

AUTOMORPHISM GROUPS IN LOTUS GRAPH AND UNIFORM BOW GRAPH

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Abstract. *This research aims to explore the connection between abstract algebra and graph theory through the study of lotus graph and uniform bow graph. The focus is on determining all automorphisms of both graphs and analyzing the algebraic structure they form. It is shown that the set of automorphisms, under composition, satisfies the group axioms, thus illustrating a natural link between group theory and graph theory.*
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1. Introduction

The origin of graph theory can be traced back to Leonhard Euler, who first introduced it while solving the famous problem of the Seven Bridges of Königsberg in 1735 [1]. Since its inception, graph theory has continuously evolved and found numerous applications in real-world problems. In addition, its concepts can also be related to other areas of mathematics, such as group theory. In addition to representing groups in the form of graphs, it is also possible to construct group based on a given graph by using automorphisms and composition functions.

The automorphism groups of several types of graphs have been studied in previous works. For example, the automorphism group of helm and book graphs was determined by [2], fan and double fan graphs by [3], some classes of graphs, such as Petersen graph by [4], certain power graphs of finite groups by [5], lollipop graphs by [6], connected bipartite irreducible graphs by [7], ladder and cycle graphs by [8], some families of bipartite graphs by [9], Hamming graph by [10], complete graph by [11], some token graphs by [12], and polyhedral graphs by [13]. These previous studies illustrate the growing interest in exploring the connections between graph theory and group theory through the concept of automorphisms.

According to previous research, the automorphism groups of lotus graph and uniform bow graph remain unexplored, as no detailed investigations addressing

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these graphs have been reported in the existing literature. Motivated by this gap, this paper aims to determine the automorphism groups of lotus graph and uniform bow graph by analyzing their automorphisms and then verifying whether these automorphisms form a group under the composition operation.

2. Some Concepts

In this section, we present some basic definitions and preliminary results that will be used throughout the paper.

2.1. Binary Operation

Definition 2.1. [14] A binary operation $*$ on a set S is a function $*$: $S \times S \mapsto S$ from the set $S \times S$ of all ordered pairs of elements in S into S .

2.2. Group

Definition 2.2. [14] Let $(G, *)$ denote a nonempty set G together with a binary operation $*$ on G . Then G is called a **group** if the following properties hold.

- (i) **Associativity:** For all $a, b, c, \in G$, we have $a * (b * c) = (a * b) * c$.
- (ii) **Identity:** There exists an **identity element** $e \in G$, that is, an element $e \in G$ such that $e * a = a$ and $a * e = a$ for all $a \in G$.
- (iii) **Inverses:** For each $a \in G$ there exists an **inverse element** $a^{-1} \in G$, that is, an element $a^{-1} \in G$ such that $a * a^{-1} = e$ and $a^{-1} * a = e$.

Definition 2.3. [14] A group G is said to be **abelian** if $ab = ba$ for all $a, b \in G$.

Definition 2.4. [14] Let G_1 and G_2 be groups, and let $\phi : G_1 \rightarrow G_2$ be a function. Then ϕ is said to be a **group isomorphism** if ϕ is one-to-one and onto and

$$\phi(ab) = \phi(a)\phi(b)$$

for all $a, b \in G_1$. In this case, G_1 is said to be **isomorphic** to G_2 , and this is denoted by $G_1 \cong G_2$.

2.3. Permutation

Definition 2.5. [14] Let S be a set. A function $\sigma : S \rightarrow S$ is called a **permutation** of S if σ is one-to-one and onto.

Definition 2.6. [14] The set of all permutations of a set S is denoted by $Sym(S)$. The set of all permutations of the set $\{1, 2, \dots, n\}$ is denoted by S_n . The group $Sym(S)$ is called **the symmetric group** on S , and S_n is called the symmetric group of degree n .

Definition 2.7. [14] To facilitate working with permutations in the symmetric group S_n , a standard notation is introduced. A permutation $\sigma \in S_n$ is completely

determined by its action on each element of the set $\{1, 2, \dots, n\}$. Such a permutation is expressed in two-line notation as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

where each element i in the top row corresponds to its image $\sigma(i)$ in the bottom row.

Definition 2.8. [14] Let S be a set, and let $\sigma \in \text{Sym}(S)$. Then σ is called a cycle of length k if there exist elements $a_1, a_2, \dots, a_k \in S$ such that $\sigma(a_1) = a_2, \sigma(a_2) = a_3, \dots, \sigma(a_{k-1}) = a_k, \sigma(a_k) = a_1$, and $\sigma(x) = x$ for all other elements $x \in S$ with $x \neq a_i$ for $i = 1, 2, \dots, k$. In this case we write $\sigma = (a_1 a_2 \dots a_k)$.

Example 2.9. [14] In S_5 the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$$

is a cycle of length 3, written $(1\ 3\ 4)$ or $(1\ 3\ 4)(2)(5)$. The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

is not a cycle, since

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix} = (1\ 3\ 4)(2\ 5)$$

is the product of two cycles.

Definition 2.10. [15] A **dihedral group** is the group of symmetries of a regular polygon. The group of symmetries of the square is denoted by D_4 because the square has 4 corners (or 4 sides). The dihedral group D_4 is a group of order 8.

2.4. Graph

Definition 2.11. [1] A simple graph G consists of a non-empty finite set $V(G)$ of elements called vertices (or nodes or points) and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called edges (or lines).

Definition 2.12. [1] Two vertices v and w of a graph are **adjacent** if there is an edge vw joining them.

Definition 2.13. [1] A walk is a ‘way of getting from one vertex to another’ in a graph and consists of a sequence of edges, one following after another. A walk in which no vertex appears more than once, except for the beginning and end vertices which coincide, is called a **cycle**. The cycle graph on n vertices is denoted by C_n .

3. Automorphism of a Graph

In this section, the connection between graph theory and group theory is explored through the study of graph automorphisms. An automorphism of a graph is a permutation of its vertices that preserves adjacency. The set of all such automorphisms forms a group under composition, known as the **automorphism group** of the graph.

Definition 3.1. *An automorphism φ of a simple graph G is a one-to-one mapping of the vertex-set of G onto itself with the property that $\varphi(v)$ and $\varphi(w)$ are adjacent whenever v and w are. The automorphism group $\Gamma(G)$ of G is the group of automorphisms of G under composition.*

4. Lotus Graph

In order to present lotus graph, it is essential to provide a description of shell graph, since lotus graph is constructed based on shell graph.

Definition 4.1. [16] *A **chord** is an edge between two vertices of a cycle that is not an edge on the cycle.*

Definition 4.2. [17] *A **shell graph** is a cycle C_n with $(n - 3)$ chords sharing a common end point called the **apex**. Shell graphs are denoted as $C(n, n - 3)$.*

Definition 4.3. [18] *A graph is obtained from shell graph by adding a vertex in between each pair of adjacent vertices on the cycle and adding an edge in apex and two or more chords is known as **Lotus graph**.*

In this paper, lotus graph will be denoted by $L(n)$ for $n \geq 2$, see Figure 1.

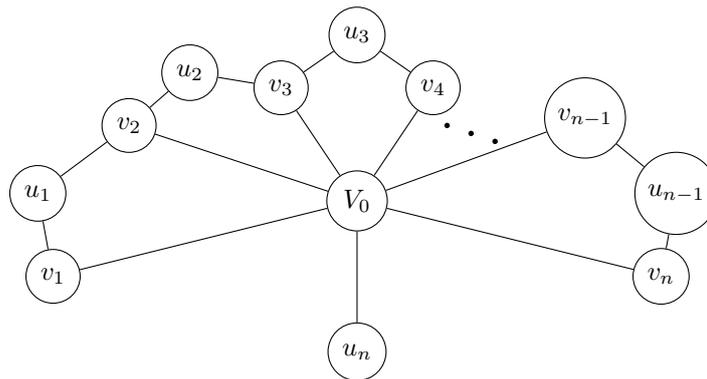


Figure. 1. Lotus graph $L(n)$.

4.1. Automorphism Group in Lotus Graph

Before presenting the automorphism group in lotus graph for $n \geq 2$, we first provide some illustrative examples.

Example 4.4. Let $n = 2$ for lotus graph $L(2)$. We have $u_1, u_2, v_0, v_1, v_2 \in V(L(2))$ and $v_0u_2, v_0v_1, v_0v_2, v_1u_1, v_2u_1 \in E(L(2))$. The automorphisms in $L(2)$ are

- (1) $\theta_1 = (u_1)(u_2)(v_0)(v_1)(v_2)$.
- (2) $\theta_2 = (u_1)(u_2)(v_0)(v_1 v_2)$.

Then, $\Gamma(L(2)) = \{\theta_1, \theta_2\}$. It will be shown that $\Gamma(L(2))$ is a group under the composition operation. The following table shows the result of the composition operation on the elements of $\Gamma(L(2))$.

Table 1. $(\Gamma(L(2)), \circ)$

\circ	θ_1	θ_2
θ_1	θ_1	θ_2
θ_2	θ_2	θ_1

Based on Table 1, we can see that:

- (i) The composition operation is well-defined on $\Gamma(L(2))$, as the result of composing any two elements in $\Gamma(L(2))$ is uniquely determined and lies within $\Gamma(L(2))$.
- (ii) The operation of composition on $\Gamma(L(2))$ satisfies the associative property.
- (iii) There exists θ_1 as an identity element.
- (iv) Every element in $\Gamma(L(2))$ is self-inverse.
- (v) The composition operation on $\Gamma(L(2))$ is commutative.

Hence, all group axioms are satisfied. In addition, it is abelian. We conclude that $\Gamma(L(2))$ forms an abelian group under composition. Thus, $(\Gamma(L(2)), \circ)$ is the automorphism group in $L(2)$.

Example 4.5. Let $n = 3$ for lotus graph $L(3)$. We have $u_1, u_2, u_3, v_0, v_1, v_2, v_3 \in V(L(3))$ and $v_0u_3, v_0v_1, v_0v_2, v_0v_3, v_1u_1, v_2u_1, v_2u_2, v_3u_2 \in E(L(3))$. The automorphisms in $L(3)$ are:

- (1) $\theta_1 = (u_1)(u_2)(u_3)(v_0)(v_1)(v_2)(v_3)$.
- (2) $\theta_2 = (u_3)(v_0)(v_2)(u_1 u_2)(v_1 v_3)$.

Then, $\Gamma(L(3)) = \{\theta_1, \theta_2\}$ is an abelian group under the composition operation, whose group properties can be verified in the same manner as for the $(\Gamma(L(2), \circ))$ in the previous example. Thus, $(\Gamma(L(3)), \circ)$ is the automorphism group in $L(3)$.

Theorem 4.6. Let $L(n)$ be the lotus graph for $n \geq 2$. Then, the automorphism group in $L(n)$ contains exactly two elements and forms an abelian group.

Proof. For $n = 2$ and $n = 3$, as shown in Examples 4.4 and 4.5 respectively, each case has an automorphism group of order 2, and each group is abelian. Now, for $n \geq 3$, we proceed to identify all automorphisms of $L(n)$ generally.

$$\begin{aligned}
 \text{(i) } \theta_1 &= (u_1)(u_2)(u_3) \cdots (u_n)(v_0)(v_1)(v_2)(v_3) \cdots (v_n) \\
 \text{(ii) } \theta_2 &= \begin{cases} (u_n)(v_0)(v_{\frac{n+1}{2}})(u_1 u_{n-1})(u_2 u_{n-2}) \cdots (u_{\frac{n-1}{2}} u_{\frac{n+1}{2}}) \\ (v_1 v_n)(v_2 v_{n-1}) \cdots (v_{\frac{n-1}{2}} v_{\frac{n+3}{2}}) & \text{for odd } n \\ (u_{\frac{n}{2}})(u_n)(v_0)(u_1 u_{n-1})(u_2 u_{n-2}) \cdots (u_{\frac{n-1}{2}} u_{\frac{n+2}{2}}) \\ (v_1 v_n)(v_2 v_{n-1}) \cdots (v_{\frac{n}{2}} v_{\frac{n+2}{2}}) & \text{for even } n \end{cases}
 \end{aligned}$$

Thus, we can construct the set of automorphisms $\Gamma(L(n)) = \{\theta_1, \theta_2\}$. It can be shown that $\Gamma(L(n)) = \{\theta_1, \theta_2\}$ forms an abelian group under the operation of composition. The proof follows the same reasoning as in the group verification provided in Example 4.4. Therefore, we have shown that the automorphism group of $L(n)$ contains exactly two elements and forms an abelian group. \square

5. Uniform Bow Graph

We now introduce uniform bow graph, which is related to shell graph as mentioned earlier in Definition 4.2. In addition, uniform bow graph is also associated with bow graph, which is derived from shell graph itself.

Definition 5.1. [17] A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. Hence a **double shell** consists of two edge disjoint shells with a common apex.

Definition 5.2. [17] A bow graph is a double shell in which each shell has any order. A bow graph in which each shell has the same order 'l' is a **uniform bow graph**. A uniform bow graph is denoted as $B(n)$, see Figure 2.

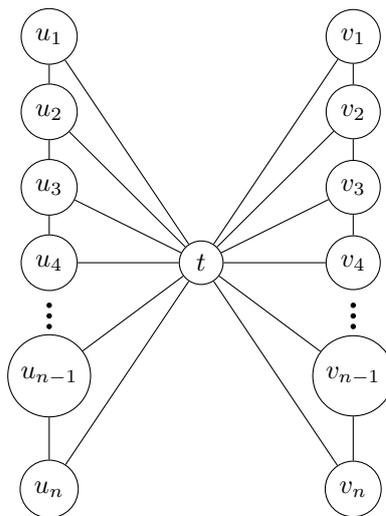


Figure. 2. Uniform bow graph $B(n)$.

5.1. Automorphism Group in Uniform Bow Graph

Before identifying the group in uniform bow graph, an example is first provided.

Example 5.3. Let $n = 2$ for uniform bow graph $B(3)$. We have $u_1, u_2, v_1, v_2, t \in V(B(2))$ and $tu_1, tu_2, tv_1, tv_2, u_1u_2, v_1, v_2 \in E(B(2))$. The automorphisms in $B(2)$ are

- (1) $\theta_1 = (u_1)(u_2)(v_1)(v_2)(t)$.
- (2) $\theta_2 = (u_1 u_2)(v_1)(v_2)(t)$.
- (3) $\theta_3 = (u_1)(u_2)(v_1 v_2)(t)$.
- (4) $\theta_4 = (u_1 u_2)(v_1 v_2)(t)$.
- (5) $\theta_5 = (u_1 v_2 u_2 v_1)(t)$.
- (6) $\theta_6 = (u_1 v_2)(u_2 v_1)(t)$.
- (7) $\theta_7 = (u_1 v_1 u_2 v_2)(t)$.
- (8) $\theta_8 = (u_1 v_1)(u_2 v_2)(t)$.

Then $\Gamma(B(2)) = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8\}$. It will be shown that $\Gamma(B(2))$ is a group under the composition operation. The following table shows the result of the composition operation on the elements of $\Gamma(B(2))$. Based on Table 2, we can see

Table 2. $(\Gamma(B(2)), \circ)$

\circ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
θ_1	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
θ_2	θ_2	θ_1	θ_4	θ_3	θ_6	θ_5	θ_8	θ_7
θ_3	θ_3	θ_4	θ_1	θ_2	θ_8	θ_7	θ_6	θ_5
θ_4	θ_4	θ_3	θ_2	θ_1	θ_7	θ_8	θ_5	θ_6
θ_5	θ_5	θ_8	θ_6	θ_7	θ_4	θ_2	θ_1	θ_3
θ_6	θ_6	θ_7	θ_5	θ_8	θ_3	θ_1	θ_2	θ_4
θ_7	θ_7	θ_6	θ_8	θ_5	θ_1	θ_3	θ_4	θ_2
θ_8	θ_8	θ_5	θ_7	θ_6	θ_2	θ_4	θ_3	θ_1

that

- (i) The composition operation is well-defined on $\Gamma(B(2))$, as the result of composing any two elements in $\Gamma(B(2))$ is uniquely determined and lies within $\Gamma(B(2))$.
- (ii) The operation of composition on $\Gamma(B(2))$ satisfies the associative property.
- (iii) There exists θ_1 as an identity element.
- (iv) For every element in $\Gamma(B(2))$, there exists an inverse with respect to the operation, satisfying the group inverse property.

Hence, all group axioms are satisfied. However, the group is not abelian, as the group operation is not commutative. Furthermore, it is isomorphic to the dihedral group D_4 . Thus, $(\Gamma(B(2)), \circ)$ is the automorphism group in $B(2)$.

Theorem 5.4. *Let $B(n)$ be the uniform bow graph for $n \geq 2$. Then, the automorphism group in $B(n)$ is isomorphic to the dihedral group D_4 of order 8.*

Proof. It has already been shown that for $n = 2$, the automorphism group of $B(2)$ is isomorphic to the dihedral group D_4 in Example 5.3. We now present a generalization of the result for $n \geq 3$. The following are all automorphisms of $B(n)$:

$$\begin{aligned}
 (1) \quad & \theta_1 = (u_1)(u_2) \cdots (u_n)(v_1)(v_2) \cdots (v_n)(t), \\
 (2) \quad & \theta_2 = \begin{cases} (u_1 \ u_n)(u_2 \ u_{n-1}) \cdots (u_{\frac{n-1}{2}} \ u_{\frac{n+3}{2}})(u_{\frac{n+1}{2}}) \\ (v_1)(v_2) \cdots (v_n)(t) & \text{for odd } n, \\ (u_1 \ u_n)(u_2 \ u_{n-1}) \cdots (u_{\frac{n}{2}} \ u_{\frac{n+2}{2}})(v_1)(v_2) \cdots (v_n)(t) & \text{for even } n, \end{cases} \\
 (3) \quad & \theta_3 = \begin{cases} (u_1)(u_2) \cdots (u_n) \\ (v_1 \ v_n)(v_2 \ v_{n-1}) \cdots (v_{\frac{n-1}{2}} \ v_{\frac{n+3}{2}})(v_{\frac{n+1}{2}})(t) & \text{for odd } n, \\ (u_1)(u_2) \cdots (u_n)(v_1 \ v_n)(v_2 \ v_{n-1}) \cdots (v_{\frac{n}{2}} \ v_{\frac{n+2}{2}})(t) & \text{for even } n, \end{cases} \\
 (4) \quad & \theta_4 = \begin{cases} (u_1 \ u_n)(u_2 \ u_{n-1}) \cdots (u_{\frac{n-1}{2}} \ u_{\frac{n+3}{2}})(u_{\frac{n+1}{2}}) \\ (v_1 \ v_n)(v_2 \ v_{n-1}) \cdots (v_{\frac{n-1}{2}} \ v_{\frac{n+3}{2}})(v_{\frac{n+1}{2}})(t) & \text{for odd } n, \\ (u_1 \ u_n)(u_2 \ u_{n-1}) \cdots (u_{\frac{n}{2}} \ u_{\frac{n+2}{2}}) \\ (v_1 \ v_n)(v_2 \ v_{n-1}) \cdots (v_{\frac{n}{2}} \ v_{\frac{n+2}{2}})(t) & \text{for even } n, \end{cases} \\
 (5) \quad & \theta_5 = (u_1 \ v_n \ u_n \ v_1)(u_2 \ v_2)(u_3 \ v_3) \cdots (u_{n-1} \ v_{n-1})(t). \\
 (6) \quad & \theta_6 = (u_1 \ v_n)(u_2 \ v_2)(u_3 \ v_3) \cdots (u_{n-1} \ v_{n-1})(u_n \ v_1)(t). \\
 (7) \quad & \theta_7 = (u_1 \ v_1 \ u_n \ v_n)(u_2 \ v_2)(u_3 \ v_3) \cdots (u_{n-1} \ v_{n-1})(t). \\
 (8) \quad & \theta_8 = (u_1 \ v_1)(u_2 \ v_2) \cdots (u_n \ v_n)(t).
 \end{aligned}$$

Based on these automorphisms, we construct

$$\Gamma(B(n)) = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8\}.$$

$\Gamma(B(n))$ forms a group, as can be shown by the same method used in Example 5.3. Furthermore, the group is non-abelian, as its operation is not commutative, and it is isomorphic to a dihedral group D_4 of order 8. Thus, we have shown that the automorphism group of $B(n)$ is isomorphic to D_4 of order 8. \square

6. Conclusion

In conclusion, we obtain the following two results:

- (1) The abelian automorphism group in lotus graph $L(n)$, for $n \geq 2$ contains exactly two elements.
- (2) The automorphism group in uniform bow graph $B(n)$, for $n \geq 2$ is isomorphic to dihedral group D_4 of order 8.

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