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ANALYSIS OF THE BROWNIAN MOTION ON THE MATRIX LIE GROUP SO(2) FOR DETERMINING A SHORT-TERM INTEREST RATE MODEL: A SIMULATION APPROACH

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Abstract. *In this paper, we observe the special orthogonal matrix Lie group containing of all 2×2 real matrices, denoted by* SO(2)*, which can be geometrically visualized as the one-dimensional torus* $\mathbb{S}^1$ *which is nothing but the unit circle. A Brownian motion on* SO(2) *can be constructed and represented by a stochastic differential equation defined over a dynamic state space. The research aims to derive a short-term interest rate model on* SO(2) *through Brownian motion analysis which is a geometric approach. We employ a qualitative methodology, including a literature review of Brownian motion, stochastic differential equations, and dynamical state-space techniques on* SO(2)*. Firstly, we prove the isomorphism* $\mathrm{SO}(2) \cong \mathbb{S}^1$*, secondly, we determine Brownian motion on* SO(2) *and its equivalent, and thirdly, we formulate the corresponding stochastic differential equation, and the last, determine the short-term interest rate equation on* SO(2)*. In this study, it is confirmed Lim and Privault's work that the interest rate equation on* SO(2) *is given by* $r_t = \beta + 2\gamma\cos(W_t)$ *with* β, γ *is constant and* W_t *is standard Brownian motion. To clarify the obtained results, this study also gave a quantitative approach that is Python simulation of interest rate calculation using the matrix Lie group interest rate and other equations. The interest rate equation uses the matrix Lie group* $\mathrm{SO}(n)$ *with* $n \geq 3$ *still open to further research that can be applied to long-term interest rates.*

Keywords: Brownian Motion; Lie group matrix SO(2); Short-term interest rates;

1. Introduction

Interest rate is defined as the amount of charge on the principal loan that must be repaid by the debtor [1]. This interest rate can be formulated in mathematical models and is subsequently referred to as an interest rate model. For instance,

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interest rate models can be modeled using linear regression or machine learning approaches [2]. In general, the prevailing interest rate models used in the banking sector are still based on standard stochastic processes.

Interest rate models play a crucial role in a country's economic instruments because they can function as a tool for maintaining stability [3]. This important role is evident in instruments such as bonds and deposits [4]. Furthermore, interest rates are also essential in green economics. In the 2022 World Economic Forum, the Indonesian government committed to improving the national economy through the synergy of human resources, economic growth, and environmental quality sustainability. One example of this economic improvement is the implementation of green economics, which is expected to support the government in achieving the Sustainable Development Goals (SDGs) ([5],[6],[7],[8]). However, green financing remains a challenge not only in Indonesia but also in many countries, such as China and India ([9],[10]). A primary barrier to accessing such green financing is an interest rate structure that inadequately considers sustainability and long-term risk dynamics. Current interest rate models remain conventional and insufficiently adaptive to economic policies. Moreover, long-term projects are not yet the primary preference for financial institutions, resulting in relatively high offered interest rates ([11],[12]).

Mathematically, prevailing interest rates still follow standard stochastic process models such as Vasicek and Cox-Ingersoll-Ross ([13],[14],[15]). Additionally, conventional interest rate models – including those using linear regression – often fail to accurately represent the random behavior of real-world interest rates. The need for adaptive interest rate models extends beyond green economic policies to encompass all economic types. This necessitates innovation in conventional interest rates that have long dominated financial markets.

Two critical factors requiring attention are the stochastic behavior of interest rates and financing complexity. First, random behavior in data—particularly concerning interest rates—can be addressed by implementing the concept of Brownian motion [16]. This Brownian motion originates from the random movement of microscopic particles in liquid or gaseous media [17]. Mathematically, Brownian motion can be represented as a stochastic process [18], with its state space constructible from vector spaces [19]. Second, financing complexity can be resolved by involving matrix Lie groups which applies the geometric approach.

One example of a matrix Lie group is the special orthogonal group which is denoted by $\mathrm{SO}(n)$. The matrix Lie group $\mathrm{SO}(n)$ consists of all orthogonal matrices with determinant equal to one. This group is called "special" because geometrically, $\mathrm{SO}(n)$ represents rotation transformations. Furthermore, $\mathrm{SO}(n)$ is a Lie subgroup of $\mathrm{O}(n)$ - the orthogonal matrix Lie group that includes both rotations and reflections in an $\frac{n}{2}(n-1)$-dimensional space. In particular, for $n = 2$, the matrix Lie group $\mathrm{SO}(2)$ contains rotation matrices in the plane and can be defined as $\mathrm{SO}(2) = \left\{A \in M(2, \mathbb{R}) \mid A^T = A^{-1},\ \det(A) = 1\right\}$.The group $\mathrm{SO}(2)$ is isomorphic to the unit circle $\mathbb{S}^1 = \left\{x \in \mathbb{R}^2 \mid \|x\| = 1\right\} \subseteq \mathbb{R}^2$. Brownian motion on $\mathrm{SO}(2)$ can be more easily represented on the circle $\mathbb{S}^1$ and used as a model for constructing short-term interest rate models. For the case $n = 3$, the matrix Lie group $\mathrm{SO}(3)$ contains rotation matrices in 3D space and can be defined as $\mathrm{SO}(3) = \{A \in M(3,\mathbb{R}) \mid A^T = A^{-1},\ \det(A) = 1\}$. Geometric visualizations of SO(2) and SO(3) are shown in Figure 1 and Figure 2 respectively.

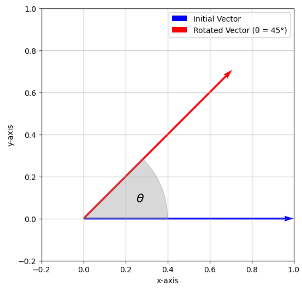

Figure. 1. Illustration of Vector Rotation with SO(2)

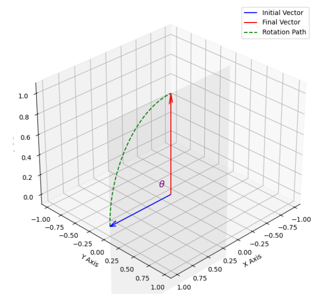

Figure. 2. Illustration of Vector Rotation with SO(3)

Research on matrix Lie groups and their applications has been extensively conducted by various researchers ([20,21,22,23,24,25,26,27,28,29,30]). Therefore, matrix Lie groups are highly relevant as powerful tools in various scientific fields, particularly in constructing interest rate models [30]. On the other hand, several researchers have conducted studies on short-term interest rates. Pintoux [31] investigated Brownian motion on the matrix Lie groups SO(2) and SO(3) but did not utilize it to determine interest rate models. Bildirici, Ucan, and Lousada [32] tested interest rate structures using Lie algebras and the SO(3) matrix Lie group, considering external factors such as economic crises, US military intervention in Iraq, and the COVID-19 pandemic during economic lockdowns. Park et al. [33] studied the non-linear and random behavior of interest rates represented by stochastic differential equations evolving in dynamic state spaces. Nevertheless, research related to the analysis of Brownian motion on SO(2) for determining short-term interest rate models.

Thus, the analysis of Brownian motion on the SO(2) matrix Lie group is considered capable of "reforming" conventional interest rates to make them more adaptive in financing schemes for the general public. In this research, we demonstrate a geometric approach to short-term interest rate processes on SO(2). Starting from the proof of the isomorphism $\mathbb{S}^1 \cong \mathrm{SO}(2)$, we examine Brownian motion on

SO(2) and its equivalences, determine stochastic differential equations, and confirm Lim and Privault's work about the short-term interest rate formula on SO(2). Subsequently, the derived formula is simulated using Python to clarify the obtained results.

2. Preliminaries

This section presents a discussion on some important materials which shall be applied in this research: Lie matrix groups, Brownian motion, and models of short-term interest rates.

2.1. *Matrix Lie Groups*

Definition 2.1. *[34] A subgroup G of* $\mathrm{GL}(n,\mathbb{C})$ *is called a* matrix Lie group *if for every sequence of matrices $\{A_m\} \in G$ converging to some matrix $A \in \mathrm{GL}(n,\mathbb{C})$, exactly one of the following holds $A \in G$ or A is non–invertible.*

Definition 2.2. *[30] A matrix Lie group G is said to be* compact *if it satisfies:*

(1) Whenever a sequence $\{A_n\} \in G$ converges to a matrix A, then $A \in G$.
(2) There exists a constant $C > 0$ such that for every $A = (A_{ij}) \in G$ and all $1 \le i, j \le n$, then $|A_{ij}| \le C$.

Subsequently we will discuss the matrix Lie group of orthogonal matrices, denoted by $\mathrm{SO}(n,\mathbb{R})$.

Proposition 2.3. *[30] Let*

$$\mathrm{SO}(n,\mathbb{R}) \;=\; \big\{\, A \in M(n,\mathbb{R}) \mid A^T = A^{-1},\ \det(A) = 1 \big\}.$$

Then $\mathrm{SO}(n,\mathbb{R})$ *is a compact matrix Lie group.*

Proof. First, $(\mathrm{SO}(n,\mathbb{R}),\times)$ is clearly a group. We show it is a closed subgroup of $\mathrm{GL}(n,\mathbb{C})$. Let $\{A_n\} \in \mathrm{SO}(n,\mathbb{R})$ be any sequence converging to $A \in \mathrm{GL}(n,\mathbb{C})$. Since each A_n satisfies $A_n^T A_n = I$ and $\det(A_n) = 1$, taking limits gives

$$\lim_{n\to\infty} A_n^T A_n = A^T A = I, \qquad \lim_{n\to\infty} \det(A_n) = \det(A) = 1.$$

Hence $A \in \mathrm{SO}(n,\mathbb{R})$. Thus $\mathrm{SO}(n,\mathbb{R})$ is closed in $\mathrm{GL}(n,\mathbb{C})$ and so is a Lie matrix group. To prove compactness, observe that for any $A = A_{ij}) \in \mathrm{SO}(n,\mathbb{R})$, orthogonality ($A^T A = I$) forces each entry to satisfy $|A_{ij}| \le 1$. Hence one may take $C = 1$ in Definition2.2, and conclude that $\mathrm{SO}(n,\mathbb{R})$ is compact. □

Example 2.4. For $n = 2$, the matrix Lie group SO(2) can be written as

$$\mathrm{SO}(2) \;=\; \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \;\Big|\; \theta \in \mathbb{R} \ (\mathrm{mod}\ 2\pi) \right\}. \tag{2.1}$$

Algebra of the Lie group $\mathrm{SO}(n)$ is given by

$$\mathfrak{so}(n) \;=\; \{\, A \in M(n,\mathbb{R}) \mid A^T = -A\}. \tag{2.2}$$

Example 2.5. The elements of $\mathfrak{so}(2)$ are exactly those 2×2 real matrices satisfying $A^T = -A$. In particular one finds

$$A = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad a \in \mathbb{R}.$$

2.2. *Brownian Motion*

Definition 2.6. *[18] A stochastic process $\{X(t)\}_{t\geq 0}$ is called a* Brownian motion *if it satisfies:*

(1) $X(0) = 0$.
(2) $\{X(t)\}$ has independent and stationary increments.
(3) For each $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

Definition 2.7. *[18] A stochastic process $\{X(t)\}_{t\geq 0}$ is called a* Brownian motion with drift *μ and variance parameter σ^2 if it satisfies:*

(1) $X(0) = 0$.
(2) $\{X(t)\}$ has independent and stationary increments.
(3) For each $t > 0$, $X(t)$ is normally distributed with mean μt and variance $\sigma^2 t$.

Theorem 2.8. *[18] A stochastic process $\{W_t\}_{t\geq 0}$ is a* standard Brownian motion *if and only if the process*

$$X(t) \;=\; \mu\, t \;+\; \sigma\, W_t \tag{2.3}$$

satisfies the state-equation for Brownian motion with drift μ and diffusion σ.

2.3. *Short-Term Interest Rate Models*

A short-term interest rate model is a mathematical framework that describes how short-term interest rates evolve over time, enabling predictions of future interest rate behavior [35].

Definition 2.9. *[36] The model for the short-term interest rate process on matrix Lie groups is given by the following equation:*

$$r_t = \beta + \gamma\, \mathrm{tr}(g_t), \tag{2.4}$$

where $(g_t)_{t\in\mathbb{R}_+}$ is Brownian motion on a matrix Lie group.

Other commonly used interest rate models include the Vasicek model and the Cox-Ingersoll-Ross (CIR) model. According to [37], the Vasicek model is given by:

$$dr_t = \kappa(\varphi - r_t)dt + \sigma dW_t, \tag{2.5}$$

while the CIR model is given by:

$$dr_t = \kappa(\varphi - r_t)dt + \sigma\sqrt{r_t}dW_t, \tag{2.6}$$

where κ represents the speed of mean reversion, φ is the long-term interest rate level, and σ denotes volatility. The parameter estimation formulas for κ, φ, and σ are as follows.

$$\kappa = \frac{n^2 - 2n + 1 + \sum_{i=1}^{n-1} r_{t+1} \sum_{i=1}^{n-1} \frac{1}{r_t} - \sum_{i=1}^{n-1} r_t \sum_{i=1}^{n-1} \frac{1}{r_t} - (n-1) \sum_{i=1}^{n-1} \frac{r_{t+1}}{r_t}}{(n^2 - 2n + 1 - \sum_{i=1}^{n-1} r_t \sum_{i=1}^{n-1} \frac{1}{r_t})\Delta t}, \tag{2.7}$$

$$\varphi = \frac{(n-1) \sum_{i=1}^{n-1} r_{t+1} - \sum_{i=1}^{n-1} \frac{r_{t+1}}{r_t} \sum_{i=1}^{n-1} r_t}{(n^2 - 2n + 1 + \sum_{i=1}^{n-1} r_{t+1} \sum_{i=1}^{n-1} \frac{1}{r_t} - \sum_{i=1}^{n-1} r_t \sum_{i=1}^{n-1} \frac{1}{r_t} - (n-1) \sum_{i=1}^{n-1} \frac{r_{t+1}}{r_t})}, \tag{2.8}$$

$$\sigma = \sqrt{\frac{1}{n-2} \sum_{i=1}^{n-1} \left(\frac{r_{t+1} - r_t}{\sqrt{r_t}} - \frac{\theta}{\sqrt{r_t}} + c\sqrt{r_t} \right)^2}. \tag{2.9}$$

3. Result and Discussion

We state here some new results of short rate model through the matrix Lie group SO(2). The short rate model which is given by SO(2) is obtained by Lim and Privault but for own interest, we shall give the complete proof. Indeed, to strengthen the results obtained we also demonstrate this models compared with ordinary least square (OLS), Vasicek, and CIR.

Proposition 3.1. *The Lie matrix group* SO(2) *defined in* (2.1) *is isomorphic to*

$$\mathbb{S}^1 = \{ x \in \mathbb{R}^2 \mid \|x\| = 1 \}.$$

Proof. Consider the mapping $f : \mathrm{SO}(2) \to \mathbb{S}^1$ defined by the (3.1)

$$f\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right) = (\cos\theta, \sin\theta). \tag{3.1}$$

First, we prove that f is a mapping. We will show that for all $A \in \mathrm{SO}(2)$, it holds that $f(A) \in \mathbb{S}^1$. Suppose $A \in \mathrm{SO}(2)$ with

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Based on Eq. (3.1), the value of

$$f(A) = f\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right) = (\cos\theta, \sin\theta) \in \mathbb{S}^1.$$

Next, we must prove the injectivity of the map, that is,

$$\forall A \in \mathrm{SO}(2), \exists!\ (\cos\theta, \sin\theta) \in \mathbb{S}^1 \Rightarrow f(A) = (\cos\theta, \sin\theta).$$

Suppose $f(A) = y_1$ and $f(A) = y_2$. We will prove that $y_1 = y_2$. By contradiction, assume that $y_1 \neq y_2$, then we have $(\cos\theta, \sin\theta) \neq (\cos\theta, \sin\theta)$. This implies $\cos\theta \neq \cos\theta$ and $\sin\theta \neq \sin\theta$, which is a contradiction. Therefore, we must have $y_1 = y_2$. Thus, f is a well-defined function. Second, we prove that f is a group homomorphism. Assume $A, B \in \mathrm{SO}(2)$ such that

$$A = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}.$$

Then,

$$\begin{aligned} f(AB) &= f\left(\begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}\begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}\right),\\ &= f\left(\begin{bmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{bmatrix}\right),\\ &= (\cos(\theta_1+\theta_2), \sin(\theta_1+\theta_2)),\\ &= f\left(\begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}\right) f\left(\begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}\right),\\ &= f(A)f(B). \end{aligned}$$

Since $f(AB) = f(A)f(B)$, f is a group homomorphism. Third, we prove that f is injective. Suppose $A, B \in \mathrm{SO}(2)$ with $f(A) = f(B)$. We must prove $A = B$.

$$f\left(\begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}\right) = f\left(\begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}\right) \Rightarrow (\cos\theta_1, \sin\theta_1) = (\cos\theta_2, \sin\theta_2).$$

Therefore, $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$, implying $A = B$.So f is injective. Fourth, we prove that f is surjective. Let $z = (\cos\theta, \sin\theta) \in S^1$. Choose $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \in \mathrm{SO}(2)$. Then, $f(A) = f\left(\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\right) = (\cos\theta, \sin\theta) = z$. Therefore, f is surjective. Since f is a group homomorphism, injective, and surjective, it follows that f is a group isomorphism. Thus, $\mathrm{SO}(2) \cong \mathbb{S}^1$. □

Next, we discuss Brownian motion on SO(2). Since SO(2) is isomorphic to the circle group $\mathbb{S}^1$, Brownian motion on SO(2) can be visualized as follows.

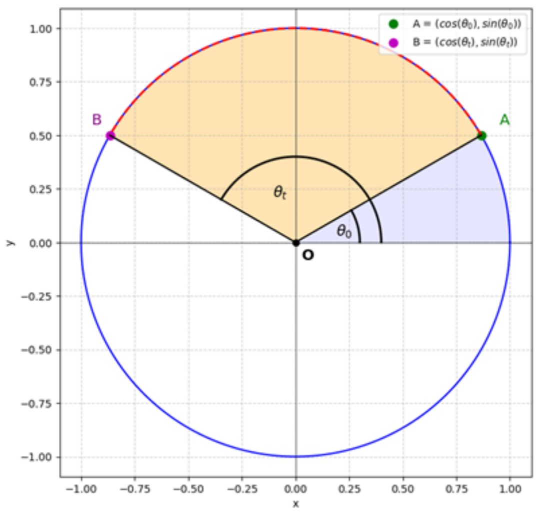

Figure. 3. Visualization of Brownian Motion on $\mathbb{S}^1$

Figure 3 illustrates the random movement from an initial rotation (θ_0) to a rotation at time t (θ_t) along the arc of $\mathbb{S}^1$. The starting point and the point at time

t along the $\mathbb{S}^1$ arc are represented as $(\cos\theta_0, \sin\theta_0)$ and $(\cos\theta_t, \sin\theta_t)$ respectively. This random motion can be interpreted as Brownian motion on $\mathbb{S}^1$, with details provided in the following Proposition 3.2.

Proposition 3.2. *Brownian motion on* $\mathbb{S}^1$ *is given by*

$$\theta_t \;=\; \theta_0 \;+\; \sigma\, W_t, \tag{3.2}$$

where θ_t *is the angle at time* t, θ_0 *is the initial angle,* σ *is the diffusion coefficient, and* W_t *is a standard Brownian motion.*

Proof. Referring to Figure 3, let θ_0 be the initial angle at $t = 0$ and θ_t the angle at time t. The arc on the circle from θ_0 to θ_t has central angle $\theta_t = \theta_0 + \sigma W_t$. Define $g_t := \theta_t - \theta_0$. Then from Eq. (3.2) we get:

$$\begin{aligned} g_t &= \sigma W_t, \\ g_t &= 0 \cdot t + \sigma W_t. \end{aligned}$$

Eq. (3.2) is exactly the state equation $X(t) = \mu t + \sigma W_t$ in Eq. (2.3) with drift $\mu = 0$. Hence $\{\theta_t\}$ is a Brownian motion on $\mathbb{S}^1$. □

Next, we discuss Brownian motion related to stochastic differential equations (SDEs) that will be used to analyze short-term interest rate processes. Let g_t be Brownian motion on SO(2) as described in Eq. (3.2). The stochastic differential equation for this Brownian motion can be stated in Proposition 3.3 below.

Proposition 3.3. *Brownian motion Eq.* (3.2) *with* $\sigma = 1$ *is equivalent to the Brownian motion of Eq.* (3.3) *below.*

$$dg_t = g_t A \circ dW_t, \tag{3.3}$$

with g_t *being Brownian motion on* SO(2), W_t *being standard Brownian motion, and* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. □

Proof. Since SO(2) is isomorphic to $\mathbb{S}^1$, we define a rotation mapping R by angle θ_t as follows.

$$R : \theta_t \mapsto g(\theta_t) = \begin{pmatrix} \cos\theta_t & -\sin\theta_t \\ \sin\theta_t & \cos\theta_t \end{pmatrix} \in \mathrm{SO}(2). \tag{3.4}$$

Differentiating $g(\theta_t)$ with respect to t gives:

$$\frac{dg(\theta_t)}{dt} = \dot{\theta}_t g'(\theta_t), \tag{3.5}$$

where $\dot{\theta}_t = d\theta_t/dt$. Consider $g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta \in \mathbb{R} \pmod{2\pi}$. Its derivative is:

$$\begin{aligned}\frac{dg(\theta)}{d\theta} &= \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}, \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ &= g(\theta)A.\end{aligned}$$

Therefore,

$$\begin{aligned}dg_t &= \frac{dg(\theta_t)}{dt}dt, \\ &= \dot{\theta}_t g'(\theta_t)dt, \\ &= g_t A\dot{\theta}_t dt, \\ &= g_t A \circ d\theta_t.\end{aligned}$$

From Eq. (4.2) $\theta_t = \theta_0 + W_t$, we have $d\theta_t = dW_t$. Substituting yields:

$$dg_t = g_t A \circ dW_t,$$

which completes the proof. □

Corollary 3.4. *[36] The stochastic differential equation in Eq.* (3.3) *can be transformed into Eq.* (3.6)*:*

$$dg_t = g_t \circ d\begin{pmatrix} 0 & -W_t \\ W_t & 0 \end{pmatrix}. \tag{3.6}$$

Proof. Differentiating Eq. (4.3) from Proposition 4.3 gives:

$$\begin{aligned}dg_t &= g_t A \circ dW_t, \\ &= g_t \circ (AdW_t), \\ &= g_t \circ d(AW_t), \\ &= g_t \circ d\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} W_t\right), \\ &= g_t \circ d\begin{pmatrix} 0 & -W_t \\ W_t & 0 \end{pmatrix}.\end{aligned}$$

□

Proposition 3.5. *[33] The Brownian motion in equation* (2.3) *on* SO(2) *with nonzero drift μ can be transformed into the following stochastic differential equation:*

$$dg_t \;=\; g_t\, B\, dt \;+\; g_t\, dW_t, \tag{3.7}$$

with

$$B = \begin{bmatrix} 0 & -\mu \\ \mu & 0 \end{bmatrix}, \quad dW = \begin{bmatrix} 0 & -\,dW_t \\ dW_t & 0 \end{bmatrix} \;\in\; \mathfrak{so}(2), \quad \mu \in \mathbb{R}.$$

Proof. Let $g_t \in \mathrm{SO}(2)$. Then

$$g_t = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Take $\sigma = 1$ and set $\theta = X(t)$. Hence the Brownian motion in (2.3) becomes

$$\theta = \mu t + W_t, \qquad \frac{d\theta}{dt} = \mu\, dt + dW_t.$$

Differentiate g_t with respect to θ:

$$\frac{dg_t}{d\theta} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}.$$

By the chain rule,

$$\frac{dg_t}{dt} = \frac{dg_t}{d\theta}\,\frac{d\theta}{dt} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \big(\mu\, dt + dW_t\big).$$

Distributing, we obtain:

$$\begin{aligned}
\frac{dg_t}{dt} &= \frac{dg_t}{d\theta}\,\frac{d\theta}{dt}, \\
&= \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \big(\mu\, dt + dW_t\big), \\
&= \begin{bmatrix} -\mu\sin\theta\, dt - \sin\theta\, dW_t & -\mu\cos\theta\, dt - \cos\theta\, dW_t \\ \mu\cos\theta\, dt + \cos\theta\, dW_t & -\mu\sin\theta\, dt - \sin\theta\, dW_t \end{bmatrix}, \\
&= \underbrace{\begin{bmatrix} -\sin\theta\, dW_t & -\cos\theta\, dW_t \\ \cos\theta\, dW_t & -\sin\theta\, dW_t \end{bmatrix}}_{g_t\, dW} + \underbrace{\begin{bmatrix} -\mu\sin\theta & -\mu\cos\theta \\ \mu\cos\theta & -\mu\sin\theta \end{bmatrix}}_{g_t\, B} dt, \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -dW_t \\ dW_t & 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} dt, \\
&= g_t\, dW + g_t\, B\, dt.
\end{aligned}$$

Since $B, dW \in \mathfrak{so}(2)$ and $\mu \in \mathbb{R}$, then Eq. (3.7) follows, proving the claim. □

Proposition 3.6. *The explicit solution to Eq.* (3.6) *is given by:*

$$g_t = \begin{pmatrix} \cos W_t & -\sin W_t \\ \sin W_t & \cos W_t \end{pmatrix}, \tag{3.8}$$

where W_t is standard Brownian motion.

Proof. Based on Eq. (3.6),

$$dg_t = g_t \circ d \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix},$$

$$g_t^{-1} \circ dg_t = g_t^{-1} \left(g_t \left(d \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix} \right) \right).$$

Since g_t is Brownian motion on SO(2) so $g_t \in \mathrm{SO}(2)$, therefore g_t has an inverse and g_t has entries that are greater than or equal to 0. □

Then,

$$\begin{aligned} \frac{dg_t}{g_t} &= d \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix}, \\ \int \frac{dg_t}{g_t} &= \int d \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix}, \\ \ln g_t &= \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix}, \\ g_t &= \exp \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix}. \end{aligned}$$

Compute the value of $g_t = \exp \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix}$ by setting $A = \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix} \in M(n, \mathbb{C})$. The exponential of matrix A can be calculated using the following steps:

(1) Find the eigenvalues of A with the characteristic polynomial:

$$\begin{aligned} C_A(\lambda) &= \det\big(\lambda I_2 - A\big), \\ &= \det\Big(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix} \Big), \\ &= \det\Big(\begin{bmatrix} \lambda & W_t \\ -W_t & \lambda \end{bmatrix} \Big), \\ &= \lambda \cdot \lambda \; - \; W_t \cdot (-W_t), \\ &= \lambda^2 + W_t^2. \end{aligned}$$

The $C_A(\lambda)$ obtained is $C_A(\lambda) = \lambda^2 + W_t^2$. The eigenvalues of matrix A can be obtained by finding all roots of the equation $C_A(\lambda) = 0$, which are $\lambda_1 = iW_t$ and $\lambda_2 = -iW_t$ with i being the imaginary number.

(2) Find the eigenvectors corresponding to $\lambda_1 = iW_t$ or $\lambda_2 = -iW_t$. For the case $\lambda_1 = iW_t$, it can be obtained by determining the solution to the homogeneous system $(iW_t I_2 - A)v = 0$, with $v_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The augmented matrix $[iW_t I_2 - A : 0]$ gives the solution to the homogeneous system:

$$v_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} i \\ 1 \end{bmatrix}, s \in \mathbb{R}.$$

Therefore, the eigenvector corresponding to eigenvalue $\lambda_1 = iW_t$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$. Hence,

$S_1 = \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$ is a basis for eigenspace E_1. Next for the case $\lambda_2 = -iW_t$, it can be obtained by determining the solution to the homogeneous system $(-iW_t I_2 - A)v = 0$, with $v_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If the augmented matrix is $[-iW_t I_2 - A : 0]$, then the solution to the homogeneous system is:

$$v_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} -i \\ 1 \end{bmatrix}, s \in \mathbb{R}.$$

Therefore the eigenvector corresponding to eigenvalue $\lambda_2 = -iW_t$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. Hence, $S_2 = \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$ is a basis for eigenspace E_2. The matrices obtained are $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} iW_t & 0 \\ 0 & -iW_t \end{bmatrix}$.

(3) The exponential of D are given by:

$$e^D = \text{diag}\{e^{iW_t}, e^{-iW_t}\} = \begin{bmatrix} e^{iW_t} & 0 \\ 0 & e^{-iW_t} \end{bmatrix}.$$

(4) The exponential of A can be calculated with the help of Euler's formula for complex numbers, namely $e^{iW_t} = \cos W_t + i \sin W_t$ and $e^{-iW_t} = \cos W_t - i \sin W_t$. The calculation of the exponential of A are given by:

$$\begin{aligned}
e^A &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \text{diag}\{e^{iW_t}, e^{-iW_t}\} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}, \\
&= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{iW_t} & 0 \\ 0 & e^{-iW_t} \end{bmatrix} \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}, \\
&= \frac{1}{2i} \begin{bmatrix} ie^{iW_t} & -ie^{-iW_t} \\ e^{iW_t} & e^{-iW_t} \end{bmatrix} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}, \\
&= \frac{1}{2i} \begin{bmatrix} ie^{iW_t} + ie^{-iW_t} & -e^{iW_t} + e^{-iW_t} \\ e^{iW_t} - e^{-iW_t} & ie^{iW_t} + ie^{-iW_t} \end{bmatrix}, \\
&= \frac{1}{2} \begin{bmatrix} e^{iW_t} + e^{-iW_t} & ie^{iW_t} - ie^{-iW_t} \\ -ie^{iW_t} + ie^{-iW_t} & e^{iW_t} + e^{-iW_t} \end{bmatrix}, \\
&= \frac{1}{2} \begin{bmatrix} 2\cos W_t & -2\sin W_t \\ 2\sin W_t & 2\cos W_t \end{bmatrix}, \\
&= \begin{bmatrix} \cos W_t & -\sin W_t \\ \sin W_t & \cos W_t \end{bmatrix}.
\end{aligned}$$

Therefore, $g_t = \exp \begin{bmatrix} 0 & -W_t \\ W_t & 0 \end{bmatrix} = \begin{bmatrix} \cos W_t & -\sin W_t \\ \sin W_t & \cos W_t \end{bmatrix}$.

Proposition 3.7. *[36] The short-term interest rate model for Brownian motion on* SO(2) *is given by Eq.* (3.9) *as follows.*

$$r_t = \beta + 2\gamma \cos W_t, \tag{3.9}$$

where the condition $\beta \geq 2|\gamma|$ *ensures* $r_t \geq 0$ *for all* $t \in \mathbb{R}_+$.

Proof. From Eq. (2.4) and the explicit solution in Eq. (3.8), we have:

$$\begin{aligned} r_t &= \beta + \gamma \cdot \mathrm{tr}(g_t), \\ &= \beta + \gamma \cdot \mathrm{tr}\left(\begin{pmatrix} \cos W_t & -\sin W_t \\ \sin W_t & \cos W_t \end{pmatrix}\right), \\ &= \beta + 2\gamma \cos W_t. \end{aligned}$$

□

The simulation in this study uses U.S. interest rate data. The data set consists of monthly interest rates from January 2023 to May 2025. Data was accessed on June 1, 2025, from `https://fred.stlouisfed.org/`.

Table 1. US Interest Rate Data

Date	US interest rate	Date	US interest rate
01/01/2023	1.7831	01/04/2024	1.9382
01/02/2023	1.4202	01/05/2024	2.1034
01/03/2023	2.0596	01/06/2024	1.9996
01/04/2023	1.4439	01/07/2024	2.0484
01/05/2023	1.5369	01/08/2024	1.6642
01/06/2023	1.0606	01/09/2024	1.5816
01/07/2023	1.4260	01/10/2024	1.4801
01/08/2023	1.6021	01/11/2024	1.9584
01/09/2023	1.6997	01/12/2024	1.8244
01/10/2023	2.0825	01/01/2025	2.0555
01/11/2023	2.0943	01/02/2025	2.0277
01/12/2023	1.6809	01/03/2025	1.8592
01/01/2024	1.6809	01/04/2025	1.6705
01/02/2024	1.6186	01/05/2025	1.6676
01/03/2024	1.9260		

In Table 1, it can be seen that there are data fluctuations every year. These interest rates will be simulated by several models, namely ordinary least square (OLS), SO(2), Vasicek, and CIR. The simulation results will be compared across each model.

The OLS calculation with Python produces the equation $y = 0.1x + 1.6187$. Meanwhile, for the SO(2) interest rate model, we use the assumptions $\beta = \bar{x} = 1.758$ and $\gamma = \frac{s}{2} = 0.1302$ with x being the data mean and s the data standard deviation. The SO(2)-based model, according to Eq. (3.9), yields $r_t = 1.758 + 0.2604 \cos W_t$.

Parameter estimation for the Vasicek and CIR models using Eq. (2.7), Eq. (2.8), and Eq. (2.9) gives $\kappa = 0.0149$, $\varphi = 0.0283$, $\sigma = 0.2268$. Substituting these parameters, the resulting Vasicek model is $dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t = 0.0149\big(0.0283 - r_t\big)\,dt + 0.2268\,dW_t$, and the CIR model obtained is $dr_t = 0.0149\big(0.0283 - r_t\big)\,dt + 0.2268\sqrt{r_t}\,dW_t$. Simulations are performed up to 5000 iterations and each iteration is evaluated using MAPE. The model selected is the one with the smallest MAPE. The simulation results of these four best models can be seen in Table 2 and Figure 4.

Table 2: Simulation Results of the Four Models

Date	Actual Data	SO(2) Model	Vasicek Model	CIR Model	OLS Model
01/01/2023	1.7831	1.7360	1.7831	1.7831	1.6187
01/02/2023	1.4202	1.6648	1.5257	1.6440	1.6286
01/03/2023	2.0596	2.0126	1.5527	1.9953	1.6386
01/04/2023	1.4439	1.5085	1.3411	1.6210	1.6486
01/05/2023	1.5369	1.7016	1.1006	1.5756	1.6586
01/06/2023	1.0606	1.7255	0.9813	1.5623	1.6686
01/07/2023	1.4260	1.7715	1.2115	1.6441	1.6785
01/08/2023	1.6021	1.8857	1.8663	1.7156	1.6885
01/09/2023	1.6997	1.9042	1.9941	1.7830	1.6985
01/10/2023	2.0825	1.9904	1.8689	2.0946	1.7085
01/11/2023	2.0943	2.0188	1.9585	1.9484	1.7184
01/12/2023	1.6809	2.0021	1.6536	1.8269	1.7284
01/01/2024	1.6809	2.0177	1.6187	2.1401	1.7384
01/02/2024	1.6168	1.6298	1.7038	1.7644	1.7484
01/03/2024	1.9260	1.7297	1.6100	1.8092	1.7584
01/04/2024	1.9382	2.0184	1.8116	1.8706	1.7683
01/05/2024	2.1034	1.8736	2.0274	1.7530	1.7783
01/06/2024	1.9996	1.9370	2.1378	2.0069	1.7883
01/07/2024	2.0484	1.9643	1.8857	1.4164	1.7983
01/08/2024	1.6642	1.5738	1.7302	1.5306	1.8082
01/09/2024	1.5816	1.7545	1.9004	1.8110	1.8182
01/10/2024	1.4801	1.9968	1.7543	1.9839	1.8282
01/11/2024	1.9584	1.9790	1.8678	1.9303	1.8382
01/12/2024	1.8244	1.9886	2.2332	1.6519	1.8482
01/01/2025	2.0555	2.0172	2.1608	2.0202	1.8581
01/02/2025	2.0277	1.8882	1.9228	1.8251	1.8681
01/03/2025	1.8592	1.9049	1.5098	2.3646	1.8781
01/04/2025	1.6705	2.0164	1.4547	2.0340	1.8881
01/05/2025	1.6676	1.8045	1.7315	1.6670	1.8981

Figure 4 demonstrates that the SO(2), Vasicek, and CIR models closely follow the fluctuations in the data, while the OLS model produces only a straight line with an upward trend. The MAPE (Mean Absolute Percentage Error) values are

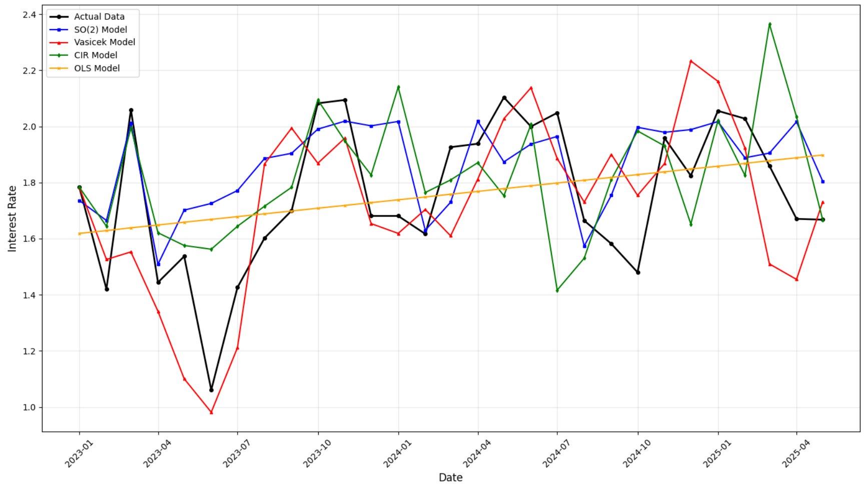

Figure. 4. Comparison of Four Interest Rate Models

11.58% for the SO(2)-based model, 10.63% for Vasicek, 11.95% for CIR, and 12.17% for OLS. The SO(2)-based model provides a better approximation of the actual data compared to the CIR and OLS models. Although the Vasicek model performs slightly better, the SO(2) model effectively captures the data fluctuations.

4. Conclusion

Brownian motion on the matrix Lie group SO(2) is given by $\theta_t = \theta_0 + \sigma W_t$, where θ_t is the angle at time t, θ_0 is the angle at time 0, σ is the diffusion coefficient, and W_t is standard Brownian motion. When $\sigma = 1$, it is equivalent to $dg_t = g_t A \circ dW_t$ with g_t being Brownian motion on SO(2) and $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and thus equivalent to $dg_t = g_t A \circ dW_t$. For $\sigma \neq 1$, $dg_t = g_t B dt + g_t dW$ with $B = \begin{bmatrix} 0 & -\mu \\ \mu & 0 \end{bmatrix}$, $dW = \begin{bmatrix} 0 & -dW_t \\ dW_t & 0 \end{bmatrix} \in \mathfrak{so}(2)$ and $\mu \in \mathbb{R}$.

Finally, it is equivalent to $g_t = \begin{bmatrix} \cos W_t & -\sin W_t \\ \sin W_t & \cos W_t \end{bmatrix}$. The matrix Lie group approach implements the Brownian motion function on SO(2) to model the short-term interest rate process. The explicit formula for the short-term interest rate on SO(2) is $r_t = \beta + 2\gamma \cos W_t$ where β, γ are constants and W_t is standard Brownian motion. Simulation of the short-term interest rate model based on the matrix Lie group SO(2) using Python shows model accuracy with a MAPE of 11.58% and follows the data fluctuations.

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Bibliography

[1] H. A. Alafif, Interest Rate and Some of Its Applications, *Journal of Applied Mathematics and Physics*, vol. **11**, no. 06, pp. 1557 – 1569, 2023, doi: 10.4236/jamp.2023.116102.

[2] A. A. M. Salem and A. J. A. Albourawi, Predictive Models for Interest Rate Forecasting Using Machine Learning: A Comparative Analysis and Practical Application , *Brilliance: Research of Artificial Intelligence*, vol. **4**, no. 2, pp. 764 – 770, Dec. 2024, doi: 10.47709/brilliance.v4i2.4983.

[3] S. Mala, F. Hatim, and D. Hasim, The Effect of Exchange Rates, Interest Rates, Inflation, Net Exports on Economic Growth in Indonesia , *Global Economic, Social, and Development Review*, vol. **28**, no. 2, pp. 48 – 62, Feb. 2025, doi: 10.24123/gesdr.v28i2.7031.

[4] Josua Panatap Soehaditama, Sustainability in Bank: Deposits, Investment and Interest Rate , *Formosa Journal of Sustainable Research*, vol. **2**, no. 5, pp. 1069 – 1078, May 2023, doi: 10.55927/fjsr.v2i5.3912.

[5] N. Liu, C. Liu, Y. Xia, Y. Ren, and J. Liang, Examining the Coordination Between Green Finance and Green Economy Aiming for Sustainable Development: A Case Study of China , *Sustainability*, vol. **12**, no. 9, p. 3717, May 2020, doi: 10.3390/su12093717.

[6] L. Xing, B. H. Chang, and S. H. Aldawsari, Green Finance Mechanisms for Sustainable Development: Evidence from Panel Data , *Sustainability*, vol. **16**, no. 22, p. 9762, Nov. 2024, doi: 10.3390/su16229762.

[7] M. S. Alam, H. Dinçer, K. M. Kisswani, M. A. I. Khan, S. Yüksel, and M. Alsharif, Analysis of green energy-oriented sustainable development goals for emerging economies , *Journal of Open Innovation: Technology, Market, and Complexity*, vol. **10**, no. 3, p. 100368, Sep. 2024, doi: 10.1016/j.joitmc.2024.100368.

[8] A. Ali, F. Faisal, A. Zhakanova Isiksal, and I. S. A. AL Maktoumi, Do green finance and health expenditures lessen the ecological footprint to ensure sustainable development?, *Innovation and Green Development*, vol. **4**, no. 2, p. 100223, Apr. 2025, doi: 10.1016/j.igd.2025.100223.

[9] S. Bhatnagar, D. Sharma, and V. V. Singh, Green finance and investment in India: Unveiling enablers and barriers for a sustainable future , *J Clean Prod*, vol. **493**, p. 144908, Feb. 2025, doi: 10.1016/j.jclepro.2025.144908.

[10] S. Zhang and A. (Waikong) Cheung, Are green finance and inclusive finance complements or substitutes for MSMEs? – Evidence from China's green finance reform and innovation pilot zone , *Energy Econ*, vol. **141**, p. 108125, Jan. 2025, doi: 10.1016/j.eneco.2024.108125.

[11] V. Le Coz and J.-P. Bouchaud, Revisiting elastic string models of forward interest rates , *Quant Finance*, vol. **24**, no. 11, pp. 1561 – 1578, Nov. 2024, doi: 10.1080/14697688.2024.2401450.

[12] R. M. Verschuren, Stochastic interest rate modelling using a single or multiple curves: an empirical performance analysis of the Lévy forward price model , *Quant Finance*, vol. **20**, no. 7, pp. 1123 – 1148, Jul. 2020, doi: 10.1080/14697688.2020.1722318.

[13] X. Xu, J. Li, and D. Yin, How and Why Does Green Bond Have Lower Issuance Interest Rate? Evidence from China , *Emerging Markets Finance and Trade*, vol. **60**, no. 1, pp. 138 – 148, Jan. 2024, doi: 10.1080/1540496X.2023.2207702.

[14] B. Akhtari and H. Li, The Cox-Ingersoll-Ross process under volatility uncertainty , *J Math Anal Appl*, vol. **531**, no. 1, p. 127867, Mar. 2024, doi: 10.1016/j.jmaa.2023.127867.

[15] Y. Karadede, A novel stochastic ProFiVaS model based on decomposition of stochastic Vasicek differential equation for modeling and simulating of financial indicators , *Expert Syst Appl*, vol. **258**, p. 125073, Dec. 2024, doi: 10.1016/j.eswa.2024.125073.

[16] P. Tassopoulos and Y. Protonotarios, Brownian Motion the Stochastic Behavior of Stocks , *Journal of Mathematical Finance*, vol. **12**, no. 01, pp. 138 – 149, 2022, doi: 10.4236/jmf.2022.121009.

[17] D. Paksi, N. Annuš, I. Štempeľová, and D. Dancsa, Random number-based Brownian motion and practical examples of its implementing in high school computer science lessons , *International Journal of Advanced Natural Sciences and Engineering Researches*, vol. **7**, no. 4, pp. 463 – 467, May 2023, doi: 10.59287/ijanser.792.

[18] S. M. Ross, *Introduction to probability models, tenth edition*, 2010.

[19] T. Pierce, R. Rajkumar, A. Stine, D. Weisbart, and A. M. Yassine, Brownian Motion in a Vector Space over a Local Field is a Scaling Limit , May 2024.

[20] E. Zhang and L. Noakes, Riemannian cubics in quadratic matrix Lie groups , *Appl Math Comput*, vol. **375**, p. 125082, Jun. 2020, doi: 10.1016/j.amc.2020.125082.

[21] K. S. Phogat and D. E. Chang, Invariant extended Kalman filter on matrix Lie groups , *Automatica*, vol. **114**, p. 108812, Apr. 2020, doi: 10.1016/j.automatica.2020.108812.

[22] M. G. Cowling, Decay estimates for matrix coefficients of unitary representations of semisimple Lie groups , *J Funct Anal*, vol. **285**, no. 8, p. 110061, Oct. 2023, doi: 10.1016/j.jfa.2023.110061.

[23] S. Labsir, A. Renaux, J. Vilà-Valls, and E. Chaumette, Barankin, McAulay–Seidman and Cramér–Rao bounds on matrix Lie groups , *Automatica*, vol. **156**, p. 111199, Oct. 2023, doi: 10.1016/j.automatica.2023.111199.

[24] E. Kurniadi and A. Nurlaelasari, *Grup Lie Matriks dan Aljabar Lie-nya*, 1st ed., vol. I. Sukoharjo: CV Tahta Media, 2023.

[25] E. Kurniadi, N. Gusriani, and B. Subartini, *Analisis Harmonik Grup Lie Semi-Direct Produk Aff(2)=R2+GL2*, 1st ed., vol. I. Sukoharjo: CV Tahta Media, 2023.

[26] E. Kurniadi and K. Parmikanti, The Lie Algebra su(3) Representation with Respect to Its Basis , *Jurnal Matematika UNAND*, vol. **13**, no. 3, pp. 163 – 169, Jul. 2024, doi: 10.25077/jmua.13.3.163-169.2024.

[27] E. Kurniadi, H. Henti, and E. Carnia, The Levi Decomposition of The Lie Algebra M_2 (R)⋊gl_2 (R) , *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. **18**, no. 2, pp. 0717 – 0724, May 2024, doi:

10.30598/barekengvol18iss2pp0717-0724.

[28] T. Li and J. Wang, Variational unscented Kalman filter on matrix Lie groups, *Automatica*, vol. **172**, p. 111995, Feb. 2025, doi: 10.1016/j.automatica.2024.111995.

[29] S. Shanbhag and D. E. Chang, Globally exponentially convergent observer for systems evolving on matrix Lie groups , *Math Comput Simul*, vol. **232**, pp. 475 – 482, Jun. 2025, doi: 10.1016/j.matcom.2025.01.013.

[30] E. Kurniadi and A. Nurlaelasari, Grup Lie matriks dan aljabar Lie-nya , 2023.

[31] C. Pintoux, *Calculs stochastique et de Malliavin appliqués aux modèles de taux d'intérêt engendrant des formules fermées*, 2011.

[32] M. Bildirici, Y. Ucan, and S. Lousada, interest rate based on the Lie group SO(3) in the evidence of chaos , *Mathematics*, vol. **10**, no. 21, p. 3998, Oct. 2022, doi: 10.3390/math10213998.

[33] F. C. Park, C. M. Chun, C. W. Han, and N. Webber, Interest rate models on lie groups , *Quant Finance*, vol. **11**, no. 4, pp. 559 – 572, Apr. 2011, doi: 10.1080/14697680903468963.

[34] B. C. Hall, *Lie groups, Lie algebras, and representations*, vol. 222. Cham: Springer International Publishing, 2015. doi: 10.1007/978-3-319-13467-3.

[35] M. A. Pureheart-Irikefe, B. O. Anthony, and J. O. Oniore, Analysis of Short-Term Interest Rate Models with Stochastic Volatility , *International Journal of Systems Science and Applied Mathematics 2025*, vol. **10**, no. 2, pp. 12 – 26, 2025, doi: 10.11648/j.ijssam.20251002.11.

[36] N. Lim and N. Privault, Analytic bond pricing for short rate dynamics evolving on matrix Lie groups , *Quant Finance*, vol. **16**, no. 1, pp. 119 – 129, Jan. 2016, doi: 10.1080/14697688.2014.990497.

[37] S. Artika, I. G. P. Purnaba, and D. C. Lesmana, Penentuan Premi Asuransi Jiwa Berjangka Menggunakan Model Vasicek Dan Model Cox-Ingersoll-Ross (CIR), *Journal of Mathematics and Its Applications*, vol. **17**, no. 2, pp. 129 – 139, Dec. 2018, doi: 10.29244/jmap.17.2.129-139.