

## SOFT GRAPHS OF THE BARBELL STAR GRAPH

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**Abstract.** Let  $G = (V(G), E(G))$  be a simple graph and  $A$  be a nonempty set of parameters. Let  $R \subseteq A \times V(G)$  be an arbitrary relation from  $A$  to  $V(G)$ . A mapping  $F : A \rightarrow P(V(G))$  can be defined as  $F(x) = \{y \in V \mid xRy\}$  and a mapping  $K : A \rightarrow P(E(G))$  can be defined as  $K(x) = \{uv \in E \mid \{u, v\} \subseteq F(x)\}$ . A pair  $(F, A)$  and  $(K, A)$  are soft sets on  $V(G)$  and  $E(G)$ , respectively, then  $(F(a), K(a))$  is a subgraph of  $G$ . The 4-tuple  $SG = (G, F, K, A)$  is called a soft graph of  $G$ . In this paper, we will construct a soft graph for the barbell star graph  $BS_{m,n}$ , a graph obtained from the amalgamation of a path graph  $P_m$  with  $m$  vertices and two star graphs  $S_n$  with  $n + 1$  vertices.

*Keywords:* Soft Set, Soft Graph, Barbell Star Graph

### 1. Introduction

In 1965, Zadeh [1] introduced the concept of fuzzy sets that could be used to handle problems involving uncertainty, vagueness, or subjectivity in various fields such as artificial intelligence, control systems, and decision-making. Fuzzy sets discuss the membership of an element represented by a value between 0 and 1. However, the theory faces practical limitations, especially when it comes to combining membership functions for specific cases. To overcome this, Molodstov [2] introduced soft set theory in 1999 as a mathematical tool designed to address uncertainty and handle multiple parameters without requiring the exact specification of each parameter.

Furthermore, Maji et al. [3] described some fundamental algebraic operations on soft sets and provided an analytical framework for soft set theory. Ali et al. [4] proposed several operations on soft sets and formulated the notion of their complement, showing its compliance with De Morgan's laws. Later, Maji et al. [5] explored the application of soft sets in decision-making problems. In their subsequent work,

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Maji et al. [6] combined soft sets with other mathematical structures, presenting a hybrid model known as fuzzy soft sets, a fuzzy extension of soft sets. They also investigated several results related to this model. The concept of soft sets is also modified and extended into the graph domain to form soft graph theory.

The soft graph theory was first developed by K. Thumbakara and Bobin George in 2014 [7]. This research conducted a study on the definition of soft graph homomorphism, complete soft graph, soft tree, and its properties. Then in 2015, Akram modified the definition of a soft graph and explained some soft graph operations [8]. In 2019, Abbood provided a soft graph characterization for null graphs, complete graphs, cycle graphs, bipartite graphs, and star graphs [9]. Thenge then provides the definitions of connected and degree for soft graphs [10]. In 2022, George provided definitions for the Hamilton soft graph and examined some of its properties [11]. Significant research has been conducted to establish the soft graphs corresponding to certain special graphs. In 2021 and 2022, Palani constructed soft graphs for path, cycle, complete bipartite, and combinations of cycle graphs with complete graphs [12,13]. Once the soft graph representation of a given graph is established, it becomes possible to extend the analysis by incorporating additional graph-theoretic notions, such as the domination number or the metric dimension, within the framework of the soft graph. In this research, we construct a soft graph for the barbell star graph  $BS_{m,n}$  with several different soft-set definitions.

## 2. Some Concepts

For the basic concepts of graph theory, we refer to [14] and [15]. A **graph** is defined as the set of ordered pairs  $(V, E)$  denoted  $G = (V(G), E(G))$  where  $V(G)$  is a nonempty set of vertices and  $E(G)$  is the set of possibly empty edges. The number of vertices in the graph  $G$  is called **orde** which is denoted by  $|V(G)|$ . The number of edges in the graph  $G$  is called **the size** which is denoted by  $|E(G)|$ . Two vertices  $u$  and  $v$  in a graph  $G$  are **adjacent** if  $u$  and  $v$  are connected by an edge. A **simple graph** is a graph without loops or multiple edges. A **loop** is an edge that connects a vertex to itself, while **multiple edges** are two or more edges that connect the same two vertices. A graph  $H$  is said to be a **subgraph** of a graph  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . A subgraph can be denoted by  $H \subseteq G$ . A **path** from vertex  $v_0$  to the vertex  $v_n$  in the graph  $G$  is a finite sequence of vertices and edges in  $G$  such that  $v_{i-1}v_i \in E(G)$  for  $i = 1, 2, 3, \dots, n$ . The vertex  $v_0$  is called the initial vertex, and the vertex  $v_n$  is called the end vertex. The **length** of a path is the number of edges involved in the path. The length of the shortest path from vertex  $u$  to  $v$  is called the **distance** between two vertices  $u$  and  $v$  denoted by  $d(u, v)$ . A graph  $G$  is said to be **connected** if there is a path connecting every vertex to every other vertex. The **eccentricity** of a vertex  $v$  in  $G$  is defined as the maximum distance of one vertex from another vertex in  $G$  denoted by  $e(v)$ . The vertex  $v$  is said to be an **eccentric vertex** of  $u$  if the distance from  $u$  to  $v$  is equal to the eccentricity of  $u$  or  $d(u, v) = e(u)$ . The **radius** of  $G$  is the minimum eccentricity of all vertices in  $G^*$  denoted by  $rad(G)$ . The **diameter** of  $G^*$  is the maximum eccentricity of all vertices in  $G$  denoted by  $diam(G)$ . A **null graph** is a graph with no edges. A **star graph**

$S_n$  is a graph with  $n + 1$  vertices where there is a center vertex of degree  $n$  and  $n$  other vertices of degree one called leaves. A **path graph**  $P_n$  has two vertices of degree one and the other  $n - 2$  vertices of degree two. For  $t \in N$  and  $i \in 1, 2, \dots, t$ . Let  $G_i$  be a simple connected graph, and  $|V(G_i)| = k_i \geq 2$ , for some  $k_i \in N$ . For  $t \geq 2$ , let  $G_1, G_2, \dots, G_t$  be a finite collection of graphs, and each  $G_i, i \in 1, 2, \dots, t$  has a fixed vertex  $v_{0i}$  called a terminal. The **amalgamation**  $Amal(G_i, v_{0i})$  is a graph formed by taking all  $G_i$ 's and identifying their terminals. The definition of amalgamation refers to [16].

**Definition 2.1.** [2] Let  $U$  be a finite nonempty set of objects called the universe,  $P(U)$  be a power set over  $U$ , and let  $E$  be a nonempty set called parameters. An ordered pair  $(F, E)$  is said to be a soft set over  $U$ , where  $F$  is a mapping from  $E$  into the set of all subsets of the set  $U$ , that is,  $F : E \rightarrow P(U)$ .

**Definition 2.2.** [8] Let  $G = (V, E)$  be a simple graph and  $A$  be any nonempty set of parameters. Let the subset  $R$  of  $A \times V$  be an arbitrary relation from  $A$  to  $V$ . A mapping  $F : A \rightarrow P(V(G))$  can be defined as  $F(x) = \{y \in V(G) \mid xRy\}$  and a mapping  $K : A \rightarrow P(E(G))$  can be defined as  $K(x) = \{uv \in E(G) \mid \{u, v\} \subseteq F(x)\}$ . The pair  $(F, A)$  is a soft set over  $V$  and the pair  $(K, A)$  is a soft set over  $E$ . A 4-tuple  $SG = (G, F, K, A)$  is called a soft graph of  $G$  if it satisfies the following conditions:

1.  $G = (V(G), E(G))$  is simple graph,
2.  $A$  is nonempty set of parameters,
3.  $(F, A)$  is soft set over  $V$ ,
4.  $(K, A)$  is soft set over  $E$ ,
5.  $(F(x), K(x))$  is subgraph of  $G$  for all  $x \in A$ .

The subgraph  $(F(x), K(x))$  is denoted by  $H(x)$ . A soft graph can also be represented by  $SG = \langle F, K, A \rangle = \{H(x) \mid x \in A\}$ .

### 3. Result and Discussion

Suppose that two-star graphs and a path graph,  $G_1 = S_n, G_2 = S_n$ , and  $G_3 = P_m$ . The star graph  $S_n$ , has one center vertex  $v_{0,j}$  and other vertices  $v_{i,j}$  of degree one called leaves. The path graph has two terminal vertices,  $a_1$  and  $a_m$  (vertices of degree one), and the other vertices are of degree two. Furthermore, an amalgamation graph is constructed from two star graphs and a path graph by linking a leaf from each star graph to the initial and final vertices of the path graph. We designated vertex  $v_{1,1}$  in the first star graph, vertices  $a_1$  and  $a_m$  in the path graph, and vertex  $v_{1,2}$  in the second star graph as terminal vertices. Then attached vertices  $v_{1,1}$  and  $a_1$  to a fixed vertex  $a_1(a_1 = v_{1,1})$  and also attached vertices  $a_m$  and  $v_{1,2}$  to a fixed vertex  $a_m(a_m = v_{1,2})$ . Thus, an amalgamation graph of two star graphs and a path graph results in a barbell star graph, and is denoted by  $BS_{m,n}$  for  $m, n \in N$  with  $m \geq 1, n \geq 2$ .

The set of vertices and edges of graph  $BS_{m,n}$  are defined as follows.

$$\begin{aligned}
 V(BS_{m,n}) &= \{ v_{0,j}, v_{i,j}, a_l \mid i \in \{2, 3, \dots, n\}, j \in \{1, 2\}, l \in \{1, 2, \dots, m\} \}, \\
 E(BS_{m,n}) &= \{ v_{0,j}v_{i,j} \mid i \in \{2, 3, \dots, n\}, j \in \{1, 2\} \} \\
 &\cup \{ v_{0,j}a_l \mid j \in \{1, 2\}, l \in \{1, \dots, m\} \} \\
 &\cup \{ a_l a_{l+1} \mid l \in \{1, 2, \dots, m\} \},
 \end{aligned}$$

with  $|V(BS_{m,n})| = 2(n + 1) + m - 2,$   
 $|E(BS_{m,n})| = 2(n) + m - 1$  (3.1)

The barbell star graph  $BS_{m,n}$  is shown in Figure 1.

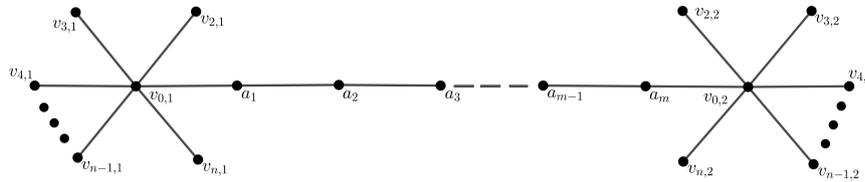


Figure. 1. Barbell Star Graph  $BS_{m,n}$

The eccentricity for all vertices of  $BS_{m,n}$  graph is shown in Table 2.

Table 1. Eccentricity of Vertices of  $BS_{m,n}$

$m$ is Odd Number		$m$ is Even Number	
$x$	$e(x)$	$x$	$e(x)$
$v_{i,1}$	$m + 3$	$v_{i,1}$	$m + 3$
$v_{0,1}$	$m + 2$	$v_{0,1}$	$m + 2$
$a_1$	$m + 1$	$a_1$	$m + 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{\frac{m+1}{2}}$	$\lceil \frac{m+3}{2} \rceil$	$a_{\frac{m}{2}}$	$\lceil \frac{m+3}{2} \rceil$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_m$	$m + 1$	$a_m$	$m + 1$
$v_{0,2}$	$m + 2$	$v_{0,2}$	$m + 2$
$v_{i,2}$	$m + 3$	$v_{i,2}$	$m + 3$

**Theorem 3.1.** Let  $G = (V(G), E(G))$  be a barbell star graph  $BS_{m,n}$  graph, and let  $A$  be a nonempty set of parameters where  $A = V(G)$ . If  $(F, A)$  is a soft set over  $V(G)$  defined by a mapping  $F : A \rightarrow P(V(G))$  such that  $F(x) = \{y \in V \mid d(x, y) = k\}$  for all  $x \in A$ , and  $(K, A)$  is a soft set over  $E(G)$  defined by a mapping  $K : A \rightarrow P(E(G))$  such that  $K(x) = \{uv \in E \mid \{u, v\} \subseteq F(x)\}$  for all  $x \in A$ , then  $SG =$

$(G, F, K, A)$  is a soft graph represented by  $\{H(x) \mid x \in A\}$  where  $H(x)$  is a null graph for  $0 \leq k \leq \text{diam}(G)$ . Furthermore, no such soft graph of  $G$  exist when  $k > \text{diam}(G)$ .

**Proof.** Let  $G = (V(G), E(G))$  be a  $BS_{m,n}$  graph and let  $A$  be a set of all vertices in  $G$ , so  $A = V(G)$ . Define a function  $F(x) = \{y \in V \mid d(x, y) = k\}$ , and  $K(x) = \{uv \in E \mid \{u, v\} \subseteq F(x)\}$  for every  $x \in A$ .

Since  $0 \leq k \leq \text{diam}(G)$  and  $F(x) = \{y \in V \mid d(x, y) = k\}$  for all  $x \in A$ , then there exist  $y \in V(G^*)$  such that  $d(x, y) = k$ . Consequently,  $F(x) \neq \emptyset$  and contains vertices with no adjacent vertices. Thus  $K(x) = \emptyset$ . Therefore, the subgraph  $H(x) = ((F(x), K(x)))$  is a null graph.

Conversely, for  $k > \text{diam}(G)$ , it is clear that there does not exist  $y \in V(G)$  such that  $d(x, y) > \text{diam}(G)$  then  $F(x) = \emptyset$ , consequently  $K(x) = \emptyset$ . Therefore soft graph of  $BS_{m,n}$  does not exist. □

**Theorem 3.2.** Let  $G = (V(G), E(G))$  be a barbel star graph  $BS_{m,n}$ , and let  $A$  be a set of parameters where  $A = V(G)$ . If  $(F, A)$  is a soft set over  $V(G)$  defined by a mapping  $F : A \rightarrow P(V(G))$  such that  $F(x) = \{y \in V \mid d(x, y) \leq 1\}$  for all  $x \in A$ , and  $(K, A)$  is a soft set over  $E(G)$  defined by a mapping  $K : A \rightarrow P(E(G))$  such that  $K(x) = \{uv \in E \mid \{u, v\} \subseteq F(x)\}$  for all  $x \in A$ , then  $SG = (G, F, K, A)$  is a soft graph represented by  $\{H(x) \mid x \in A\}$ , where:

$$H(x) = \begin{cases} P_2, & \text{for } x = v_{i,j}; i = 2, 3, \dots, n, j = 1, 2, \\ P_3, & \text{for } x = a_l; l = 1, 2, \dots, m, \\ S_n, & \text{for } x = v_{0,1}; l = 1 \text{ or } x = v_{0,2}; l = m. \end{cases}$$

with

- a)  $P_2$  is a path graph with two vertices,
- b)  $P_3$  is a path graph with three vertices,
- c)  $S_n$  is a star graph with  $n + 1$  vertices.

**Proof.** Let  $G = (V(G), E(G))$  be a  $BS_{m,n}$  graph and  $A$  be the set of all vertex in  $G$  that is  $A = V(G)$ . Define a function  $F(x) = \{y \in V \mid d(x, y) \leq 1\}$  and  $K(x) = \{uv \in E \mid \{u, v\} \subseteq F(x)\}$  for every  $x \in A$ .

Consider the following cases.

**Case 1.** For  $x = v_{i,j}; i = 2, 3, \dots, n, j = 1, 2$ .

Since  $F(x) = \{y \in V \mid d(x, y) \leq 1\}$  and  $x = v_{i,j}; i = 2, 3, \dots, n, j = 1, 2$ , then  $F(x)$  contains two adjacent vertices  $\{v_{i,j}, v_{0,j}\}$  and  $K(x)$  is the set an edge connected the two vertices in  $F(x)$ . Thus,  $H(x) = (F(x), K(x))$  is isomorphic to  $P_2$ .

**Case 2.** For  $x = a_l; l = 1, 2, \dots, m$ .

Since  $F(x) = \{y \in V \mid d(x, y) \leq 1\}$  and  $x = a_l; l = 1, 2, \dots, m$ , then  $F(x) = \{a_{l-1}, a_l, a_{l+1} \mid l = 1, 2, \dots, m\}$ , contains three consecutive vertices in  $P_m$ , and

$K(x) = \{a_{l-1}a_l, a_la_{l+1} \mid l \in \{1, 2, \dots, m\}\}$  contains edges in  $P_m$  that connect the vertices in  $F(x)$ . Thus,  $H(x) = (F(x), K(x))$  is path graph with length 3,  $P_3$ .

**Case 3.** For  $x = v_{0,1}$ ;  $l = 1$  or  $x = v_{0,2}$ ;  $l = m$ .

Since  $F(x) = \{y \in V \mid d(x, y) \leq 1\}$  and  $x = v_{0,1}$ ;  $l = 1$  or  $x = v_{0,2}$ ;  $l = m$  then  $F(x) = \{a_1, v_{0,1}, v_{i,1} \mid i = 1, 2, \dots, n\}$  for  $x = v_{0,1}$  or  $F(x) = \{a_m, v_{0,2}, v_{i,2} \mid i = 1, 2, \dots, n\}$  for  $x = v_{0,1}$ , contains all vertices in  $S_n$  and  $K(x)$  contains all edges in  $S_n$ . Thus,  $H(x) = (F(x), K(x))$  is isomorphic to  $S_n$ .  $\square$

**Theorem 3.3.** Let  $G = (V(G), E(G))$  be a  $BS_{m,n}$  graph.  $A$  is an arbitrary non-empty set of parameters where  $A \subseteq V(G^*)$ . If  $(F, A)$  is a soft set over  $V(G^*)$  defined by a mapping  $F : A \rightarrow P(V(G))$  such that  $F(x) = \{y \in V \mid d(x, y) \leq k\}$  with  $k \geq 2$  for every  $x \in A$ , and  $(K, A)$  is a soft set over  $E(G)$  defined by a mapping  $K : A \rightarrow P(E(G))$  such that  $K(x) = \{uv \in E \mid \{u, v\} \subseteq F(x)\}$  for all  $x \in A$ , then  $SG = (G, F, K, A)$  is a soft graph represented by  $\{H(x) \mid x \in A\}$  where:

$$H(x) = \begin{cases} BS_{m,n}, & \text{for } k \geq e(x), x \in A, \\ P_{2k+1}, & \text{for } k \leq m + 2 - e(x), x \in A, \\ Amal(S_n, P_{m+k+2-e(x)}, a_l), & \text{otherwise.} \end{cases}$$

**Proof.** Let  $G = (V(G), E(G))$  be a simple graph and  $A$  be a set of all vertices in  $G$  that is  $A = V(G)$ . Define a function  $F(x) = \{y \in V \mid d(x, y) \leq k\}$  with  $k \geq 2$  and  $K(x) = \{uv \in E \mid \{u, v\} \subseteq F(x)\}$ .

Consider the following cases.

**Case 1.** For  $k \geq e(x)$ ;  $x \in A$ .

Since  $F(x) = \{y \in V \mid d(x, y) \leq k\}$  and  $k \geq e(x)$  for  $k \geq 2$ , then  $F(x) = \{v_{0,j}, v_{i,j}, a_l \mid j = 1, 2; i = 2, 3, \dots, n; l = 1, 2, \dots, m\}$  contains all vertices in  $BS_{m,n}$  and  $K(x)$  contains all edges of  $BS_{m,n}$ . Thus,  $H(x) = (F(x), K(x))$  is isomorphic to  $BS_{m,n}$ .

**Case 2.** For  $k \leq m + 2 - e(x)$ ;  $x \in A$ .

Since  $F(x) = \{y \in V \mid d(x, y) \leq k\}$  and  $k \leq m + 2 - e(x)$ , then the vertices having eccentricity  $m, m - 1, m - 2, \dots, \lceil \frac{m+3}{2} \rceil$  satisfy the given conditions. They are precisely the vertices of the path graph  $P_m$  excluding its final vertex (see Table 1), so  $F(x) = \{a_{l-k}, \dots, a_{l-1}, a_l, a_{l+1}, \dots, a_{l+k}\}$  contains  $2k+1$  consecutive vertices of  $P_m$  and  $K(x)$  contains the edges of  $P_m$  that connect the vertices in  $F(x)$ . Thus,  $H(x) = (F(x), K(x))$  is isomorphic to  $P_{2k+1}$ .

**Case 3.** For  $k < e(x)$ ;  $x \in A$  or  $k > m + 2 - e(x)$ ;  $x \in A$ .

**Sub-case 3.1** For  $k < e(x)$ ;  $x \in A$

Since  $k < e(x)$  and  $F(x) = \{y \in V \mid d(x, y) \leq k\}$  for  $k \geq 2$ , then

$F(x) = \{v_{0,1}, v_{i,1}, a_l \mid i = 2, 3, \dots, n; l = 1, 2, \dots, m\}$  or

$F(x) = \{v_{0,2}, v_{i,2}, a_l \mid i = 2, 3, \dots, n; l = 1, 2, \dots, m\}$  contains all vertices from a star graph  $S_n$  and all the vertices of the path graph  $P_m$  and  $K(x)$  contains all edges in  $S_n$  and  $P_m$ . Thus,  $H(x) = (F(x), K(x))$  is isomorphic to  $Amal(S_n, P_m)$ .

**Sub-case 3.2** For  $k > m + 2 - e(x)$ ;  $x \in A$ .

Since  $k > m + 2 - e(x)$  and  $F(x) = \{y \in V \mid d(x, y) \leq k\}$  for  $k \geq 2$ , then

$F(x) = \{v_{0,1}, v_{i,1}, a_l \mid i = 2, 3, \dots, n; l = 1, 2, \dots, m\}$  or  
 $F(x) = \{v_{0,2}, v_{i,2}, a_l \mid i = 2, 3, \dots, n; l = 1, 2, \dots, m\}$  contains all vertices from a star graph  $S_n$  and all the vertices of the path graph  $P_m$  and  $K(x)$  contains all edges in  $S_n$  and  $P_m$ . Thus,  $H(x) = (F(x), K(x))$  is isomorphic to  $Amal(S_n, P_m)$ .  $\square$

We will give the soft graph of barbell star graph  $BS_{m,n}$  with  $m = 2, 3, 4, 5$  for every  $x \in A$  as shown in Table 3 – Table 6 below. The symbol  $\hat{A}$  in the table denotes the amalgamation.

Table 2. Soft Graph of  $BS_{2,n}$

$x$	$e(x)$	$H(x)$			
		$d(x, y) \leq 2$	$d(x, y) \leq 3$	$d(x, y) \leq 4$	$d(x, y) \leq k; k \geq 5$
$v_{i,1}$	5	$\hat{A}(S_n, P_1, a_1)$	$\hat{A}(S_n, P_2, a_1)$	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(2S_n, P_2, a_1, a_2)$
$v_{0,1}$	4	$\hat{A}(S_n, P_2, a_1)$	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(2S_n, P_2, a_1, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$
$a_1$	3	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(2S_n, P_2, a_1, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$
$a_2$	3	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$
$v_{0,2}$	4	$\hat{A}(S_n, P_2, a_2)$	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$
$v_{i,2}$	5	$\hat{A}(S_n, P_1, a_2)$	$\hat{A}(S_n, P_2, a_2)$	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(2S_n, P_2, a_1, a_2)$

Table 3. Soft Graph of  $BS_{3,n}$

$x$	$e(x)$	$H(x)$				
		$d(x, y) \leq 2$	$d(x, y) \leq 3$	$d(x, y) \leq 4$	$d(x, y) \leq 5$	$d(x, y) \leq k; k \geq 6$
$v_{i,1}$	6	$\hat{A}(S_n, P_1, a_1)$	$\hat{A}(S_n, P_2, a_1)$	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(2S_n, P_3, a_1, a_3)$
$v_{0,1}$	5	$\hat{A}(S_n, P_2, a_1)$	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(2S_n, P_3, a_1, a_3)$	$\hat{A}(2S_n, P_3, a_1, a_3)$
$a_1$	4	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(2S_n, P_3, a_1, a_3)$	$\hat{A}(2S_n, P_3, a_1, a_3)$	$\hat{A}(2S_n, P_3, a_1, a_3)$
$a_2$	3	$P_5$	$\hat{A}(2S_n, P_3, a_1, a_3)$			
$a_3$	4	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(2S_n, P_3, a_1, a_3)$	$\hat{A}(2S_n, P_3, a_1, a_3)$	$\hat{A}(2S_n, P_3, a_1, a_3)$
$v_{0,2}$	5	$\hat{A}(S_n, P_2, a_2)$	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(2S_n, P_3, a_1, a_3)$	$\hat{A}(2S_n, P_3, a_1, a_3)$
$v_{i,2}$	6	$\hat{A}(S_n, P_1, a_2)$	$\hat{A}(S_n, P_2, a_2)$	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(2S_n, P_3, a_1, a_3)$

Table 4. Soft Graph of  $BS_{4,n}$

$x$	$e(x)$	$H(x)$					
		$d(x, y) \leq 2$	$d(x, y) \leq 3$	$d(x, y) \leq 4$	$d(x, y) \leq 5$	$d(x, y) \leq 6$	$d(x, y) \leq k; k \geq 7$
$v_{i,1}$	7	$\hat{A}(S_n, P_1, a_1)$	$\hat{A}(S_n, P_2, a_1)$	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(S_n, P_5, a_1)$	$\hat{A}(2S_n, P_4, a_1, a_4)$
$v_{0,1}$	6	$\hat{A}(S_n, P_2, a_1)$	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(S_n, P_5, a_1)$	$\hat{A}(2S_n, P_4, a_1, a_4)$	$\hat{A}(2S_n, P_4, a_1, a_4)$
$a_1$	5	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(S_n, P_5, a_1)$	$\hat{A}(2S_n, P_4, a_1, a_4)$	$\hat{A}(2S_n, P_4, a_1, a_4)$	$\hat{A}(2S_n, P_4, a_1, a_4)$
$a_2$	4	$P_5$	$\hat{A}(S_n, P_5, a_1)$	$\hat{A}(2S_n, P_4, a_1, a_4)$			
$a_3$	4	$P_5$	$\hat{A}(S_n, P_5, a_2)$	$\hat{A}(2S_n, P_4, a_1, a_4)$			
$a_4$	5	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(S_n, P_5, a_2)$	$\hat{A}(2S_n, P_4, a_1, a_4)$	$\hat{A}(2S_n, P_4, a_1, a_4)$	$\hat{A}(2S_n, P_4, a_1, a_4)$
$v_{0,2}$	6	$\hat{A}(S_n, P_2, a_2)$	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(S_n, P_5, a_2)$	$\hat{A}(2S_n, P_4, a_1, a_4)$	$\hat{A}(2S_n, P_4, a_1, a_4)$
$v_{i,2}$	7	$\hat{A}(S_n, P_1, a_2)$	$\hat{A}(S_n, P_2, a_2)$	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(S_n, P_5, a_2)$	$\hat{A}(2S_n, P_4, a_1, a_4)$

Table 5. Soft Graph of  $BS_{5,n}$

$x$	$e(x)$	$H(x)$						
		$d(x,y) \leq 2$	$d(x,y) \leq 3$	$d(x,y) \leq 4$	$d(x,y) \leq 5$	$d(x,y) \leq 6$	$d(x,y) \leq 7$	$d(x,y) \leq k; k \geq 8$
$v_{i,1}$	8	$\hat{A}(S_n, P_1, a_1)$	$\hat{A}(S_n, P_2, a_1)$	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(S_n, P_5, a_1)$	$\hat{A}(S_n, P_6, a_1)$	$\hat{A}(2S_n, P_5, a_1, a_5)$
$v_{0,1}$	7	$\hat{A}(S_n, P_2, a_1)$	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(S_n, P_5, a_1)$	$\hat{A}(S_n, P_6, a_1)$	$\hat{A}(2S_n, P_5, a_1, a_5)$	$\hat{A}(2S_n, P_5, a_1, a_5)$
$a_1$	6	$\hat{A}(S_n, P_3, a_1)$	$\hat{A}(S_n, P_4, a_1)$	$\hat{A}(S_n, P_5, a_1)$	$\hat{A}(S_n, P_6, a_1)$	$\hat{A}(2S_n, P_5, a_1, a_5)$	$\hat{A}(2S_n, P_5, a_1, a_5)$	$\hat{A}(2S_n, P_5, a_1, a_5)$
$a_2$	5	$P_5$	$\hat{A}(S_n, P_5, a_1)$	$\hat{A}(S_n, P_6, a_1)$	$\hat{A}(2S_n, P_5, a_1, a_5)$			
$a_3$	4	$P_5$	$P_7$	$\hat{A}(2S_n, P_5, a_1, a_2)$	$\hat{A}(2S_n, P_5, a_1, a_5)$			
$a_4$	5	$P_5$	$\hat{A}(S_n, P_5, a_2)$	$\hat{A}(S_n, P_6, a_2)$	$\hat{A}(2S_n, P_5, a_1, a_5)$			
$a_5$	6	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(S_n, P_5, a_2)$	$\hat{A}(S_n, P_6, a_2)$	$\hat{A}(2S_n, P_5, a_1, a_5)$	$\hat{A}(2S_n, P_5, a_1, a_5)$	$\hat{A}(2S_n, P_5, a_1, a_5)$
$v_{0,2}$	7	$\hat{A}(S_n, P_2, a_2)$	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(S_n, P_5, a_2)$	$\hat{A}(S_n, P_6, a_2)$	$\hat{A}(2S_n, P_5, a_1, a_5)$	$\hat{A}(2S_n, P_5, a_1, a_5)$
$v_{i,2}$	8	$\hat{A}(S_n, P_1, a_2)$	$\hat{A}(S_n, P_2, a_2)$	$\hat{A}(S_n, P_3, a_2)$	$\hat{A}(S_n, P_4, a_2)$	$\hat{A}(S_n, P_5, a_2)$	$\hat{A}(S_n, P_6, a_2)$	$\hat{A}(2S_n, P_5, a_1, a_5)$

From the subgraph patterns formed above, if given a function  $F(x) = \{y \in V \mid d(x, y) \leq k\}$  with  $k \geq 2$  and a function  $K(x) = \{uv \in E \mid \{u, v\} \subseteq F(x)\}$  for every  $x \in A$ , then three types of subgraphs  $H(x)$  of  $BS_{m,n}$  will be formed as follows.

(1) The subgraph  $H(x)$  will be of the form  $BS_{m,n}$  if:

- a)  $m = 2$  with
  - $k = 3$  and  $e(x) = 3$ ,
  - $k = 4$  and  $e(x) = 3, 4$ ,
  - $k = 5$  and  $e(x) = 3, 4, 5$ ,
- b)  $m = 3$  with
  - $k = 3$  and  $e(x) = 3$ ,
  - $k = 4$  and  $e(x) = 3, 4$ ,
  - $k = 5$  and  $e(x) = 3, 4, 5$ ,
  - $k = 6$  and  $e(x) = 3, 4, 5, 6$ ,
- c)  $m = 4$  with
  - $k = 4$  and  $e(x) = 4$ ,
  - $k = 5$  and  $e(x) = 4, 5$ ,
  - $k = 6$  and  $e(x) = 4, 5, 6$ ,
  - $k = 7$  and  $e(x) = 4, 5, 6, 7$ .

From the pattern above, we concluded that if  $k \geq e(x)$  then the subgraph  $H(x)$  for an  $x \in A$  will be of the form  $BS_{m,n}$ .

(2) The subgraph  $H(x)$  will be of the form  $P_{2k+1}$  if

- a)  $m = 3$  with  $k = 2$  and  $e(x) = 3$ ,
- b)  $m = 4$  with  $k = 2$  and  $e(x) = 4$ ,
- c)  $m = 5$  with  $k = 2$  and  $e(x) = 4, 5$ ,
- d)  $m = 5$  with  $k = 3$  and  $e(x) = 4$ .

From the pattern above, we concluded that if  $k+e(x) \leq m+2$  or  $k \leq m+2-e(x)$  then, the subgraph  $H(x)$  for an  $x \in A$  will be of the form  $P_{2k+1}$ .

(3) The subgraph  $H(x)$  will be of the form  $BS_{m,n}$  if

- a)  $m = 2$  with
  - $k = 2$  and  $e(x) = 3$  then  $H(x) = Amal(S_n, P_3, a_l) ; l = 1, 2$ ,
  - $k = 2$  and  $e(x) = 4$  then  $H(x) = Amal(S_n, P_2, a_l) ; l = 1, 2$ ,
  - $k = 2$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_1, a_l) ; l = 1, 2$ ,

- $k = 3$  and  $e(x) = 4$  then  $H(x) = Amal(S_n, P_3, a_l)$  ;  $l = 1, 2$ ,  
 $k = 3$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_2, a_l)$  ;  $l = 1, 2$ ,  
 $k = 4$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_3, a_l)$  ;  $l = 1, 2$ ,  
 b)  $m = 3$  with  
 $k = 2$  and  $e(x) = 4$  then  $H(x) = Amal(S_n, P_3, a_l)$  ;  $l = 1, 2$ ,  
 $k = 2$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_2, a_l)$  ;  $l = 1, 2$ ,  
 $k = 2$  and  $e(x) = 6$  then  $H(x) = Amal(S_n, P_1, a_l)$  ;  $l = 1, 2$ ,  
 $k = 3$  and  $e(x) = 4$  then  $H(x) = Amal(S_n, P_4, a_l)$  ;  $l = 1, 2$ ,  
 $k = 3$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_3, a_l)$  ;  $l = 1, 2$ ,  
 $k = 3$  and  $e(x) = 6$  then  $H(x) = Amal(S_n, P_2, a_l)$  ;  $l = 1, 2$ ,  
 $k = 4$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_4, a_l)$  ;  $l = 1, 2$ ,  
 $k = 4$  and  $e(x) = 6$  then  $H(x) = Amal(S_n, P_3, a_l)$  ;  $l = 1, 2$ ,  
 $k = 5$  and  $e(x) = 6$  then  $H(x) = Amal(S_n, P_4, a_l)$  ;  $l = 1, 2$ ,  
 c)  $m = 4$  with  
 $k = 2$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_3, a_l)$  ;  $l = 1, 2$ ,  
 $k = 2$  and  $e(x) = 6$  then  $H(x) = Amal(S_n, P_2, a_l)$  ;  $l = 1, 2$ ,  
 $k = 2$  and  $e(x) = 7$  then  $H(x) = Amal(S_n, P_1, a_l)$  ;  $l = 1, 2$ ,  
 $k = 3$  and  $e(x) = 4$  then  $H(x) = Amal(S_n, P_5, a_l)$  ;  $l = 1, 2$ ,  
 $k = 3$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_4, a_l)$  ;  $l = 1, 2$ ,  
 $k = 3$  and  $e(x) = 6$  then  $H(x) = Amal(S_n, P_3, a_l)$  ;  $l = 1, 2$ ,  
 $k = 3$  and  $e(x) = 7$  then  $H(x) = Amal(S_n, P_2, a_l)$  ;  $l = 1, 2$ ,  
 $k = 4$  and  $e(x) = 5$  then  $H(x) = Amal(S_n, P_5, a_l)$  ;  $l = 1, 2$ ,  
 $k = 4$  and  $e(x) = 6$  then  $H(x) = Amal(S_n, P_4, a_l)$  ;  $l = 1, 2$ ,  
 $k = 4$  and  $e(x) = 7$  then  $H(x) = Amal(S_n, P_3, a_l)$  ;  $l = 1, 2$ ,  
 $k = 5$  and  $e(x) = 6$  then  $H(x) = Amal(S_n, P_5, a_l)$  ;  $l = 1, 2$ ,  
 $k = 5$  and  $e(x) = 7$  then  $H(x) = Amal(S_n, P_4, a_l)$  ;  $l = 1, 2$ ,  
 $k = 6$  and  $e(x) = 7$  then  $H(x) = Amal(S_n, P_5, a_l)$  ;  $l = 1, 2$ .

From the pattern above, we concluded that for otherwise  $k < e(x)$  or  $k > m + 2 - e(x)$  the subgraph  $H(x)$  for an  $x \in A$  will be of the form  $Amal(S_n, P_{m+k+2-e(x)}, a_l)$  for  $l = 1, 2$ .

#### 4. Conclusion

In this paper, we constructed the barbell star graph from the amalgamation of the star graph and the path graph, denoted by  $BS_{m,n}$ , and also enumerated the soft graph forms of this amalgamation graph for different definitions of  $F(x)$ .

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