{a_3, b_3, \; y_1, y_2, \cdots, y_{r_2}\}$, $S_i = \{a_i, b_i\}$ for $i = 4, 5, \cdots, m-1$, and $S_m = \{b_1\}$. Now, we must show that the representations of all vertices in S_1, S_2, S_3 and S_i for $4 \leq i \leq m-1$ are distinct. It is clearly that for $i < j$, either $d(x_i, S_5) \neq d(x_j, S_5)$ or $d(x_i, S_2) \neq d(x_j, S_2)$ holds; and so x_i and x_j can be distinguished. Note that $d(x_i, S_2) \geq 2$, $d(a_1, S_2) = d(a_m, S_2) = 1$, $d(a_1, S_m) \geq 2$ and $d(a_m, S_m) = 1$. Since $d(a_1, S_2) = d(a_m, S_2) = 1$ and $d(b_n, S_2) = 2$, then the vertices x_i, a_1, a_m, and b_n can be distinguished for all i. By a similar argument, the representations of a_3, b_3, and y_i for all i are also distinct. Then $d(a_i, S_2) = 1$ and $d(b_i, S_2) = 2$. So, the vertices a_i and b_i can be distinguished for all i ($4 \leq i \leq m-1$). So, Π is a resolving partition of G_5. Therefore, $pd(G_5) \leq m$. By Lemma 1.1, all vertices a_i for all $i \in [2, m]$ must be in different partition classes. If vertex a_1 is in the same partition class with other a_i for some i then all b_js for all $j \in [1, n]$ must be in different partition classes. Therefore, $pd(G_5) \geq m$. This concludes that $pd(G_5) = m$.

Now, consider if $m = n + 1$ and $n \geq 4$. Define a partition $\Pi = \{S_1, S_2, \cdots, S_{m-1}, S_m\}$ of $V(G_5)$ with $S_1 = \{a_1, a_m, b_n, x_1, x_2, \cdots, x_{r_1}\}$, $S_2 = \{a_2, b_2, y_1, y_2, \cdots, y_{r_2}\}$, for $i = 3, 4, \cdots, m-2$ define $S_i = \{a_i, b_i\}$, and $S_{m-1} = \{a_{m-1}\}$ and $S_m = \{b_1\}$. Since, any two vertices can be distinguished by either S_m or S_{m-1} then Π is a resolving partition of G_5. Therefore, $pd(G_5) \leq m$. By Lemma 1.1, all a_i for $i \in [2, n]$ must be in different partition classes. If a_1 is contained in the same partition class with other a_i for some $i \in [2, m]$ then all b_j must be in different partition classes. However, one of these partition classes must be different with the other ones containing a_i. Therefore, $pd(G_5) \geq m$. This concludes that $pd(G_5) = m$.
Case 2. $m \geq n + 2$.
Define a partition $\Pi = \{S_1, S_2, S_3, \cdots, S_{m-1}\}$ of $V(G_5)$ with $S_1 = \{a_1, a_2, x_1, x_2, \cdots, x_{r_1}\}$, $S_2 = \{a_3, b_1\}$, $S_3 = \{a_4, b_2, y_1, y_2, \cdots, y_{r_2}\}$, for $i =$

$4, 5, \cdots, n+1$ define $S_i = \{a_{i+1}, b_{i-1}\}$, and for $i = n+2, n+3, \cdots, m-1$ define $S_i = \{a_{i+1}\}$. Now, we must show that the representations of all vertices in S_1, S_2, S_3 and S_i for $4 \leq i \leq n+1$ are distinct. It is clearly that for $i < j$, either $d(x_i, S_2) \neq d(x_j, S_2)$ or $d(x_i, S_m) \neq d(x_j, S_m)$ holds; and so x_i and x_j can be distinguished. Note that $d(x_i, S_3) \geq 2$, $d(a_1, S_3) = d(a_2, S_3) = 1$, $d(a_1, S_2) \geq 2$ and $d(a_2, S_2) = 1$. So, the vertices x_i, a_1, and a_2 can be distinguished for all i. By a similar argument, the representations of a_4, b_2, and y_i for all i are also distinct. Then $d(a_3, S_m) = 2$ and $d(b_1, S_m) = 1$. So, the vertices a_3 and b_1 can be distinguished. So, Π is a resolving partition of G_5. Therefore, $pd(G_5) \leq m-1$. By Lemma 1.1, all a_i for $i \in [2, m]$ must be in different partition classes. Therefore, $pd(G_5) \geq m-1$ □

Now, consider the subdivision of a complete bipartite graph on its three edges. For $m \geq n \geq 3; r_1, r_2, r_3 \geq 1$, define $G_6 = S(K_{m,n}(\{e_1, e_2, e_3\}; r_1, r_2, r_3))$ with $V(G_5) = \{a_1, a_2, \cdots, a_m, \ b_1, b_2, \cdots, b_n, \ x_1, x_2, \cdots, x_{r_1}, y_1, y_2, \cdots, y_{r_2}, z_1, z_2, \cdots, z_{r_3}\}$. We obtain the following theorem.

Theorem 2.6. *If e_1, e_2 and e_3 form a star $K_{1,3}$ with its center in the partite set of order m in $K_{m,n}$, then:*

$$pd(G_6) = \begin{cases} 4, & (m,n) = (3,3), \\ m, & m = n, n+1, \\ m-1, & m \geq n+2. \end{cases}$$

Proof. Let $e_1 = a_1b_1$, $e_2 = a_1b_2$, $e_3 = a_1b_3$. For the case of $(m,n) = (3,3)$ it is easy to prove. Now, consider the following two cases.

Case 1. $m = n$, or $n+1$.

If $m = n$ then define a partition $\Pi = \{S_1, S_2, \cdots, S_m\}$ of $V(G_6)$ with $S_1 = \{a_2, b_2, y_1, y_2, \cdots, y_{r_2}\}$, $S_2 = \{a_3, b_3, z_1, z_2, \cdots, z_{r_3}\}$, $S_3 = \{a_1, a_4, b_4, x_1, x_2, \cdots, x_{r_1}\}$, $S_i = \{a_{i+1}, b_i\}$, for $i = 4, 5, \cdots, m-1$ and $S_m = \{b_1\}$. It can be verified that Π is a resolving partition of G_6. So, $pd(G_6) \leq m$. By Lemma 1.1, all a_i for $i \in [2, m]$ must be in different partition classes. If a_1 is in the same partition class with other a_i for some $i \in [2, m]$ then all b_j for $j \in [1, n]$ must be in different partition classes. So, $pd(G_6) \geq m$. This concludes the proof.

Now consider if $m = n+1$. Define a partition $\Pi = \{S_1, S_2, \cdots, S_m\}$ of $V(G_6)$ with $S_1 = \{a_2, b_2, y_1, y_2, \cdots, y_{r_2}\}$, $S_2 = \{a_3, b_3, z_1, z_2, \cdots, z_{r_3}\}$, $S_3 = \{a_1, a_4, b_4, x_1, x_2, \cdots, x_{r_1}\}$, $S_i = \{a_{i+1}, b_i\}$, for $i = 4, 5, \cdots, m-1$ and $S_m = \{b_1\}$. Now, we must show that the representations of all vertices in S_1, S_2, S_3, and S_i for $4 \leq i \leq m-1$ are distinct. It is clearly that for $i < j$, either $d(y_i, S_2) \neq d(y_j, S_2)$ or $d(y_i, S_m) \neq d(y_j, S_m)$ holds; and so y_i and y_j can be distinguished. Note that $d(y_i, S_2) \geq 2$, $d(a_2, S_2) = d(b_2, S_2) = 1$, $d(a_2, S_3) = 2$ and $d(b_2, S_3) = 1$. So, the vertices y_i, a_2, and b_2 can be distinguished for all i. By a similar argument, the representations of a_3, b_3, and z_i for all i are also distinct. Then, the representations of a_1, a_4, and x_i for all i are also distinct. Note that $d(a_{i+1}, S_m) = 1$ and $d(b_i, S_m) = 2$. So, for $i = 4, 5, \cdots, m-1$ the vertices a_{i+1} and b_i can be distinguished. So, Π is a resolving partition of G_6. This shows that $pd(G_6) \leq m$. Lemma 1.1 requires that all a_i for $i \in [2, m]$ must be in different partition classes. If a_1 is in the same partition class with other a_i for some $i \in [2, m]$ then all b_j for $j \in [1, n]$ must be in different partition classes. However, one of these partition classes must be different with the other ones containing a_i. Therefore, $pd(G_6) \geq m$.

Case 2. $m \geq n+2$. Define a partition $\Pi = \{S_1, S_2, \cdots, S_{m-1}\}$ of $V(G_6)$ with $S_1 = \{a_2, b_1\}$, $S_2 = \{a_3, b_2, y_1, y_2, \cdots, y_{r_2}\}$, $S_3 = \{a_4, b_3, z_1, z_2, \cdots, z_{r_3}\}$, $S_4 = \{a_1, a_5, b_4, x_1, x_2, \cdots, x_{r_1}\}$, $S_i = \{a_{i+1}, b_i\}$, for $i = 5, 6, \cdots, n$ and $S_i = \{a_{i+1}\}$, for $i = n+1, \cdots, m-1$. we must show that the representations of all vertices in S_1, S_2, S_3, S_4, and S_i for $5 \leq i \leq n$ are distinct. Note that $d(a_2, S_2) = 2$ and $d(b_1, S_m) = 1$. So, the vertices a_2 and b_1 can be distinguished. It is clearly that for $i < j$, either $d(y_i, S_4) \neq d(y_j, S_4)$ or $d(y_i, S_m) \neq d(y_j, S_m)$ holds; and so y_i and y_j can be distinguished. Since $d(y_i, S_3) \geq 2$, $d(a_3, S_3) = d(b_2, S_3) = 1$, $d(a_3, S_4) = 2$ and $d(b_2, S_4) = 1$. So, the vertices y_i, a_3, and b_2 can be distinguished for all i. By a similar argument, the representations of a_4, b_3, and z_i for all i are also distinct. Then, the representations of a_1, a_5, b_4, and x_i for all i are also distinct. Note that $d(a_{i+1}, S_m) = 2$ and $d(b_i, S_m) = 1$. So, for $i = 5, 6, \cdots, n$ the vertices a_{i+1} and b_i can be distinguished.This implies that $pd(G_6) \leq m-1$. Lemma 1.1 requires that all a_i for $2 \leq i \leq m$ must be in different partition classes. So, $pd(G_6) \geq m-1$. This concludes that $pd(G_6) = m-1$. □

References

[1] Slater, P.J., 1975, Leaves of trees, Proc. 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, *Congressus Numerantium* Vol. **14**: 549 – 559

[2] Harary, F., Melter, R.A., 1976, On the metric dimension of a graph, *Ars Combinatoria* Vol. **2**: 191 – 195

[3] Chartrand G., Eroh L., Johnson M., and Oellermann O.R., 2000, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* Vol. **105**: 99 – 113

[4] Caceres J., Hernando C., Mora M., Pelayo I.M., Puertas M.L., Seara C., and Wood D.R., 2007, On the metric dimension of Cartesian products of graphs, *SIAM J. Discrete Math.* Vol. **21**(2): 423 – 441

[5] Hasibuan, I.M, Salman, A.N.M., Saputro, S.W., 2022, The Non-isolated Resolving Number of a Graph Cartesian Product with a Complete Graph, *Acta Math. Univ. Comenianae* Vol. **91**(3): 191 – 204

[6] Chartrand, G., Salehi, E., Zhang, P., 2000, The partition dimension of a graph, *Aequationes Mathematicae* Vol. **59**: 45 – 54

[7] Chartrand, G., Zhang, P., Salehi, E., 1998, On the partition dimension of a graph, *Aequationes Math* Vol. **130**: 157 – 160

[8] Darmaji, Uttunggadewa, S., Simanjuntak, R., Baskoro, E.T., 2009, The partition dimension of complete multipartite graph, a special caterpillar and a windmill, *The Journal of Combinatorial Mathematics and Combinatorial Computing (JCMCC)* Vol. **71**: 209 – 215

[9] Fehr, G.S., Oellermann, O.R., 2006, The partition dimension of Cayley digraphs *Aequationes Mathematicae* Vol. **71**: 1 – 18

[10] Haryeni, D.O., Baskoro, E.T., Saputro, S.W., 2024, Family of Graphs with Partition Dimension Three, *Indonesian Journal of Combinatorics* Vol. **8**(2): 64

– 75

[11] Haspika, Hasmawati, Aris, N, 2023, The Partition Dimension on the Grid Graph, *Jurnal Matematika, Statistika, dan Komputasi* Vol. **19**(2): 351 – 358

[12] Tomescu, I., 2000, Discrepancies between metric and partition dimension of a connected graph, *Discrete Mathematics* Vol. **308**: 5026 – 5031

[13] Amrullah, Azmi, S., Soeprianto, H., Turmuzi, M., Anwar, Y.S., 2019, The partition dimension of subdivision graph on the star, *Journal of Physics: Conference Series* **1280**(2): 022037

[14] Amrullah, Baskoro, E.T., Simanjuntak, R., Uttunggadewa, S., 2015, The Partition Dimension of a Subdivision of a Complete Graph, *Procedia Computer Science* Vol. **74**: 53 – 59