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THE LOCAL ANTIMAGIC TOTAL CHROMATIC NUMBERS ON BARBELL WHEEL GRAPHS

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Abstract. *Let $G = (V, e)$ be a graph with a finite non-empty vertex set $V(G)$ and a edge set $E(G)$. A local antimagic total labeling on graph G defined as a bijective mapping f from a union of the vertex set and the edges set of G to a set of integers $\{1, 2, \cdots, |V(G)|+|E(G)|\}$ such as for all two adjacent vertices u and v we have $w_t(u) \neq w_t(v)$, where $w_t(u) = f(u) + \sum_{e\in E(u)} f(e)$ is a weight of vertex u, and $E(u)$ is a set of adjacent edges on the vertex u. Each distinct vertex weight in local antimagic total labeling can be considered as distinct colors, so that local antimagic total labeling on graph G induces vertex coloring on graph G, with minimum numbers of colors or its chromatic number denoted as $\chi_{lat}(G)$. The barbell wheel graph $BW_{n,k}$, with $n \geq 3$ and $k \geq 2$, is defined as a graph with two subgraphs of wheels W_n that are connected by the path subgraph P_k at each center vertex. In this paper, we prove that the barbell wheel graph $BW_{n,k}$ has local antimagic total labeling. We also determine its local antimagic total chromatic number.*

Keywords: Barbell wheel graph, local antimagic total chromatic number, local antimagic total labeling

1. Introduction

Let $G = (V, E)$ be a simple connected finite graph. A graph G has the vertex set $V(G)$ and the edge set $E(G)$. A graph H is called a subgraph of a graph G if $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$ [1]. A union graph $G \cup H$ is a graph with a set of vertex $V(G) \cup V(H)$ and a set of edges $E(G) \cup E(H)$. A join graph $G + H$ is a $V(G) \cup V(H)$ with edges that connect every vertices of G to every vertices of H [1]. We refer to Chartrand and Zhang [1] for basic graph terminologies.

Labeling in G is mapping an integer to vertices or/and edges of G [2]. One of its kind is antimagic labeling. Hartsfield and Ringel [3] introduced antimagic labeling, which is a bijective mapping $f : E(G) \to \{1, 2, \cdots, |E(G)|\}$ such that the sum

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of all labels of each edge incident to a vertex is different from every other vertex. Hartsfield and Ringel [3] gave two conjectures about antimagic labeling, one written as follows.

Conjecture 1.1. *[3] Every connected graph different from K_2 is antimagic.*

Arumugam *et al* [4] introduced the concept of local antimagic labeling. A bijection $f : E(G) \to \{1, 2, \cdots, |E(G)|\}$ is called local antimagic labeling if for all $u, v \in V(G)$, applies $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of all edges incident to u. Arumugam *et al* [4] proposed the following conjecture.

Conjecture 1.2. *[4] Every connected graph other than K_2 is local antimagic.*

Putri *et al* [5] initiated the study of local vertex antimagic total labeling or, for short, local antimagic total labeling. A bijection $f : V(G) \cup E(G) \to \{1, 2, \cdots, |V(G)| + |E(G)|\}$ is called local antimagic total labeling if for any two adjacent vertices u and v, $w_t(u) \neq w_t(v)$, where $w_t(u) = f(u) + \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of all edges incident to u. Lau *et al.* [6] gave a theorem as following.

Theorem 1.3. *[6] Every graph G is a local antimagic total.*

Vertex coloring on G is a color assignment to each vertex of G so that no adjacent vertices have the same color. The minimum of color used in the coloring of the vertex in G is called the chromatic number, which is denoted by $\chi(G)$ [3]. If for any vertex u, we see $w_t(u)$ as a color, then clearly any local antimagic total labeling on G induced a proper vertex coloring on G. The minimum colors induced by local antimagic total labeling or the local antimagic total chromatic number of G is denoted by $\chi_{lat}(G)$, with "*lat*" is short for local antimagic total. Putri *et al.* [5] presented an observation as follows.

Observation 1.4. *[5] If $\chi(G)$ is the chromatic number vertex coloring, then $\chi_{lat}(G) \geq \chi(G)$.*

The research carried out by Slamin and Hasan [7], Amalia and Masruroh [8], and Yang *et al.* [9] explained the local antimagic total labeling and its chromatic number for the wheel graph and for some wheel-related graphs. Some of the graphs are fan graph, bowknot graph, Dutch windmill graph, analogous Dutch windmill graph, and flower graph.

Herbster and Pontil [10] and Ghosh *et al.* [11] defined barbell graph independently. Herbster and Pontil [10] said that two n-cliques connected by a single edge is a barbell graph, while Ghosh *et al.* [11] said a barbell graph is a $K_n - K_n$ graph. In another word barbell graph, in this paper, denoted by B_n, there are two complete graphs K_n connected by an edge on one of each vertex.

In this paper we show what is a barbell wheel graph $BW_{n,k}$ and explain a construction of the local antimagic total labeling in $BW_{n,k}$ to get its local antimagic total chromatic number.

2. Barbell Wheel Graph $BW_{n,k}$

The barbell wheel graph $BW_{n,k}$ $(n \geq 3, k \geq 2)$ is obtained by connecting two wheel graphs W_n with one path graph P_k at each center. $BW_{n,k}$ has vertex set $V(BW_{n,k}) = \{x_i : i = 1, 2, \cdots, n\} \cup \{y_i : i = 1, 2, \cdots, n\} \cup \{z_i : i = 1, 2, \cdots, k\}$ and the edge set $E(BW_{n,k}) = \{x_i x_{i+1}, x_i z_1 : i = 1, 2, \cdots, n\} \cup \{y_i y_{i+1}, y_i z_k : i = 1, 2, \cdots, n\} \cup \{z_i z_{i+1} : i = 1, 2, \cdots, k-1\}$, where $x_{n+1} = x_1$ and $y_{n+1} = y_1$. The order of $BW_{n,k}$ is $2n + k$, and its total of vertices and edges is $6n + 2k - 1$. Figure 1 shows the general shape of the barbell wheel graph $BW_{n,k}$.

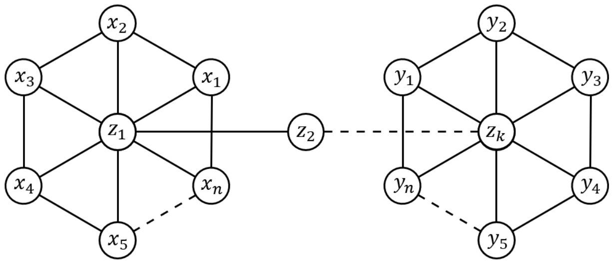

Figure. 1. Barbell wheel graph $BW_{n,k}$

3. Local Antimagic Total Chromatic Number of Barbell Wheel Graph

For the barbell wheel $BW_{3,2}$ we have $\chi_{lat}(BW_{3,2}) = 4$. By vertex coloring of the barbell wheel graph and based on Observation 1.4, it is clear that $\chi_{lat}(BW_{3,2}) \geq 4$. For the upper bound, we have the total local antimagic labeling shown in Figure 2. Since $4 \leq \chi_{lat}(BW_{3,2}) \leq 4$, it is clear that $\chi_{lat}(BW_{3,2}) = 4$.

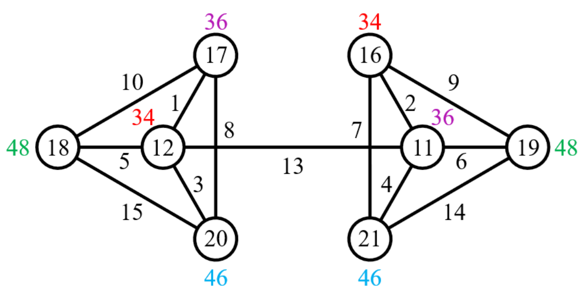

Figure. 2. Local antimagic total labeling on $BW_{3,2}$ with 4 colors

Theorem 3.1. *For the barbell wheel $BW_{n,2}$ and $n > 3$, we have:*

$$3 \leq \chi_{lat}(BW_{n,2}) \leq 4, \text{ for even } n,$$
$$4 \leq \chi_{lat}(BW_{n,2}) \leq 5, \text{ for odd } n.$$

Proof. It is clear that $\chi_{lat}(BW_{n,2}) \geq 4$, for $n > 3$. For the upper bound, we define local antimagic labeling for barbell wheel graph $BW_{n,2}$, for $n > 3$ in two cases for n.

Case 1. For even n. Let $f : V(BW_{n,2}) \cup E(BW_{n,2}) \to \{1, 2, \cdots, (6n + 3)\}$ be a labeling on barbell wheel graph $BW_{n,2}$, for even n and $n > 3$, defined as follows.

$$f(x_i z_1) = \begin{cases} i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 2n+1-i, & \text{for even } i \text{ and } 2 \leq i \leq n. \end{cases}$$

$$f(y_i z_2) = \begin{cases} 1+i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 2n+2-i, & \text{for even } i \text{ and } 2 \leq i \leq n. \end{cases}$$

$$f(x_i x_{i+1}) = \begin{cases} 4n+4-i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 2n+2+i, & \text{for even } i \text{ and } 2 \leq i \leq n-2, \\ 2n+2, & i = n. \end{cases}$$

$$f(y_i y_{i+1}) = \begin{cases} 4n+3-i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 2n+1+i, & \text{for even } i \text{ and } 2 \leq i \leq n-2, \\ 2n+1, & i = n. \end{cases}$$

$$f(z_1 z_2) = 3n+2.$$

$$f(z_1) = 3n+1.$$

$$f(z_2) = 3n+3.$$

$$f(x_i) = \begin{cases} 6n+1-i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 4n+2+i, & \text{for even } i \text{ and } 2 \leq i \leq n-2, \\ 6n+2, & i = n. \end{cases}$$

$$f(y_i) = \begin{cases} 6n+2-i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 4n+3+i, & \text{for even } i \text{ and } 2 \leq i \leq n-2, \\ 6n+3, & i = n. \end{cases}$$

It is clear that f is bijective. Observe that the labeling f induced a weight function as follows.

$$w_t(x_i) = w_t(y_i) = \begin{cases} 12n+6, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 12n+10, & \text{for even } i \text{ and } 2 \leq i \leq n. \end{cases}$$

$$w_t(z_1) = n^2 + 6n + 3.$$

$$w_t(z_2) = n^2 + 7n + 5.$$

It is easy to see that any adjacent vertices have distinct weights. Hence, f is a local antimagic total labeling on the barbell wheel graph $BW_{n,2}$, for even n and $n > 3$. With this local antimagic total labeling f, we have $\chi_{lat}(BW_{n,2}) \leq 4$, for even n and $n > 3$. So it is proven that $3 \leq \chi_{lat}(BW_{n,2}) \leq 4$, for even n and $n > 3$.

Case 2. For odd n. Let $f : V(BW_{n,2}) \cup E(BW_{n,2}) \to \{1, 2, \cdots, (6n + 3)\}$ be a labeling on barbell wheel graph $BW_{n,2}$, for odd n and $n > 3$, defined as follows.

$$f(x_iz_1) = \begin{cases} i, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ n+i, & \text{for even } i \text{ and } 2 \leq i \leq n-1. \end{cases}$$

$$f(y_iz_2) = \begin{cases} 1+i, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ n+1+i, & \text{for even } i \text{ and } 2 \leq i \leq n-1. \end{cases}$$

$$f(x_ix_{i+1}) = \begin{cases} 4n+4-i, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ 3n-i, & \text{for even } i \text{ and } 2 \leq i \leq n-1. \end{cases}$$

$$f(y_iy_{i+1}) = \begin{cases} 4n+3-i, & \text{for odd } i \text{ and } 1 \leq i \leq n, \\ 3n+1-i, & \text{for even } i \text{ and } 2 \leq i \leq n-1. \end{cases}$$

$$f(z_1z_2) = 3n+1.$$

$$f(z_1) = 3n.$$

$$f(z_2) = 3n+2.$$

$$f(x_i) = \begin{cases} 5n+3, & i = 1, \\ 5n+4+i, & \text{for even } i \text{ and } 2 \leq i \leq n-1, \\ 4n+2+i, & \text{for odd } i \text{ and } 3 \leq i \leq n. \end{cases}$$

$$f(y_i) = \begin{cases} 5n+4, & i = 1, \\ 5n+3+i, & \text{for even } i \text{ and } 2 \leq i \leq n-1, \\ 4n+1+i, & \text{for odd } i \text{ and } 3 \leq i \leq n. \end{cases}$$

It is clear that f is bijective. Observe that the labeling f induced a weight function as follows.

$$w_t(x_i) = w_t(y_i) = \begin{cases} 12n+11, & i = 1, \\ 11n+7, & \text{for even } i \text{ and } 2 \leq i \leq n-1, \\ 13n+9, & \text{for odd } i \text{ and } 3 \leq i \leq n. \end{cases}$$

$$w_t(z_1) = n^2 + 6n + 1.$$

$$w_t(z_2) = n^2 + 7n + 3.$$

It is easy to see that any adjacent vertices have distinct weights. Hence, f is a local antimagic total labeling on the barbell wheel graph $BW_{n,2}$, for odd n and $n > 3$. With this local antimagic total labeling f, we have $\chi_{lat}(BW_{n,2}) \leq 5$, for odd n and $n > 3$. Thus, we obtain that $4 \leq \chi_{lat}(BW_{n,2}) \leq 5$, for odd n and $n > 3$. □

Figure 3 shows the local antimagic total labeling on $BW_{6,2}$ and $BW_{7,2}$. We could see that local antimagic total labeling on $BW_{6,2}$ resulted in 4 colors, and local antimagic total labeling on $BW_{7,2}$ resulting 5 colors.

Theorem 3.2. *For the barbell wheel $BW_{n,3}$, for even n and $n \geq 4$, we have $\chi_{lat}(BW_{n,3}) = 3$.*

Proof. By proper vertex coloring, we can see that $\chi_{lat}(BW_{n,3}) \geq 3$ for even n. Let $f : V(BW_{n,3}) \cup E(BW_{n,3}) \to \{1, 2, \cdots, (6n+5)\}$ be a labeling on barbell wheel

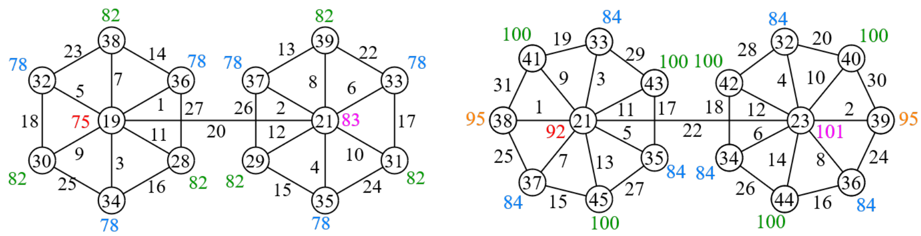

Figure. 3. Local antimagic labeling total on $BW_{6,2}$ and $BW_{7,2}$

graph $BW_{n,3}$, for even n and $n \geq 4$, defined as follows.

$$f(x_i z_1) = \begin{cases} n+i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 2n+i, & \text{for even } i \text{ and } 2 \leq i \leq n. \end{cases}$$

$$f(y_i z_3) = \begin{cases} n+1+i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 2n-1+i, & \text{for even } i \text{ and } 2 \leq i \leq n. \end{cases}$$

$$f(x_i x_{i+1}) = \begin{cases} n-i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 3n+4+i, & \text{for even } i \text{ and } 2 \leq i \leq n. \end{cases}$$

$$f(x_i x_{i+1}) = \begin{cases} n-i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1, \\ 3n+3+i, & \text{for even } i \text{ and } 2 \leq i \leq n. \end{cases}$$

$$f(z_i z_{i+1}) = 3n+6-2i, i = 1, 2.$$

$$f(z_i) = \begin{cases} 3n+i, & i = 1, 3, \\ 4n+5, & i = 2. \end{cases}$$

$$f(x_i) = \begin{cases} 4n+7, & i = 1, \\ 6n+6-i, & \text{for even } i \text{ and } 2 \leq i \leq n, \\ 5n+8-i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1. \end{cases}$$

$$f(y_i) = \begin{cases} 4n+6, & i = 1, \\ 6n+7-i, & \text{for even } i \text{ and } 2 \leq i \leq n, \\ 5n+7-i, & \text{for odd } i \text{ and } 1 \leq i \leq n-1. \end{cases}$$

It is clear that f is bijective. Observe that the labeling f induced a weight function as follows.

$$w_t(x_i) = w_t(y_i) = \begin{cases} 12n+11, & \text{for even } i, \\ 10n+11, & \text{for odd } i. \end{cases}$$

$$w_t(z_1) = w_t(z_3) = 2n^2 + \frac{13n}{2} + 5.$$

$$w_t(z_2) = 10n + 11.$$

Note that $w_t(z_2) = w_t(x_i) = w_t(y_i)$ for odd i. It is easy to see that any adjacent vertices have distinct weights. Hence, f is a local antimagic total labeling on the barbell wheel graph $BW_{n,3}$, for even n and $n \geq 4$. With this local antimagic total labeling f, we have $\chi_{lat}(BW_{n,3}) \leq 5$, for even n and $n \geq 4$. Thus, we prove that $\chi_{lat}(BW_{n,3}) = 3$, for even n and $n \geq 4$. □

Figure 4 shows local antimagic total labeling on $BW_{6,3}$. We could see that local antimagic total labeling on $BW_{6,3}$ resulted in three colors.

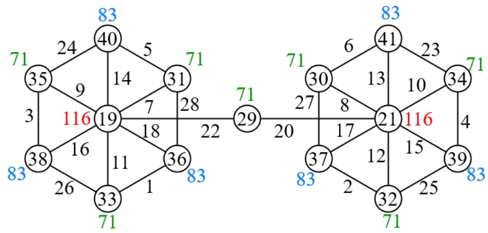

Figure. 4. Local antimagic total labeling on $BW_{6,3}$

4. Conclusion

In this paper, we have shown that $\chi_{lat}(BW_{3,2}) = 4$, $3 \leq \chi_{lat}(BW_{n,2}) \leq 4$ for even n and $n > 3$, and $4 \leq \chi_{lat}(BW_{n,2}) \leq 5$ for odd n and $n > 3$. Note that for $n > 3$, the local antimagic total chromatic number is still in the form of an interval, which leads to the following problem.

Problem 4.1. *Determine the exact value of $\chi_{lat}(BW_{n,k})$, for $n > 3$ and $k = 2$, and for $n \geq 3$ and $k \geq 3$.*

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