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THE SPACE OF CONTINUOUS FUNCTIONS WITH 2-NORMS

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Abstract. *The purpose of this paper is to study the space of continuous functions, $C[a, b]$ as a 2-normed space. In particular, we show that the space is complete with respect to some linearly independent set.*

Keywords: Completeness; 2-normed spaces; space of continuous functions

1. Introduction

Let X be a (real) vector space (of dimension at least 2). Then a mapping $\|\cdot, \cdot\| : X^2 \to \mathbb{R}$ satisfying the following properties:

(1.1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent,

(1.2) $\|x_1, x_2\|$ is invariant under permutation,

(1.3) $\|\alpha x_1, x_2\| = |\alpha| \, \|x_1, x_2\|$ for every $x_1, x_2 \in X$ and $\alpha \in \mathbb{R}$,

(1.4) $\|x + y, x_2\| \leq \|x, x_2\| + \|y, x_2\|$ for every $x, y, x_2 \in X$,

is called a 2-*norm* on X, and the pair $(X, \|\cdot, \cdot\|)$ is called a 2-*normed space.*

The notion of 2-normed spaces was first introduced by Gähler [1] in 1960's and its generalization can be found in [2,3,4]. Related works may be found in [5,6,7,8,9,10,11,12,13,14].

In 2-normed spaces, the definition of Cauchy sequence, convergence and completeness are given by as follows.

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Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A sequence (x_k) in X is said to *converge* to an $x \in X$ (in 2-norm $\|\cdot,\cdot\|$) if

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for every $y \in X$. Also, a sequence (x_k) in X is called a *Cauchy* sequence if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for every $y \in X$.

If every Cauchy sequence (x_k) in X converges to an $x \in X$, then X is said to be *complete*. A complete 2-normed space is called a 2-*Banach space*.

By the above definition of convergent (or Cauchy) sequences, the area of parallelogram which was spanned by $(x_k - x)$ and y goes to zero whenever the value of k goes to infinity, for every y in X. This condition is too strong.

In 2011, Junaeti [15] studied the space of continuous functions, $C[a, b]$. It is shown that $C[a, b]$ can be equipped with a 2-norm $\|\cdot,\cdot\|_\infty$, where for every $f_1, f_2 \in C[a, b]$, we have:

$$\|f_1, f_2\|_\infty := \max_{a\le x_1,x_2\le b} \left| \det \begin{pmatrix} f_1(x_1) & f_1(x_2) \\ f_2(x_1) & f_2(x_2) \end{pmatrix} \right|. \tag{1.1}$$

In [15], Junaeti also studied the completeness of the space with respect to the following linearly independent set $\{a_1, a_2\}$, where the graphs of a_1 and a_2 are in Figure 1 as follows.

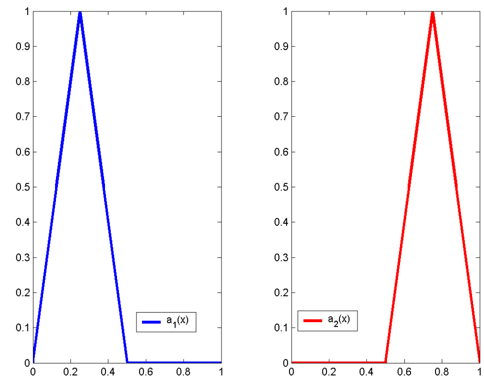

Figure. 1. The graphs of a_1 and a_2

The purpose of this paper is to investigate the completeness of $(C[a, b], \|\cdot,\cdot\|_\infty)$ with respect to some linearly independent sets, namely $\{e_1, e_2\}$, with $e_1(x) = x$, $e_2(x) = 1 - x$, and $\{b_1, b_2\}$ whose the graphs are in Figure 2 as follows.

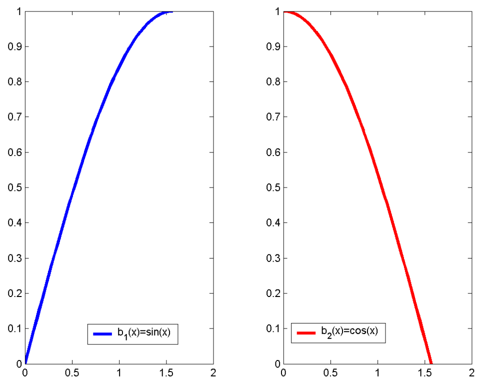

Figure. 2. The graphs of b_1 and b_2

Observe that, for x is close to zero, we have $\sin x \approx x$. By first establishing the completeness of $(C[a,b], \|\cdot,\cdot\|_\infty)$ with respect to $\{b_1, b_2\}$, we get an idea to show the completeness of $(C[a,b], \|\cdot,\cdot\|_\infty)$ with respect to $\{e_1, e_2\}$. Moreover, we can extend the result to any linearly independent set $\{\alpha_1 + \beta_1 x, \alpha_2 + \beta_2 x\}$.

2. Result and Discussion

In $C[a,b]$, the norm $\|\cdot\|_\infty$ defined by $\|f\|_\infty := \max\limits_{a \le x \le b} |f(x)|$ makes $C[a,b]$ a complete space. To show that $(C[a,b], \|\cdot,\cdot\|_\infty)$ is complete with respect to $\{b_1, b_2\}$, as well as to $\{e_1, e_2\}$, we shall derive a norm from the 2-norm.

Before we present the main results, we have the following useful lemma which we shall apply quite often.

Lemma 2.1. *For every $f_1, f_2 \in C[a,b]$, we have*

$$\|f_1, f_2\|_\infty \le 2\|f_1\|_\infty \|f_2\|_\infty.$$

Proof. By using triangle inequality for real numbers, observe that

$$\begin{aligned}
\|f_1, f_2\|_\infty &= \max_{a \le x_1, x_2 \le b} \left| \det \begin{pmatrix} f_1(x_1) & f_1(x_2) \\ f_2(x_1) & f_2(x_2) \end{pmatrix} \right|, \\
&= \max_{a \le x_1, x_2 \le b} \left| f_1(x_1) f_2(x_2) - f_2(x_1) f_1(x_2) \right|, \\
&\le \max_{a \le x_1, x_2 \le b} \left(|f_1(x_1) f_2(x_2)| + |f_2(x_1) f_1(x_2)| \right), \\
&\le \max_{a \le x_1, x_2 \le b} |f_1(x_1) f_2(x_2)| + \max_{a \le x_1, x_2 \le b} |f_2(x_1) f_1(x_2)|, \\
&= 2\|f_1\|_\infty \|f_2\|_\infty.
\end{aligned}$$

This completes the proof. □

Now, we come to the main results.

2.1. *The completeness with respect to* $\{b_1, b_2\}$

As shown in [15], we can derive a norm from the 2-norm $\|\cdot,\cdot\|_\infty$ in $C[a,b]$. The following definition is connecting between $C[a,b]$ as a 2-normed space and $C[a,b]$ as a normed space. In this part, we shall study $C[a,b]$, especially for $a = 0$ and $b = \frac{\pi}{2}$. Let $\{b_1, b_2\}$ be a linearly independent set in $C\left[0, \frac{\pi}{2}\right]$ (see Figure 2). Then the following function

$$\|f\|_\infty^\circ := \max\{\|f, b_1\|_\infty, \|f, b_2\|_\infty\}, \tag{2.1}$$

defines a norm on $C\left[0, \frac{\pi}{2}\right]$. Generally, for $C[a,b]$, the proof of this statement can be seen easily in [15].

On infinite-dimensional vector spaces there is no guarantee that every two norms are equivalent. Nevertheless, for $\|\cdot\|_\infty^\circ$ and $\|\cdot\|_\infty$, we have an important result as follows.

Proposition 2.2. *Let* $\{b_1, b_2\}$ *be a linearly independent set in* $C\left[0, \frac{\pi}{2}\right]$ *(see Figure 2). The derived norm* $\|\cdot\|_\infty^\circ$ *defined by Eq. (2.1) is equivalent to the usual norm* $\|\cdot\|_\infty$*. Precisely, we have*

$$\frac{1}{2}\|f\|_\infty^\circ \leq \|f\|_\infty \leq 6\|f\|_\infty^\circ,$$

for every $f \in C\left[0, \frac{\pi}{2}\right]$.

Proof. Suppose that f is an element of $C\left[0, \frac{\pi}{2}\right]$. Then, for every $i \in \{1, 2\}$, the following hold (from Lemma 2.1).

$$\|f, b_i\|_\infty \leq 2\|f\|_\infty \|b_i\|_\infty.$$

As a consequence of Lemma 2.1, we have:

$$\begin{aligned}\|f\|_\infty^\circ &= \max\{\|f, b_1\|_\infty, \|f, b_2\|_\infty\}, \\ &\leq \max\{2\|f\|_\infty, 2\|f\|_\infty\}, \\ &= 2\|f\|_\infty.\end{aligned}$$

Conversely, observe that:

$$\begin{aligned}\max_{0 \leq x_1 \leq \frac{\pi}{2}} |f(x_1)| &= \max_{0 \leq x_1 \leq \frac{\pi}{2}} \left|1 \cdot f^2(x_1)\right|^{\frac{1}{2}}, \\ &= \max_{\substack{0 \,\leq\, x_1 \,\leq\, \frac{\pi}{2} \\ \frac{\pi}{4} \,\leq\, x_2 \,\leq\, \frac{\pi}{2} \\ 0 \,\leq\, x_3 \,\leq\, \frac{\pi}{4}}} \left|f^2(x_1)\left[\cos^2(x_2 - x_3) + \sin^2(x_2 - x_3)\right]\right|^{\frac{1}{2}}.\end{aligned}$$

The last equality can be written as:

$$\max_{\substack{0 \,\leq\, x_1 \,\leq\, \frac{\pi}{2} \\ \frac{\pi}{4} \,\leq\, x_2 \,\leq\, \frac{\pi}{2} \\ 0 \,\leq\, x_3 \,\leq\, \frac{\pi}{4}}} \left|f^2(x_1)\sin^2\left(\frac{\pi}{2} - (x_2 - x_3)\right) + f^2(x_1)\sin^2(x_2 - x_3)\right|^{\frac{1}{2}}. \tag{2.2}$$

Applying Ptolemy's identities for sine, the first term in Eq. (2.2) becomes:

$$\left[f(x_1)\sin\left(x_2-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}+x_3\right)-f(x_1)\cos\left(x_2-\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}+x_3\right)\right]^2. \quad (2.3)$$

By setting $y_2 = x_2 - \frac{\pi}{4}$ and $y_3 = \frac{\pi}{4} + x_3$, subtracting and adding some term on Eq. (2.3), the first term can be rewritten as:

$$\left(\cos x_2 \begin{vmatrix} f(x_1) & f(y_3) \\ \cos x_1 & \cos y_3 \end{vmatrix} + \cos x_1 \begin{vmatrix} f(y_3) & f(y_2) \\ \sin y_3 & \sin y_2 \end{vmatrix} + \sin x_3 \begin{vmatrix} f(y_2) & f(x_1) \\ \cos y_2 & \cos x_1 \end{vmatrix}\right)^2. \quad (2.4)$$

Then, observe the second term in Eq. (2.2), that is:

$$f^2(x_1)\sin^2(x_2-x_3) = \left[f(x_1)\sin x_2 \cos x_3 - f(x_1)\cos x_2 \sin x_3\right]^2. \quad (2.5)$$

Using the same method with the first term, this expression equals:

$$\left(\sin x_2 \begin{vmatrix} f(x_1) & f(x_3) \\ \cos x_1 & \cos x_3 \end{vmatrix} + \cos x_1 \begin{vmatrix} f(x_3) & f(x_2) \\ \sin x_3 & \sin x_2 \end{vmatrix} + \sin x_3 \begin{vmatrix} f(x_2) & f(x_1) \\ \cos x_2 & \cos x_1 \end{vmatrix}\right)^2. \quad (2.6)$$

By substituting Eq. (2.4) and Eq. (2.6) into Eq. (2.2) and by Minkowski's inequality, we have:

$$\begin{aligned} \|f\|_\infty &= \max_{0\le x_1 \le \frac{\pi}{2}} |f(x_1)| \\ &\le 4\|f, b_2\| + 2\|f, b_1\| \\ &\le 6\max\{\|f, b_1\|, \|f, b_2\|\} \\ &= 6\|f\|_\infty^\circ. \end{aligned}$$

This completes the proof. □

Remark 2.3. *In general, we can replace the closed interval $\left[0, \frac{\pi}{2}\right]$ in Proposition 2.2 become $[a, b]$. As a consequence, we have the positive constants depends of a and b, namely C_1 and C_2, such that:*

$$C_1\|f\|_\infty^\circ \le \|f\|_\infty \le C_2\|f\|_\infty^\circ,$$

for every $f \in C[a, b]$. We leave the proof to the reader.

Since $(C[a, b], \|\cdot\|_\infty)$ is a complete, we have the following corollary.

Corollary 2.4. *The space $(C[a, b], \|\cdot\|_\infty^\circ)$ is complete.*

In Proposition 2.2, we have shown that the derived norm $\|\cdot\|_\infty^\circ$ is equivalent with the usual norm $\|\cdot\|_\infty$ in $C[a, b]$. This proposition is useful to show the notion of completeness on the space $C[a, b]$ (which is equipped with the 2-norm $\|\cdot, \cdot\|_\infty$). The main result is given by the following theorem.

Theorem 2.5. *If a sequence $(f_k) \in C[a, b]$ converges to some $f \in C[a, b]$ with respect to $\{b_1, b_2\}$ in $\|\cdot, \cdot\|_\infty$, then it also converges to f in $\|\cdot\|_\infty$. Also, if (f_k) is a Cauchy sequence with respect to $\{b_1, b_2\}$ in $\|\cdot, \cdot\|_\infty$, then it is a Cauchy sequence in $\|\cdot\|_\infty$.*

Proof. Suppose that $\{b_1, b_2\}$ is a linearly independent set in $C[a, b]$ (see Figure 3), and $\|\cdot\|_\infty^\circ$ is defined by Eq. (2.1). If (f_k) converges to some $f \in C[a, b]$ in

$\|\cdot,\cdot\|_\infty$, then for every $i \in \{1,2\}$, we have $\|f_k - f, b_i\|_\infty \to 0$, as $k \to \infty$. Hence we obtain $\|f_k - f\|_\infty^\circ \to 0$, as $k \to \infty$. So, a sequence (f_k) converges to f in $\|\cdot\|_\infty^\circ$. By Proposition 2.2, we conclude that (f_k) also converges to f in $\|\cdot\|_\infty$. The result for the Cauchy sequence is proved in a similar way. □

The above theorem tells us about the relation of convergence (or Cauchy sequence) between $(C[a,b], \|\cdot,\cdot\|_\infty)$ and $(C[a,b], \|\cdot\|_\infty)$. From this result, we have the following corollary.

Corollary 2.6. *The space* $(C\,[a,b]\,, \|\cdot,\cdot\|_\infty)$ *is a 2-Banach space with respect to* $\{b_1, b_2\}$.

Proof. Let (f_k) be a Cauchy sequence in $(C\,[a,b]\,, \|\cdot,\cdot\|_\infty)$. Then, by Theorem 2.5, (f_k) is a Cauchy sequence in $\|\cdot\|_\infty$. Since $(C\,[a,b]\,, \|\cdot\|_\infty)$ is a Banach space and so (f_k) must converges to an element $f \in C\,[a,b]$ in $\|\cdot\|_\infty$. By Lemma 2.1, (f_k) must also converges to f in $\|\cdot\|_\infty$. Therefore, $(C\,[a,b]\,, \|\cdot,\cdot\|_\infty)$ is a 2-Banach space. □

2.2. *The completeness with respect to* $\{e_1, e_2\}$

As shown in Eq. (2.1), we shall define another formula of derived norm with respect to $\{e_1, e_2\}$ in $C[0,1]$. Suppose that $\{e_1, e_2\}$ is a linearly independent set in $C[0,1]$ and f is an element of $C[0,1]$. Then the following function:

$$\|f\|_\infty^* := \max\{\|f, e_1\|_\infty, \|f, e_2\|_\infty\}, \tag{2.7}$$

defines a norm on $C[0,1]$.

Before we show that norm $\|\cdot\|_\infty^*$ and usual norm $\|\cdot\|_\infty$ are equivalent, we have an interesting observation as follows.

Remark 2.7. *The space spanned by linearly independent set* $\{e_1, e_2\}$ *is the same as the space spanned by* $\{1, x\}$, *as well as spanned by the linearly independent set* $\{\alpha_1 + \beta_1 x, \alpha_2 + \beta_2 x\}$. *As a consequence, we have:*

$$\begin{aligned}\left|\det\begin{pmatrix}\alpha_1+\beta_1x_1 & \alpha_1+\beta_1x_2\\ \alpha_2+\beta_2x_1 & \alpha_2+\beta_2x_2\end{pmatrix}\right| &= \left|\det\begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix}\det\begin{pmatrix}1 & 1\\ x_1 & x_2\end{pmatrix}\right|,\\ &= \left|\det\begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix}\det\begin{pmatrix}e_1(x_1) & e_1(x_2)\\ e_2(x_1) & e_2(x_2)\end{pmatrix}\right|.\end{aligned}$$

By taking the maximum over all x_1, x_2 *on the both sides, we have:*

$$\|\alpha_1+\beta_1x, \alpha_2+\beta_2x\|_\infty = |K|\|1, x\|_\infty = |K|\|e_1, e_2\|_\infty,$$

where $K = \det\begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix} \neq 0.$

By different linearly independent set, we have the following proposition.

Proposition 2.8. *Let $\{e_1, e_2\}$ be a linearly independent set in $C\,[0,1]$ (see Figure 2). The derived norm $\|\cdot\|_\infty^*$ defined by (2.7) is equivalent to the usual norm $\|\cdot\|_\infty$. Precisely, we have:*

$$\frac{1}{2}\|f\|_\infty^* \le \|f\|_\infty \le 3\|f\|_\infty^*,$$

for every $f \in C\,[0,1]$.

Proof. Suppose that f is an element of $C[0,1]$. Then, for every $i \in \{1,2\}$ hold (from Lemma 2.1):

$$\|f, e_i\|_\infty \le 2\|f\|_\infty.$$

As a consequence of Lemma 2.1, we have:

$$\begin{aligned}\|f\|_\infty^* &= \max\{\|f, e_1\|_\infty, \|f, e_2\|_\infty\},\\ &\le \max\{2\|f\|_\infty, 2\|f\|_\infty\},\\ &= 2\|f\|_\infty.\end{aligned}$$

Conversely,

$$\begin{aligned}\|f\|_\infty &= \max_{0\le x_1\le 1}|f(x_1)|,\\ &= \max_{0\le x_1,x_2,x_3\le 1}(|f(x_1)||x_2 - x_3|),\\ &= \max_{0\le x_1,x_2,x_3\le 1}\left(|f(x_1)|\left|\det\begin{pmatrix} x_2 & x_3 \\ 1-x_2 & 1-x_3\end{pmatrix}\right|\right),\\ &= \max_{0\le x_1,x_2,x_3\le 1}\left(|f(x_1)|\left|\det\begin{pmatrix} e_1(x_2) & e_1(x_3) \\ e_2(x_2) & e_2(x_3)\end{pmatrix}\right|\right).\end{aligned}$$

Then, observe that:

$$\begin{aligned}&|f(x_1)|\left|\det\begin{pmatrix} e_1(x_2) & e_1(x_3) \\ e_2(x_2) & e_2(x_3)\end{pmatrix}\right|\\ &\qquad\le |e_2(x_3)|\left|\det\begin{pmatrix} f(x_1) & f(x_2) \\ e_1(x_1) & e_1(x_2)\end{pmatrix}\right| + |e_1(x_1)|\left|\det\begin{pmatrix} f(x_2) & f(x_3) \\ e_2(x_2) & e_2(x_3)\end{pmatrix}\right|\\ &\qquad\quad + |e_2(x_2)|\left|\det\begin{pmatrix} f(x_3) & f(x_1) \\ e_1(x_3) & e_1(x_1)\end{pmatrix}\right|. \qquad (2.8)\end{aligned}$$

By taking the maximum over all x_1, x_2, x_3 of the left and the right hand side in (2.8), we have:

$$\begin{aligned}\|f\|_\infty &\le 2\|f, e_1\|_\infty + \|f, e_2\|_\infty,\\ &\le 3\max\{\|f, e_1\|_\infty, \|f, e_2\|_\infty\},\\ &= 3\|f\|_\infty^*.\end{aligned}$$

The derived norm $\|\cdot\|_\infty^*$ is equivalent to the usual norm $\|\cdot\|_\infty$, as claimed. □

In Proposition 2.8, we can extend the closed interval $[0,1]$ become arbitrary $[a,b]$. Then, by replacing $\left[0,\frac{\pi}{2}\right]$ in Proposition 2.2 and $[0,1]$ in Proposition 2.8 becomes $[a,b]$, we have the following corollary.

Corollary 2.9. *The norm $\|\cdot\|_\infty^\circ$ and the norm $\|\cdot\|_\infty^*$ are equivalent.*

By completeness of the space $(C\,[a,b]\,,\|\cdot\|_\infty)$, we obtain the following corollary.

Corollary 2.10. *The space $(C\,[a,b]\,,\|\cdot\|_\infty^*)$ is complete.*

In the previous subsection, we have shown the completeness of $(C\,[a,b]\,,\|\cdot,\cdot\|_\infty)$ with respect to $\{b_1,b_2\}$. In this part, the following theorem become the most important tools for showing the completeness of $(C\,[a,b]\,,\|\cdot,\cdot\|_\infty)$ with respect to $\{e_1,e_2\}$. The details are as follows.

Theorem 2.11. *If a sequence $(f_k)\in C[a,b]$ converges to some $f\in C[a,b]$ with respect to $\{e_1,e_2\}$ in $\|\cdot,\cdot\|_\infty$, then it also converges to f in $\|\cdot\|_\infty$. Also, if (f_k) is a Cauchy sequence with respect to $\{e_1,e_2\}$ in $\|\cdot,\cdot\|_\infty$, then it is a Cauchy sequence in $\|\cdot\|_\infty$.*

Proof. By replacing $\{b_1,b_2\}$ to $\{e_1,e_2\}$ in Theorem 2.5, this theorem is proved in a similar way. $\square$

Corollary 2.12. *The space $(C[a,b],\|\cdot,\cdot\|_\infty)$ is a 2-Banach space with respect to $\{e_1,e_2\}$.*

By observing Remark 2.7, we have a more general result, that is, the completeness of $(C[a,b],\|\cdot,\cdot\|_\infty)$ with respect to the linearly independent set $\{\alpha_1+\beta_1 x,\alpha_2+\beta_2 x\}$. Before we present this result, we derive a norm from the 2-norm $\|\cdot,\cdot\|_\infty$ with respect to $\{\alpha_1+\beta_1 x,\alpha_2+\beta_2 x\}$, namely:

$$\|f\|_\infty^{**} := \max\{\|f,\alpha_1+\beta_1 x\|_\infty,\|f,\alpha_2+\beta_2 x\|_\infty\}. \tag{2.9}$$

Then, we need the following useful theorem to show our goal.

Theorem 2.13. *Let $\{\alpha_1+\beta_1 x,\alpha_2+\beta_2 x\}$ be a linearly independent set in $C\,[a,b]$. The derived norm $\|\cdot\|_\infty^{**}$ defined by Eq. (2.9) is equivalent to the derived norm $\|\cdot\|_\infty^*$, as well as to the usual norm $\|\cdot\|_\infty$.*

Proof. We shall only give the proof for $C[0,1]$ and leave that for $C[a,b]$ to the reader. Suppose that f is an element of $C[0,1]$. Observe that:

$$\|f,\alpha_1+\beta_1 x\|_\infty \le |\alpha_1|\|f,1\|_\infty + |\beta_1|\|f,x\|_\infty,$$

and

$$\|f,\alpha_2+\beta_2 x\|_\infty \le |\alpha_2|\|f,1\|_\infty + |\beta_2|\|f,x\|_\infty.$$

Hence we have:

$$\begin{aligned}
\|f\|_\infty^{**} &= \max\{\|f, \alpha_1 + \beta_1 x\|_\infty, \|f, \alpha_2 + \beta_2 x\|_\infty\},\\
&\le \max\{|\alpha_1|\|f,1\|_\infty + |\beta_1|\|f,x\|_\infty, |\alpha_2|\|f,1\|_\infty + |\beta_2|\|f,x\|_\infty\},\\
&\le \max\{|\alpha_1| + |\beta_1|, |\alpha_2| + |\beta_2|\}\max\{\|f,1\|_\infty, \|f,x\|_\infty\},\\
&\le \max\{|\alpha_1| + |\beta_1|, |\alpha_2| + |\beta_2|\}\max\{\|f,1-x\|_\infty + \|f,x\|_\infty, \|f,x\|_\infty\},\\
&\le \max\{|\alpha_1| + |\beta_1|, |\alpha_2| + |\beta_2|\}\max\{2\max\{\|f,1-x\|_\infty, \|f,x\|_\infty\}, \|f,x\|_\infty\},\\
&= 2\max\{|\alpha_1| + |\beta_1|, |\alpha_2| + |\beta_2|\}\max\{\|f,1-x\|_\infty, \|f,x\|_\infty\},\\
&= 2\max\{|\alpha_1| + |\beta_1|, |\alpha_2| + |\beta_2|\}\|f\|_\infty^{*}.
\end{aligned}$$

By Proposition 2.8, we have:

$$\|f\|_\infty^{**} \le 2\max\{|\alpha_1| + |\beta_1|, |\alpha_2| + |\beta_2|\}\|f\|_\infty^{*} \le 4\max\{|\alpha_1| + |\beta_1|, |\alpha_2| + |\beta_2|\}\|f\|_\infty.$$

Conversely,

$$\begin{aligned}
\|f\|_\infty^{*} &\le 2\|f\|_\infty,\\
&= 2\max_{0\le x_1\le 1}|f(x_1)|,\\
&= 2\max_{0\le x_1,x_2,x_3\le 1}(|f(x_1)||x_2 - x_3|),\\
&= 2\max_{0\le x_1,x_2,x_3\le 1}\left(|f(x_1)|\left|\det\begin{pmatrix}1 & 1\\ x_2 & x_3\end{pmatrix}\right|\right),\\
&= \frac{2}{|K|}\max_{0\le x_1,x_2,x_3\le 1}\left(|f(x_1)|\left|\det\begin{pmatrix}\alpha_1+\beta_1x_2 & \alpha_1+\beta_1x_3\\ \alpha_2+\beta_2x_2 & \alpha_2+\beta_2x_3\end{pmatrix}\right|\right).
\end{aligned}$$

Then, observe that:

$$\begin{aligned}
&|f(x_1)|\left|\det\begin{pmatrix}\alpha_1+\beta_1x_2 & \alpha_1+\beta_1x_3\\ \alpha_2+\beta_2x_2 & \alpha_2+\beta_2x_3\end{pmatrix}\right|\\
&\le |\alpha_2+\beta_2x_3|\left|\det\begin{pmatrix}f(x_1) & f(x_2)\\ \alpha_1+\beta_1x_1 & \alpha_1+\beta_1x_2\end{pmatrix}\right| + |\alpha_1+\beta_1x_1|\left|\det\begin{pmatrix}f(x_2) & f(x_3)\\ \alpha_2+\beta_2x_2 & \alpha_2+\beta_2x_3\end{pmatrix}\right|\\
&\qquad + |\alpha_2+\beta_2x_2|\left|\det\begin{pmatrix}f(x_3) & f(x_1)\\ \alpha_1+\beta_1x_3 & \alpha_1+\beta_1x_1\end{pmatrix}\right|. \qquad (2.10)
\end{aligned}$$

By taking the maximum over all x_1, x_2, x_3 and multiplying $\frac{1}{|K|}$ on the both sides in Eq. (2.10), we have:

$$\begin{aligned}
\|f\|_\infty^{*} &\le 2\|f\|_\infty,\\
&\le \frac{2}{|K|}\left(2|\alpha_2+\beta_2|\|f,\alpha_1+\beta_1x\|_\infty + |\alpha_1+\beta_1|\|f,\alpha_2+\beta_2x\|_\infty\right),\\
&\le \frac{6\max\{|\alpha_1+\beta_1|, |\alpha_2+\beta_2|\}}{|K|}\max\{\|f,\alpha_1+\beta_1x\|_\infty, \|f,\alpha_2+\beta_2x\|_\infty\},\\
&= \frac{6\max\{|\alpha_1+\beta_1|, |\alpha_2+\beta_2|\}}{|K|}\|f\|_\infty^{**}.
\end{aligned}$$

The derived norm $\|\cdot\|_\infty^{**}$ is equivalent to the derived norm $\|\cdot\|_\infty^{*}$, as well as to the usual norm $\|\cdot\|_\infty$, as claimed. □

Since $(C[a,b], \|\cdot\|_\infty)$ is a complete and by Theorem 2.13, we have $(C[a,b], \|\cdot\|_\infty^{**})$ is also complete. Then, by replacing $\{e_1, e_2\}$ to $\{\alpha_1+\beta_1 x, \alpha_2+\beta_2 x\}$, the analogues of Theorem 2.11 hold. Now, we come to the result.

As a generalization of Corollary 2.12, we have:

Corollary 2.14. *The space $(C[a,b], \|\cdot,\cdot\|_\infty)$ is complete with respect to the linearly independent set $\{\alpha_1+\beta_1 x, \alpha_2+\beta_2 x\}$.*

Proof. Let (f_k) be a Cauchy sequence in $(C[a,b], \|\cdot,\cdot\|_\infty)$. By Eq. (2.9) and Theorem 2.13, (f_k) is a Cauchy sequence in $\|\cdot\|_\infty$. Since $(C[a,b], \|\cdot\|_\infty)$ is a Banach space and so (f_k) must converges to an element $f \in C[a,b]$ in $\|\cdot\|_\infty$. By Lemma 2.1, (f_k) must also converges to f in $\|\cdot,\cdot\|_\infty$. The last statement completes the proof. □

3. Conclusion

We have first shown the completeness of $(C[a,b], \|\cdot,\cdot\|_\infty)$ with respect to linearly independent set $\{b_1, b_2\}$. This result gives us inspiration to show the completeness of $(C[a,b], \|\cdot,\cdot\|_\infty)$ with respect to linearly independent set $\{e_1, e_2\}$. Then, for the further result, we have $(C[a,b], \|\cdot,\cdot\|_\infty)$ is complete with respect to the linearly independent set $\{\alpha_1+\beta_1 x, \alpha_2+\beta_2 x\}$. For $C[a,b]$ as an n-normed space, we also derive a norm from the n-norm $\|\cdot,\ldots,\cdot\|_\infty$, where the definition of n-norm $\|\cdot,\ldots,\cdot\|_\infty$ is given by as follows.

$$\|f_1,\ldots,f_n\|_\infty := \max_{a\le x_1,\ldots,x_n\le b}\left|\det\begin{pmatrix} f_1(x_1) & \cdots & f_1(x_n)\\ \vdots & \ddots & \vdots\\ f_n(x_1) & \cdots & f_n(x_n)\end{pmatrix}\right|,$$

for all $f_1,\ldots,f_n \in C[a,b]$. It is unknown whether we can use the same method such that the usual norm is equivalent to the derived norm. This problem is interesting to study.

Then, we have a question, how about the completeness of $C[a,b]$ (which is equipped with the 2-norm $\|\cdot,\cdot\|_\infty$, as well as with the n-norm $\|\cdot,\ldots,\cdot\|_\infty$) with respect to arbitrary linearly independent set? This question is not easy to answer. We have a difficulty to show equivalence between the usual norm and the derived norm. The research on this problem is still ongoing at the present time.

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Bibliography

[1] S. Gähler, *Lineare 2-normierte räume*, Math. Nachr. **28** (1964), 1–43
[2] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische Räume. I*, Math. Nachr. **40** (1969), 165–189

[3] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische Räume. II*, Math. Nachr. **40** (1969), 229–264

[4] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische Räume. III*, Math. Nachr. **41** (1970), 23–26

[5] Y. J. Cho, C. Park, M. E. Gordji, *Approximate additive and quadratic mappings in 2-Banach spaces and related topics*, Int. J. Nonlinear Anal. Appl. **3** (2012) 75–81

[6] C.R. Diminnie, S. Gähler, A. White, *2-inner product spaces*, Demonstratio Math. **6** (1973), 525–536

[7] S. Ekariani, H. Gunawan, M. Idris, *On the space of p-summable sequences*, Mat. Vesnik **65** (2013), 58–63

[8] S. Gähler, *Uber 2-Banach räume*, Math. Nachr. **42** (1969), 335–347

[9] S. Gähler, A.H. Siddiqi, S.C. Gupta, *Contributions to non-archimedean functional analysis*, Math. Nachr. **69** (1975), 162–171

[10] H. Gunawan, *The space of p-summable sequences and its natural n-norms*, Bull. Austral. Math. Soc. **64** (2001), 137–147

[11] H. Gunawan, Mashadi, *On finite-dimensional 2-normed spaces*, Soochow J. Math. **27** (2001), 321–329

[12] M. Gürdal, A. Şahiner, I. Açik, *Approximation theory in 2-Banach spaces*, Nonlinear Analysis **71** (2009), 1654–1661

[13] R. Pilakkat, S. Thirumangalath, *Results in linear 2-normed spaces analogous to Baire's theorem and closed graph theorem*, Int. J. of Pure and Appl. Math. **74** (2012), 509–517

[14] A. White, *2-Banach space*, Math. Nachr. **42** (1969), 43–60.

[15] E. Junaeti, *Teorema Titik Tetap pada $C[a,b]$ sebagai Ruang Norm-n* (in Indonesian), unpublished, Master thesis (2011)