

ON HEREDITARY NOETHERIAN AND PRIME MODULES OVER LEAVITT PATH ALGEBRA: SINK AND INFINITE EMITTER

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Abstract. This study explores the structural and homological characteristics of left modules over Leavitt path algebras, focusing on those generated by specific vertex types in a directed graph, particularly sinks and infinite emitters. The paper examines the module $L_K(E)u$, where $u \in E^0$ represents either a sink or an infinite emitter, and determines whether this module exhibits the properties of being hereditary, Noetherian, and prime. Our findings indicate that $L_K(E)u$ is indeed hereditary, implying that all its submodules are projective. Additionally, the module satisfies the ascending chain condition, making it Noetherian. However, it fails to qualify as an a -prime module, since there exist nontrivial left ideals $I \subsetneq L_K(E)$ for which $IM \subsetneq M$, thus does not meet the criteria for primality. These results emphasize how the presence of sinks and infinite emitters significantly affects the module-theoretic behavior of Leavitt path algebras.

Keywords: HNP module, Leavitt path algebra, Sink

1. Introduction

The Leavitt path algebras are one of the fastest-growing fields in algebra theory, especially in studying algebraic structures and their associated modules. This algebra was introduced as an extension of the concept of path algebra. Leavitt path algebra is widely studied in the theory of noncommutative algebras and modules, especially in understanding ideal properties, primitivity, and representation structures. Using representation theory, we can see the properties of modules that are abstracted from a directed graph, which we call just a graph in this case. Leavitt Path Algebra (LPA) is a class of algebras constructed from directed graphs [1]. Leavitt path algebras were introduced as algebraic analogs of graph C^* -algebras

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and have since become a central object of study in noncommutative algebra. Given a directed graph E and a field K , the Leavitt path algebra $L_K(E)$ is constructed using generators corresponding to vertices and edges of the graph, together with relations that encode the combinatorial structure of E . These algebras provide a rich class of examples in ring theory and have deep connections with various branches of mathematics, including operator algebras, symbolic dynamics, and noncommutative geometry.

One of the key interests in studying Leavitt path algebras lies in understanding their module theory. Leavitt path algebras became a major object of study in ring theory and module representation due to their interesting properties as they developed. Researchers have been actively investigating the properties of Leavitt path algebras as algebraic structures through various papers. A bunch of papers have delved into their module theory, exploring how these algebras behave when viewed as modules over themselves or other algebraic structures ([2], [3], [4], [5], [6] [7], [8], [9], [10], [11], [12]). In particular, hereditary rings are of great importance because they ensure that all submodules of projective modules are also projective, simplifying homological considerations. A ring R is called left hereditary if its global dimension is at most one. Hereditary modules are characterized by the property that every submodule is projective. For Leavitt path algebras, it has been shown that $L_K(E)$ is hereditary when the underlying graph E satisfies certain finiteness conditions. However, the presence of infinite emitters or sinks can significantly affect this property, making it necessary to analyze how such graph features influence the hereditariness of the associated algebra.

Another important structural property in ring and module theory is the Noetherian condition. A ring R is left Noetherian if every ascending chain of left ideals stabilizes, or equivalently, if every left ideal is finitely generated. While many classical algebras satisfy this condition, Leavitt path algebras often fail to be Noetherian unless the graph is finite and acyclic. The presence of an infinite emitter, which is a vertex emitting infinitely many edges, typically leads to the failure of the Noetherian property. Understanding under what conditions a Leavitt path algebra over a graph containing infinite emitters remains Noetherian, or at least has well-behaved finitely generated modules, is an active area of research. At the same time, Noetherian modules are finitely generated modules.

Meanwhile, prime modules are closely related to prime ideals on a ring [13], making them important in studying ring structures and module representations. Prime ideals also play a crucial role in the structure theory of rings, particularly in understanding the spectrum of an algebra. In Leavitt path algebras, prime and primitive ideals have been extensively studied and are often characterized by the graph's structure, such as cycles, exits, and maximal tails. When the graph contains sinks or infinite emitters, the structure of prime ideals may change significantly. Sinks, being vertices that emit no edges, tend to generate minimal ideals, while infinite emitters often give rise to more complex ideal structures. Investigating how these elements interact with the primality of ideals provides deeper insight into the internal structure of the algebra. Suppose M is a module over the ring R (written R -Module M). Module M is said to be hereditary if all its submodules are projective.

Furthermore, a Module M is called a Noetherian module if M is a finitely generated module. Module M is called a prime module if $rRm = 0$ with $0 \neq r \in R$ and $0 \neq m \in M$ implies $rM = 0$. Research by ([14], [15], [16]) has examined a wide variety of situations and cases. Thus, a Hereditary Noetherian and Prime (HNP) module is a module that has the properties of a hereditary Module, a noetherian Module, and a prime Module. Irawati ([17], [18]) generalized the concept of the hereditary Noetherian prime ring (HNP) into the concept of the HNP module. Research on modules over Leavitt path algebras was first introduced by Chen [9]. Chen has found several classes of simple modules over Leavitt path algebra based on the shape of the graph that builds its Module. Among them are simple modules built through points that are sinks or infinite emitters. This article will characterize hereditary Noetherian and prime modules over Leavitt path algebras via graphs constructed by sinks or infinite emitters.

This research explores the properties of hereditary modules, Noetherian conditions, and prime modules in Leavitt path algebras associated with graphs containing both sinks and infinite emitters. Interestingly, in this paper we show that if $L_K(E)$ is a Leavitt path algebra where E is a graph containing sink or infinite emitters, K is a field and M is a module over the Leavitt path algebra $L_K(E)$, then M is a simple module over the Leavitt path algebra which is a hereditary, Noetherian and not a prime module.

Section 2 of this paper will provide basic definitions and notations that will be used in this paper. In Section 3, it is shown that a simple module over the Leavitt path algebra of a graph containing sinks or infinite emitters is a hereditary Noetherian and not prime Module. Section 4 is the conclusion.

2. Leavitt Path Algebra, Chen Simple Modules, Hereditary Noetherian and Prime (HNP) Modules

2.1. *Leavitt Path Algebra and Chen Simple Module by using Sink or Infinite Emitter*

In this section, the basic theory of Leavitt path algebra, prime modules and Chen simple model will be explained. All the proofs of the theorems can be found in [9], [11], [19].

Two set (E^0, E^1) , together with the maps $r, s : E^1 \rightarrow E^0$, form what is called a (directed) graph E . Vertices and edges are the names given to the components of E^0 and E^1 , respectively. If a vertex v does not emit any edges, it is referred to as a sink. If a vertex v emits a non-empty finite set of edges, it is referred to as a regular vertex and a vertex that emits an unlimited number of edges is called an infinite emitter.

The directed graph E can be enhanced by incorporating arrows that point in the reverse direction. The arrows in E^1 are referred to as the real edges, whereas the edges that point in the opposite direction from the real edges are termed ghost edges. The collection of all ghost edges within E is represented as $(E^1)^*$. A Leavitt path algebra can be recognized through the extended graph E associated with the path algebra KE . Consider $E = (E^0, E^1, r, s)$, which is transformed into a new

graph $\hat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* | e_i \in E^1\}$. The functions r' and s' are defined as follows: $r'|_{E^1} = r, s'|_{E^1} = s, r'(e_i^*) = s(e_i)$, and $s'(e_i^*) = r(e_i)$.

According to [11], the K -algebra produced by a set $\{v : v \in E^0\}$ and $\{e, e^* : e \in E^1\}$ that satisfies the following relations is the Leavitt path algebra of E with coefficients in K :

- (1) $s(e)e = e = er(e)$, for all $e \in E^1$.
- (2) $r(e)e^* = e^* = e^*s(e)$, for all $e^* \in E^1$.
- (3) (CK1) $e^*f = \delta_{ef}r(e)$ for all $e, f \in E^1$.
- (4) (CK2) $v = \sum_{\{e \in E^1 | s(e)=v\}} ee^*$ whenever $v \in E^0$ is not a sink.

This algebra is called $L_K(E)$, where E is a directed graph and K is a field. In this context, the relations (3) and (4) are particularly referred to as the Cuntz-Krieger relations, because they are essential to the algebra's structure. Assuming that this does not cause any misunderstandings, we will simply refer to the Leavitt path algebra $L_K(E)$ as L for notational convenience.

An important aspect of the algebra L is that every element can be represented in a particular form. Specifically, any element $a \in L$ can be expressed as $a = \sum_{i=1}^n k_i \alpha_i \beta_i^*$, where each $k_i \in K$ is a non-zero scalar from the field K (i.e., $k_i \neq 0, k_i \in K$), and α_i, β_i are paths in E .

As discussed in the work of Abrams [2], if we consider a left L -module, we can establish a homomorphism $\rho : L \rightarrow M$, for every element m belonging to M . This homomorphism is defined such that for any element $r \in L$, the action on m can be described by $\rho_m(r) = rm$. This construction allows for a systematic way of analyzing the interactions between the algebra and the module.

Building on this foundation, Chen introduced a significant class of simple modules over Leavitt path algebras [9]. These simple modules are derived through a method that highlights the importance of specific features within the graph E , particularly focusing on special vertices and cycles. We will provide a detailed overview of the systematic approach employed by Chen [9] as well as subsequent explorations in the work of Ranggaswamy [11] and [20] that elaborate on the construction of simple modules over L . This will include discussions on how particular configurations within the graph lead to the formation of these modules, thereby enriching our understanding of Leavitt path algebras and their representation theory.

Definition 2.1. *In a graph E , let u be a vertex that is either a sink or an infinite emitter. Assume that the set $B = \{p : p \text{ paths in } E \text{ with } r(p) = u\}$ is the basis of the K -vector space A_u . We transform A into a left L -module in this ways: For every vertex v and edge e in E , define the linear transformations P_v, S_e , and S_{e^*} on A by defining their actions on basis B as follows.*

- (1) For all $p \in B$,

$$P_v(p) = \begin{cases} p, & \text{if } v = s(p), \\ 0, & \text{others.} \end{cases}$$

(2)

$$S_e(p) = \begin{cases} ep, & \text{if } r(e) = s(p), \\ 0 & , \text{others.} \end{cases}$$

(3)

$$S_{e^*}(u) = 0,$$

(4)

$$S_{e^*}(p) = \begin{cases} p', & \text{if } p = ep', \\ 0 & , \text{others.} \end{cases}$$

The endomorphisms $\{P_u, S_e, S_e^* : u \in E^0, e \in E^1\}$ then satisfy the defining relations (1) - (4) of the Leavitt path algebra L , and it is simple to verify this. The resulting ϕ from L to $End_K(S_{v_\infty})$ is an algebra homomorphism, translating u to P_u , e to S_e , and e^* to S_{e^*} . The homomorphism ϕ can then be used to make A_u a left module over L .

Lemma 2.2. *If the vertex u is either a sink or infinite emitter, then A_u is a simple left L -module.*

In this case it is clear that the module constructed by a graph containing sinks or infinite emitters is a simple module over Leavitt's path algebra.

2.2. Hereditary Noetherian and Prime (HNP) Modules

The definitions of hereditary Noetherian and prime modules are provided below.

Definition 2.3. *If every right ideal of an algebra A is a projective module as an A -module, then the algebra is said to be right hereditary.*

The left hereditary algebra is defined as its dual. Furthermore, it is explained how a right hereditary algebra can be characterized through the following theorem.

Theorem 2.4. *The following statements are equivalent:*

- (1) A is right hereditary algebra.
- (2) Every submodule of a projective right A -module is projective.
- (3) Every submodule of a finitely generated projective right A -module is projective.
- (4) The radical of every finitely generated indecomposable right A -module is projective.

Next we give the definition of a hereditary module of any ring.

Definition 2.5. *Suppose R is an arbitrary ring. A module M over R is said to be hereditary if M and all its submodules are projective.*

Futhermore, we will see the properties of hereditary modules over Leavitt path algebra.

Theorem 2.6. *Suppose L is a Leavitt path algebra, E is a graph with cycle no exit. If a Leavitt path algebra L corresponds to a cycle-less graph E with an exit, then L*

is a hereditary algebra, and the module constructed by a Leavitt path algebra from a cycle-less graph without an exit is a hereditary module.

Furthermore, the following definition will explain about Noetherian modules.

Definition 2.7. Module M over a ring R is said to be Noetherian (or called a Noether module) if it satisfies the ascending condition for submodules (ACC).

Next, we will see the properties of Noetherian modules in the setting Leavitt path algebra.

Definition 2.8. Let E is a graph. If E is finite and acyclic, then $L_K(E)$ is a Noetherian algebra.

Based on the properties of Noetherian modules, the module over Noetherian algebra is a Noetherian module. Next we look at the definition of a prime module.

Definition 2.9. A left R -module M is called a prime module if for all $r \in R$ and $m \in M$, the condition $rRm = 0$ implies $rM = 0$.

3. Result and Discussion

This section discusses the hereditary Noetherian and prime modules over the Leavitt path algebra of a graph containing sinks and infinite emitters.

Theorem 3.1. Module A_u is a Hereditary Noetherian and not an a -prime module if u is a sink in E .

Proof. For simplicity, we consider a graph consisting of three vertices and two edges, as depicted in the figure below, in order to demonstrate this theorem without sacrificing generality. The graph E is represented as shown in Figure 1.

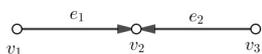


Figure. 1. Graph E with the sink

A module is called *hereditary* if every submodule is projective. Since v_2 is a sink, there are no edges $e \in E^1$ such that $s(e) = v_2$. Therefore, the only path starting at v_2 is the trivial path v_2 itself. Hence, the left ideal $L_K(E)v_2$ consists precisely of all K -linear combinations of paths ending at v_2 , which in this case is just Ku . Thus,

$$L_K(E)v_2 = \text{Span}_K\{\alpha \in \text{Path}(E) \mid r(\alpha) = v_2\} = Kv_2 \cong K.$$

Now, K as a module over any ring (including $L_K(E)$) is trivially hereditary because its submodules are either zero or the entire module, both of which are projective.

Since isomorphism preserves the hereditary property, we conclude that $L_K(E)v_2$ is also hereditary.

A module is called *Noetherian* if every ascending chain of submodules stabilizes, or equivalently, if every submodule is finitely generated. As shown above, $L_K(E)v_2 \cong K$, and K is a one-dimensional vector space over itself. Therefore, K has exactly two submodules: $\{0\}$ and K . It follows that $L_K(E)v_2$ has only finitely many submodules and thus satisfies the ascending chain condition. Hence, $L_K(E)v_2$ is a Noetherian module.

Futhermore we will proof module A_{v_2} is not a prime module. Let $r = -3v_1 + 2v_2 + 3e_1 - 2e_1^* - 2e_2^*, r \in L$ and $m = v_2 + e_1 + e_2, m \in A_{v_2}$. Then $rm = 0$. However, if we take $m_1 = e_2, m_1 \in A_{v_2}$ then $rm_1 \neq 0$. So $r \notin \text{Ann } A_{v_2}$. Therefore $\exists m \neq 0$ and $r \notin \text{Ann } A_{v_2}$ such that $rm = 0$. Thus, A_{v_2} is not prime module. So, module A_{v_2} is a Hereditary Noetherian and not prime module.

In summary, we have shown that If u is a sink in the graph E , then the left module $L_K(E)u$ is a hereditary Noetherian module, but it is not an a-prime module. This result highlights how the presence of a sink influences the homological and structural properties of modules over Leavitt path algebras. □

The next section is devoted to describing the properties of modules, specifically those that are hereditary, Noetherian, and prime that defined over Leavitt path algebras constructed from graphs containing infinite emitters.

Theorem 3.2. *Let E be a directed graph, K be a field, and $L_K(E)$ be the Leavitt path algebra of E . Suppose that $u \in E^0$ is an infinite emitter, i.e., infinitely many edges are emitted from u . Then the left module $L_K(E)u$ is a hereditary Noetherian module, but it does not satisfy the condition to be an a-prime module.*

Proof. To simplify the analysis, we consider the graph depicted in Figure 2. Using the set $B = \{p : p \text{ is a path in } E \text{ with } r(p) = v_2\}$ as a basis and considering v_2 as an infinite emitter, the simple module A_{v_2} becomes a module over the Leavitt path algebra L .

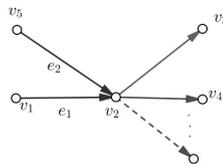


Figure. 2. Graph E with infinite emitter

Since u is an infinite emitter, there are infinitely many edges $e \in E^1$ such that $s(e) = u$. However, the structure of $L_K(E)u$ remains a projective left $L_K(E)$ -module because u is an idempotent element of $L_K(E)$, and hence $L_K(E)u$ is a direct summand of $L_K(E)$.

It is known that all projective modules over Leavitt path algebras are hereditary. Therefore, $L_K(E)u$ is a hereditary module. Next, although u emits infinitely many edges, each path ending at u must have finite length. Hence, any element in $L_K(E)u$ can be expressed as a finite linear combination of such paths. This implies that every submodule of $L_K(E)u$ is finitely generated, since only finitely many such paths are needed to generate any given submodule.

Thus, $L_K(E)u$ satisfies the ascending chain condition on submodules. Therefore $L_K(E)u$ is a Noetherian module. Furthermore we will prove that module A_{v_2} is not a prime module. The set $B = \{v_2, e_1, e_2\}$ is obvious. Let $r = v_1 + 2e_1, r \in L$ and $m = v_2 - 2e_1 + e_2, m \in A_{v_2}$, thus $rm = 0$. On the other hand, $rm_1 \neq 0$ if we assume $m_1 = e_1, m_1 \in A_{v_2}$. Consequently, $r \notin \text{Ann } A_{v_2}$. Therefore, $r \notin \text{Ann } A_{v_2}$ and there exist $m \neq 0$ such that $rm = 0$. Consequently, A_{v_2} is not a prime module.

In summary, we have shown that If u is an infinite emitter in the graph E , then the left module $L_K(E)u$ is a hereditary Noetherian module, but it is not an a-prime module. □

This result highlights how the presence of an infinite emitter influences the homological and structural properties of modules over Leavitt path algebras. The same argument applies when E has subgraph F , as shown in Figure 3.

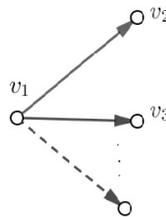


Figure 3. Subgraph F with infinite emitter

4. Conclusion

In this research, we have studied the structural and homological properties of modules over Leavitt path algebras associated with specific types of vertices in a directed graph, namely, sinks and infinite emitters. We focused on the left module $L_K(E)u$, where $u \in E^0$ is either a sink or an infinite emitter, and investigated whether this module is hereditary, Noetherian, and a-prime.

Our findings show that for both cases, the module $L_K(E)u$ is hereditary, meaning every submodule is projective. It is also Noetherian, satisfying the ascending chain condition on submodules. However, it is not an a-prime module, as there exist proper left ideals $I \subsetneq L_K(E)$ such that $IM \subsetneq M$, violating the prime module condition.

These results highlight how the combinatorial structure of the underlying graph

particularly the presence of sinks and infinite emitters significantly influences the module theoretic behavior of the corresponding modules over Leavitt path algebras.

This study contributes to the broader understanding of the interplay between graph theory and noncommutative algebra, especially in the context of module categories over Leavitt path algebras. Future work may extend these results to more general classes of modules or explore related notions such as semi-hereditary modules, torsion theories, or classification via Morita equivalence.

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