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SOME PROPERTIES OF CLEAR RINGS

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Abstract. *Let $(R, +, \cdot)$ be a ring with unity. An element in R is called a clean element if it is the sum of a unit element and an idempotent element. A ring R is called a clean ring if all elements in R are clean elements. The notion of a clean element was generalized to a clear element by replacing the idempotent element with a unit-regular element. An element in R is called a clear element if it is the sum of a unit element and a unit-regular element. A ring R is called a clear ring if all elements in R are clear elements. In this paper, we study the new properties of clear elements in a ring and clear properties in certain special rings, such as opposite rings, quotient rings, corner rings, Morita rings, and group rings.*

Keywords: Clean ring, clear ring, unit regular

1. Introduction

Let $(R, +, \cdot)$ be a ring with unity 1_R. An element $r \in R$ is said to be clean if it is the sum of a unit element and an idempotent element of R. A ring R is called a clean ring if every element of R is a clean element. This concept was first introduced by Nicholson [1]. Research on clean rings is still developing to this day. Several researchers have continued to study and expand the concept, as presented in [2,3,4].

An element $r \in R$ is said to be unit-regular if there exists a unit element $r' \in R$ such that $r \cdot r' \cdot r = r$. For every idempotent element $r \in R$, there exists a unit element $1_R \in R$ such that $r \cdot 1_R \cdot r = r$. In [5], Zabavsky, Domsha, and Romaniv introduced the notion of clear elements by generalizing the definition of clean elements through the generalization of an idempotent element to a unit-regular element. An $r \in R$ is said to be clear if it is the sum of a unit element and a unit-regular element of R. Therefore, a ring R is called a clear ring if every element of R is a clear element. In

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other words, the concept of clear rings can be viewed as a generalization of clean rings.

Several results from [5] include the following: every clean element is a clear element; every unit-regular element is a clear element; the image of a ring homomorphism from a clear ring is also a clear ring; and the direct product of clear rings is also a clear ring. In this paper, we continue the study of properties of clear elements in a ring, as well as the properties of clear rings within certain special classes of rings. The relationships between the main concepts in this study are illustrated in Figure 1.

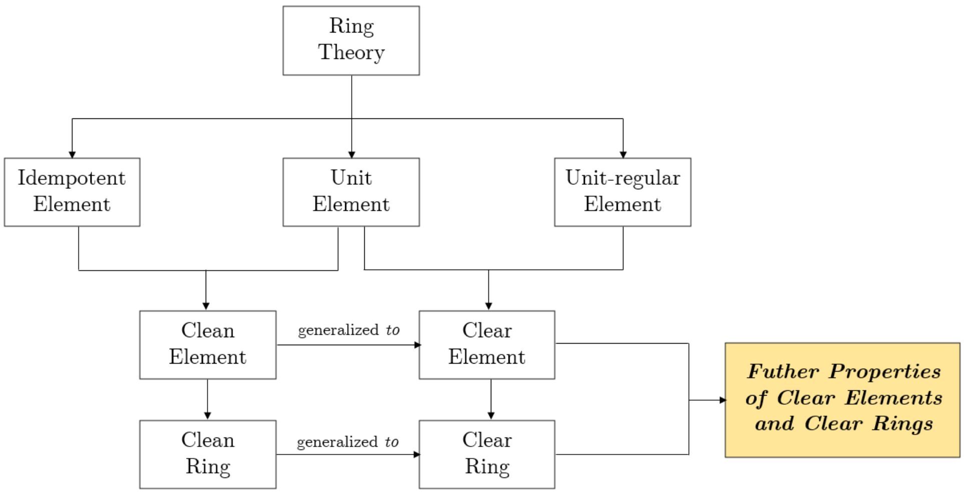

Figure. 1. The main concepts in this study

Throughout this paper, R denotes a ring with unity 1_R. The set of all unit elements of R is denoted by $U(R)$, the set of all idempotent elements of R is denoted by $Id(R)$, the set of all unit-regular elements of R is denoted by $U_{reg}(R)$, the set of all clean elements of R is denoted by $Cln(R)$, and the set of all clear elements of R is denoted by $Clr(R)$.

In this paper, we continue the study of properties of clear elements in a ring, as well as the properties of clear in several specific ring structures, such as the ring in the Morita context, the abelian ring, the opposite ring, the group ring, the Dorroh ideal, and others. The results of this research are expected to contribute to the development of the theory of algebraic structures on clear elements.

2. Some Concepts

Clear rings generalize the notion of clean rings. To understand the concept of clear rings, it is essential to first understand the notions of units, idempotent elements, clean elements, regular units, and other related concepts. This section introduces the fundamental notions relevant to this research.

Definition 2.1. *[6] Let $(R, +, \cdot)$ be a ring. An element $r \in R$ is said to be a unit if there exists $r' \in R$ such that $rr' = 1_R = r'r$.*

To facilitate the understanding of Definition 2.1, the following is an example of a unit element.

Example 2.2. Consider the ring $(\mathbb{Z}_6, +_6, \cdot_6)$. The units in $\mathbb{Z}_6$ are $\overline{1}$ and $\overline{5}$.

Definition 2.3. *[6] Let $(R, +, \cdot)$ be a ring. An element $r \in R$ is said to be idempotent if $r^2 = r$.*

To facilitate the understanding of Definition 2.3, the following is an example of an idempotent element.

Example 2.4. Consider the ring $(\mathbb{Z}_6, +_6, \cdot_6)$. The idempotent elements in $\mathbb{Z}_6$ are $\overline{0}, \overline{1}, \overline{3}$, and $\overline{4}$.

Definition 2.5. *[7] Let $(R, +, \cdot)$ be a ring. An element $r \in R$ is said to be unit-regular if there exists $r' \in U(R)$ such that $rr'r = r$.*

To facilitate the understanding of Definition 2.5, the following is an example of a unit-regular element.

Example 2.6. Consider the ring $(\mathbb{Z}_6, +_6, \cdot_6)$. The unit-regular elements in $\mathbb{Z}_6$ are $\overline{0}, \overline{1}, \overline{2}, \overline{3}$, $\overline{4}$, and $\overline{5}$ since $\overline{0} = \overline{0} \cdot_6 \overline{1} \cdot_6 \overline{0}$, $\overline{1} = \overline{1} \cdot_6 \overline{1} \cdot_6 \overline{0}$, $\overline{2} = \overline{2} \cdot_6 \overline{5} \cdot_6 \overline{2}$, $\overline{3} = \overline{3} \cdot_6 \overline{1} \cdot_6 \overline{3}$, $\overline{4} = \overline{4} \cdot_6 \overline{1} \cdot_6 \overline{4}$, and $\overline{5} = \overline{5} \cdot_6 \overline{1} \cdot_6 \overline{5}$, for some for some unit elements $\overline{1}$ and $\overline{5}$ in $\mathbb{Z}_6$.

The next basic notion required for research on clear rings is the concept of clean rings. The definition of a clean ring is presented as follows.

Definition 2.7. *[1] Let $(R, +, \cdot)$ be a ring. An element $r \in R$ is said to be clean if $r = u + e$, for some $u \in U(R)$ and $e \in Id(R)$. A ring R is called a clean ring if every element of R is a clean element.*

Definition 2.8. *[5] Let $(R, +, \cdot)$ be a ring. An element $r \in R$ is said to be clear if $r = u + v$, for some $u \in U(R)$ and $v \in U_{reg}(R)$. A ring R is called a clear ring if every element of R is a clear element.*

To facilitate the understanding of Definition 2.8, the following are examples of clear rings.

Example 2.9. Let $(\mathbb{Z}_6, +_6, \cdot_6)$ be the ring of integers modulo 6. The elements of $\mathbb{Z}_6$ are $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}$, and $\bar{5}$. We will show that $(\mathbb{Z}_6, +_6, \cdot_6)$. Consider the following decompositions.

(1) For $\bar{0} \in \mathbb{Z}_6$, $\bar{0} = \bar{1} +_6 \bar{5}$, where $\bar{1} \in U(\mathbb{Z}_6)$ and $\bar{5} \in U_{\text{reg}}(\mathbb{Z}_6)$,
(2) For $\bar{1} \in \mathbb{Z}_6$, $\bar{1} = \bar{1} +_6 \bar{0}$, where $\bar{1} \in U(\mathbb{Z}_6)$ and $\bar{0} \in U_{\text{reg}}(\mathbb{Z}_6)$,
(3) For $\bar{2} \in \mathbb{Z}_6$, $\bar{2} = \bar{1} +_6 \bar{1}$, where $\bar{1} \in U(\mathbb{Z}_6)$ and $\bar{1} \in U_{\text{reg}}(\mathbb{Z}_6)$,
(4) For $\bar{3} \in \mathbb{Z}_6$, $\bar{3} = \bar{1} +_6 \bar{2}$, where $\bar{1} \in U(\mathbb{Z}_6)$ and $\bar{2} \in U_{\text{reg}}(\mathbb{Z}_6)$,
(5) For $\bar{4} \in \mathbb{Z}_6$, $\bar{4} = \bar{1} +_6 \bar{3}$, where $\bar{1} \in U(\mathbb{Z}_6)$ and $\bar{3} \in U_{\text{reg}}(\mathbb{Z}_6)$,
(6) For $\bar{5} \in \mathbb{Z}_6$, $\bar{5} = \bar{5} +_6 \bar{0}$, where $\bar{5} \in U(\mathbb{Z}_6)$ and $\bar{0} \in U_{\text{reg}}(\mathbb{Z}_6)$.

Based on the decompositions above, it follows that every element in $\mathbb{Z}_6$ can be expressed as the sum of a unit and a unit-regular element. Hence, $(\mathbb{Z}_6, +_6, \cdot_6)$ is a clear ring.

Example 2.10. Let $(R, +, \cdot)$ be a division ring. Take any $r \in R$. Consider the following two cases:

(1) If $r = 0_R$, then $r = 1_R + (-1_R)$, where $1_R \in U(R)$ and $-1_R \in U_{\text{reg}}(R)$.
(2) If $r \neq 0_R$, then $r = r + 0_R$, where $r \in U(R)$ and $0_R \in U_{\text{reg}}(R)$.

Since every element $r \in R$ can be written as the sum of a unit and a unit-regular element, it follows that the division ring $(R, +, \cdot)$ is a clear ring.

In the following, we present several properties of clear elements and clear rings that are used in this research.

Proposition 2.11. *[5] Every clean element is a clear element.*

Proposition 2.12. *[5] Every homomorphic image of a clear ring is clear.*

Proposition 2.13. *[5] Every direct product of clear rings is a clear ring.*

Definition 2.14. *[8] Let $(R, +, \cdot)$ be a ring. An element $r \in R$ is said to be quasi-regular if there exists $r' \in R$ such that $r + r' = rr' = r'r$.*

To facilitate understanding of Definition 2.14, the following are several examples of quasi-regular elements.

Example 2.15. Let $(R, +, \cdot)$ be a ring. An element $0_R \in R$ is a quasi-reguler element since $0_R + 0_R = 0_R \cdot 0_R$.

Example 2.16. The element $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ is a quasi-regular element, since there exists $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ such that $A + B = AB = BA$.

The following proposition presents the relationship between quasi-regular elements and units in a ring.

Proposition 2.17. *[9] Let $(R, +, \cdot)$ be a ring. An element $r \in R$ is said to be quasi-regular in R if and only $1_R - r \in U(R)$.*

Proof. Let $(R, +, \cdot)$ be a ring.

($\Rightarrow$) Assume that $r \in R$ is a quasi-regular element. We will prove that $1_R - r \in U(R)$. Since $r \in R$ is a quasi-regular element, there exists $r' \in R$ such that $r + r' = rr' = r'r$. Note that:

$$(1_R - r)(1_R - r') = 1_R - r' - r + rr' = 1_R - r' - r + r + r' = 1_R,$$

and

$$(1_R - r')(1_R - r) = 1_R - r - r' + r'r = 1_R - r - r' + r + r' = 1_R.$$

Hence, $1_R - r \in U(R)$.

(⇐) Assume that $1_R - r \in R$ is a unit element. We will prove that $r \in R$ is a quasi-regular element.
Since $1_R - r \in U(R)$, then there exists $(1_R - r)^{-1} \in R$ such that

$$(1_R - r)^{-1}(1_R - r) = 1_R = (1_R - r)^{-1}(1_R - r).$$

We have $(1_R - r)^{-1}r = (1_R - r)^{-1} - 1_R$ and $r(1_R - r)^{-1} = (1_R - r)^{-1} - 1_R$. Then,

$$(1_R - r)^{-1}r = r(1_R - r)^{-1}.$$

Let $r' = 1_R - (1_R - r)^{-1}$. Note that:

$$\begin{aligned} rr' = r(1_R - (1_R - r)^{-1}) = r - r(1_R - r)^{-1} &= r - (1_R - r)^{-1}r, \\ &= (1_R - (1_R - r)^{-1})r, \\ &= r'r, \end{aligned}$$

and

$$\begin{aligned} r + r' = r + 1_R - (1_R - r)^{-1} = r - (-1_R + (1_R - r)^{-1}) &= r - (1_R - r)^{-1}r, \\ &= (1_R - (1_R - r)^{-1})r, \\ &= r'r. \end{aligned}$$

Hence, $r + r' = rr' = r'r$. Therefore, $r \in R$ is a quasi-regular element.

Thus, it is proved that an element $r \in R$ is said to be quasi-regular in R if and only $1_R - r \in U(R)$. □

Definition 2.18. *[10] Let $(R, +, \cdot)$ be a ring. A ring R is called a strongly regular ring if for every $r \in R$ there exists $r' \in R$ such that $r^2 r' = r$.*

To facilitate the understanding of Definition 2.18, the following is an example of a strongly-regular ring.

Example 2.19. Let $(\mathbb{R}, +, \cdot)$ be the ring of real numbers. The ring $\mathbb{R}$ is a strongly regular ring because for every $r \in \mathbb{R} \setminus \{0\}$, we have $r = r^2 \cdot r^{-1}$ and for $r = 0$, we have $0 = 0^2 \cdot 0$. Therefore, $\mathbb{R}$ is a strongly regular ring.

Definition 2.20. *[11] Let $(R, +, \cdot)$ be a ring. A ring R is called abelian ring if every idempotent $e \in R$ is central, i.e., $re = er$, for all $r \in R$.*

To facilitate the understanding of Definition 2.20, the following is an example of an abelian ring.

Example 2.21. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers. The set of all idempotent elements in $\mathbb{Z}$ is $Id(\mathbb{Z}) = \{0, 1\}$.

(1) If $e = 0 \in Id(\mathbb{Z})$, we have $r \cdot 0 = 0 \cdot r$, for every $r \in \mathbb{Z}$ and
(2) If $e = 1 \in Id(\mathbb{Z})$, we have $r \cdot 1 = 1 \cdot r$, for every $r \in \mathbb{Z}$.

Therefore, $(\mathbb{Z}, +, \cdot)$ is an abelian ring.

3. Result and Discussion

This study began with an examination of basic theories regarding rings and a review of previous results related to clean rings and clear rings. From this study, advanced properties related to clear elements in a ring were identified. Furthermore, this study includes a discussion of the properties of clear in several specific ring structures, such as the ring in the Morita context, the Dorroh ideal, the corner ring, the opposite ring, the group ring, and others.

3.1. *Further Properties of Clear Elements in a Ring*

Before investigating the properties of clear elements in special rings, the first step is to examine the advanced properties of clear elements in a ring. Han and Nicholson in [12] state that "If x is a clear element in R, then $1_R - x$ is also a clear element in R." This statement does not hold in clear rings. As a counterexample, consider the ring $(\mathbb{Z}, +, \cdot)$. The set of all units and regular units in $\mathbb{Z}$ are respectively $U(\mathbb{Z}) = \{-1, 1\}$ and $U_{reg}(\mathbb{Z}) = \{-1, 0, 1\}$. Consequently, the set of all clear elements in the ring $\mathbb{Z}$ is $Clr(\mathbb{Z}) = \{-2, -1, 0, 1, 2\}$. Based on the set $Clr(\mathbb{Z})$, we obtain $-2 \in Clr(\mathbb{Z})$ but $1 - (-2) = 3 \notin Clr(\mathbb{Z})$.

Furthermore, in a clean ring, the statement "If x is a clean element in R, then $-x$ is a clean element in R" does not apply. However, in a clear ring, the statement "If x is a clear element in R, then $-x$ is a clear element in R" applies.

Proposition 3.1. *Let $(R, +, \cdot)$ be a ring. If x is a clear element of R, then $-x$ is also a clear element of R.*

Proof. Let $x \in Clr(R)$. It means $x = u + v$, for some $u \in U(R)$ and $v \in U_{reg}(R)$. Since $u \in U(R)$, we have $ut = tu = 1_R$, for some $t \in R$ and since $v \in U_{reg}(R)$, we have $vyv = v$, for some $y \in U(R)$. Next, consider the following decomposition:

$$-x = -u + (-v). \tag{3.1}$$

It is clear that $-u \in U(R)$ since $(-u)(-t) = ut = tu = (-t)(-u) = 1_R$, for some $-t \in R$ and also $-v \in U_{reg}(R)$ since there exists $-y \in U(R)$ such that $(-v)(-y)(-v) = -vyv = -v$. Thus, $-x \in Clr(R)$. □

The following presents the properties of the clear element in a ring related to the quasi-regular element.

Proposition 3.2. *Let $(R, +, \cdot)$ be a ring. If x is a quasi-regular element of R, then x is also a clear element of R.*

Proof. Let $x \in R$ and x is a quasi-regular element. Therefore, by Proposition 2.17, we have $1_R - x \in U(R)$. Since $1_R - x \in U(R)$, then $x - 1_R \in U(R)$. Note that:

$$x = (x - 1_R) + 1_R, \tag{3.2}$$

for some $x - 1_R \in U(R)$ and $1_R \in U_{reg}(R)$. Thus, $x \in Clr(R)$. □

But, the converse of Proposition 3.2 is not true. As a counterexample, consider Example 3.3.

Example 3.3. In the ring $(\mathbb{Z}_4, +_4, \cdot_4)$, the element $\overline{3} \in Clr(\mathbb{Z}_4)$, but $\overline{3}$ is not a quasi-regular element because there does not exist $\overline{y} \in \mathbb{Z}_4$ such that $\overline{3} \cdot_4 \overline{y} = \overline{3} +_4 \overline{y}$.

Proposition 3.4. *Let $(R, +, \cdot)$ be a ring. If x is a strongly regular element of R, then x is also a clear element of R.*

Proof. Let $x \in R$ and x is a strongly regular element. Then, $x = xyx$ and $xy = yx$, for some $y \in R$. Next, consider the following decomposition:

$$x = (x - (1_R - xy)) + (1_R - xy). \tag{3.3}$$

Note that:

$$\begin{aligned}(x - (1_R - xy))(yxy - (1_R - xy)) &= xyxy - x + xxy - yxy + xyyxy + 1_R - xy,\\ &= xy - x + xyx - yxy + xyxyy + 1_R - xy,\\ &= xy - x + x - yxy + yxy + 1_R - xy,\\ &= 1_R,\\ (yxy - (1_R - xy))(x - (1_R - xy)) &= yxyx - yxy + yxyxy - x + xyx + 1_R - xy,\\ &= yx - yxy + yxy - x + x + 1_R - xy,\\ &= yx - yxy + yxy - x + x + 1_R - yx,\\ &= 1_R,\end{aligned}$$

and

$$\begin{aligned}(1_R - xy)1_R(1_R - xy) = (1_R - xy)(1_R - xy), &= 1_R - xy - xy + xyxy,\\ &= 1_R - xy - xy + xy,\\ &= 1_R - xy.\end{aligned}$$

As a result, we have $x - (1_R - xy) \in U(R)$ and $1_R - xy \in U_{reg}(R)$. Therefore, $x \in Clr(R)$. □

But, the converse of Proposition 3.4 is not true. As a counterexample, consider Example 3.5.

Example 3.5. In the ring $(\mathbb{Z}_4, +_4, \cdot_4)$, the element $\overline{2} \in Clr(\mathbb{Z}_4)$, but $\overline{2}$ is not a strongly regular element because there does not exist $\overline{y} \in \mathbb{Z}_4$ such that $\overline{2} \cdot_4 \overline{y} \cdot_4 \overline{2} = \overline{2}$.

3.2. *Clear Properties in Certain Special Rings*

After discussing the advanced properties of the clear element, we will now examine the properties of clarity in several types of special rings. The first discussion focuses on the properties of clear rings related to the opposite ring, as presented in the following proposition.

Proposition 3.6. *Let $(R, +, \cdot)$ be a ring. A ring R is a clear ring if and only if R^{op} is a clear ring.*

Proof. Let $(R,+,\cdot)$ be a ring.

($\Rightarrow$) Assume that R is a clear ring. We will prove that R^{op} is a clear ring.
Let $x \in R^{op}$. Since each element of R^{op} is also an element of R, we have $x \in R$ and $x = u + v$, for some $u \in U(R)$ and $v \in U_{reg}(R)$. Next, we will prove that $u \in U(R^{op})$ and $v \in U_{reg}(R^{op})$. Since $u \in U(R)$, we have $ut = tu = 1_R$, for some $t \in R$ and since $v \in U_{reg}(R)$, we have $vyv = v$, for some $y \in U(R)$. Note that:

$$u * t = tu = 1_R = ut = t * u \quad \text{and} \quad v * y * v = v * vy = vyv = v.$$

It is clear that $u \in U(R^{op})$ and $v \in U_{reg}(R^{op})$. Thus, $r \in Clr(R^{op})$.

($\Leftarrow$) The proof is analogous to the previous one.

Thus, we have proved that the ring R is a clear ring if and only if R^{op} is a clear ring. $\square$

The following is a proposition regarding the properties of clear rings related to the quotient ring.

Proposition 3.7. *Let $(R,+,\cdot)$ be a ring and an ideal $I \subseteq R$. If R is a clear ring, then the quotient ring R/I is also a clear ring.*

Proof. Let $(R,+,\cdot)$ be a clear ring and an ideal $I \subseteq R$. We will prove that R/I is also a clear ring. Let $\overline{x} = x + I \in R/I$, where $x \in R$. Since R is a clear ring, we have $x = u + v$, for some $u \in U(R)$ and $v \in U_{reg}(R)$. Next, since $u \in U(R)$, we have $ut = tu = 1_R$, for some $t \in R$ and $v \in U_{reg}(R)$, we have $vyv = v$, for some $y \in U(R)$. Consider the following decomposition:

$$x + I = (u + v) + I = (u + I) + (v + I). \tag{3.4}$$

It is clear that $(u+I) \in U\left(R/I\right)$ since $(u+I)(t+I) = (t+I)(u+I) = 1_R + I$, for some $t+I \in R/I$ and that $v+I \in U_{reg}\left(R/I\right)$ since there exists $y+I \in U\left(R/I\right)$ such that $(v+I)(y+I)(v+I) = vyv + I = v + I$. Thus, $\overline{x} \in Clr\left(R/I\right)$. $\square$

But, the converse of Proposition 3.7 is not true. As a counterexample, consider Example 3.8.

Example 3.8. Let $(\mathbb{Z},+,\cdot)$ be the ring of integers and $(6\mathbb{Z},+,\cdot)$ its ideal. By the Fundamental Homomorphism Theorem for rings, we have $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$ and $(\mathbb{Z}_6,+_6,\cdot_6)$ is a clear ring, implies that $\mathbb{Z}/6\mathbb{Z}$ is a clear ring, while $(\mathbb{Z},+,\cdot)$ is not a clear ring.

Proposition 3.9. *Let $(R,+,\cdot)$ and $(S,+,\cdot)$ be rings and $\zeta : R \longrightarrow S$ be a ring epimorphism. A ring S is a clear ring if and only if $R/\ker(\zeta)$ is a clear ring.*

Proof. Let $(R,+,\cdot)$ and $(S,+,\cdot)$ be rings and $\zeta : R \longrightarrow S$ be a ring epimorphism. Hence $\mathrm{Im}(\zeta) = S$. Next, by the Fundamental Homomorphism Theorem for rings, we have:

$$R \big/ \ker(\zeta) \cong S.$$

Therefore, we conclude that the ring S is a clear ring if and only if $R \big/ \ker(\zeta)$ is a clear ring. □

Next, we investigate a property of clear rings related to corner rings.

Proposition 3.10. *Let $(R,+,\cdot)$ be an abelian ring and $e \in Id(R)$. If R is a clear ring, then eRe is also a clear ring.*

Proof. Let $(R,+,\cdot)$ be an abelian ring, $e \in Id(R)$, and R is a clear ring. Define a function $\zeta : R \longrightarrow eRe$ by $\zeta(r) = ere$, for each $r \in R$. Let $r, r' \in R$. We will prove that ζ is an epimorphism. Note that:

$$\zeta(r + r') = e(r + r')e = ere + er'e = \zeta(r) + \zeta(r'),$$
$$\zeta(rr') = e(rr')e = e(rr')ee = er(r'e)e = er(er')e = (ere)(er'e) = \zeta(r)\zeta(r'),$$

and for every $ere \in eRe$, there exists $r \in R$ such that $\zeta(r) = ere$. Therefore, ζ is a ring epimorphism. Thus, eRe is a clear ring. □

Proposition 3.11. *Let $(R,+,\cdot)$ be an abelian ring and $e \in Id(R)$. A ring R is a clear ring if and only if eRe and $(1_R - e)R(1_R - e)$ are clear rings.*

Proof. Let $(R,+,\cdot)$ be an abelian ring and $e \in Id(R)$. Then, $eRe = eR$ and $(1_R - e)R(1_R - e) = (1_R - e)R$.

(1) We will prove that $R = eR + (1_R - e)R$.
It is clear that $eR + (1_R - e)R \subseteq R$. We will prove that $R \subseteq eR + (1_R - e)R$. Let $r \in R$, then

$$r = er + (1_R - e)r,$$

for some $er \in eR$ and $(1_R - e)r \in (1_R - e)R$. Thus, $r \in eR + (1_R - e)R$. Therefore, $R = eR + (1_R - e)R$.

(2) We will prove that $eR \cap (1_R - e)R = \{0_R\}$.
For any $x \in eR \cap (1_R - e)R$, then $x \in eR$ and $x \in (1_R - e)R$. Since $x \in eR$, then $x = er$ and $x \in (1_R - e)R$, then $x = (1_R - e)r'$, for some $r, r' \in R$. Note that:

$$\begin{aligned} er = (1_R - e)r', &\Longrightarrow eer = e(1_R - e)r', \\ &\Longrightarrow e^2 r = (e - e^2)r,' \\ &\Longrightarrow er = 0_R r', \\ &\Longrightarrow er = 0_R. \end{aligned}$$

Since $x = er = 0_R$ for an arbitrary element $x \in eR \cap (1_R - e)R$, thus $eR \cap (1_R - e)R = \{0_R\}$.

Therefore, $R = eRe \oplus (1_R - e)R(1_R - e)$. It follows that, $R \cong eRe \oplus (1_R - e)R(1_R - e)$. It has been proved that the ring R is a clear ring if and only if eRe and $(1_R - e)R(1_R - e)$ are clear rings. □

Considering that $e^2 = e$, $(1_R - e)^2 = 1_R - 2e + e = 1_R - e$, and $e(1_R - e) = e - e^2 = 0_R$. We conclude that e and $1_R - e$ are orthogonal idempotent elements in R. In the following, we generalize Proposition 3.11 for the case of n orthogonal idempotent elements in R.

Proposition 3.12. *Let $(R, +, \cdot)$ be an abelian ring and $e_1, e_2, \cdots, e_n$ are orthogonal central idempotents in R with $\sum_{i=1}^{n} e_i = 1_R$. A ring R is a clear ring if and only if e_iRe_i are clear rings, for each $i = 1, 2, \cdots, n$.*

Proof. Since $e_1 + e_2 + \cdots + e_n = 1_R$, then every $r \in R$ can be written as

$$r = r \cdot 1_R = r(e_1 + e_2 + \cdots + e_n) = re_1 + re_2 + \cdots + re_n.$$

As a result, $R = Re_1 + Re_2 + \cdots + Re_n$. Moreover, since $e_1, e_2, \cdots, e_n$ are orthogonal idempotent elements in R, it follows that $R = e_1Re_1 + e_2Re_2 + \cdots + e_nRe_n$. Let $x \in e_iRe_i \cap \sum_{j \neq i} e_jRe_j$, then $x = e_ire_i$ and $x = \sum_{j \neq i} e_jr_je_j$. So, $e_ire_i = \sum_{j \neq i} e_jr_je_j$. By multiplying e_i on the left, we obtain $e_ie_ire_i = e_i \sum_{j \neq i} e_jr_je_j \Longrightarrow e_ire_i = 0_R$. As a result $e_iRe_i \cap \sum_{j \neq i} e_jRe_j = \{0_R\}$. Thus, $R = e_1Re_1 \oplus e_2Re_2 \oplus \cdots \oplus e_nRe_n$. It has been proved that the ring R is a clear ring if and only if e_iRe_i are clear rings, for each $i = 1, 2, \cdots, n$. □

Next, the following proposition presents a property of clear rings related to diagonal matrix rings denoted by $D_n(R)$. In this paper, $\text{diag}(r_1, r_2, \cdots, r_n)$ denotes an $n \times n$ diagonal matrix whose main diagonal entries are $r_1, r_2, \cdots, r_n \in R$.

Proposition 3.13. *Let $(R, +, \cdot)$ be a ring. A ring R is a clear ring if and only if $D_n(R)$ is a clear ring.*

Proof. Let $(R, +, \cdot)$ be a ring. Define a function:

$$\zeta : R^n \longrightarrow D_n(R)$$
$$(r_1, r_2, \cdots, r_n) \longmapsto \text{diag}(r_1, r_2, \cdots, r_n),$$

for each $(r_1, r_2, \cdots, r_n) \in R^n$. Let $(r_1, r_2, \cdots, r_n), (r_1', r_2', \cdots, r_n') \in R^n$. We will prove that ζ is an isomorphism. Note that:

$$\begin{aligned}
\zeta((r_1, r_2, \cdots, r_n) + (r'_1, r'_2, \cdots, r'_n)) &= \zeta((r_1 + r'_1, r_2 + r'_2, \cdots, r_n + r'_n)), \\
&= \mathrm{diag}(r_1 + r'_1, r_2 + r'_2, \cdots, r_n + r'_n), \\
&= \mathrm{diag}(r_1, r_2, \cdots, r_n) + \mathrm{diag}(r'_1, r'_2, \cdots, r'_n), \\
&= \zeta((r_1, r_2, \cdots, r_n)) + \zeta((r'_1, r'_2, \cdots, r'_n)), \\
\zeta((r_1, r_2, \cdots, r_n)(r'_1, r'_2, \cdots, r'_n)) &= \zeta((r_1 r'_1, r_2 r'_2, \cdots, r_n r'_n)), \\
&= \mathrm{diag}(r_1 r'_1, r_2 r'_2, \cdots, r_n r'_n), \\
&= \mathrm{diag}(r_1, r_2, \cdots, r_n)\mathrm{diag}(r'_1, r'_2, \cdots, r'_n), \\
&= \zeta((r_1, r_2, \cdots, r_n))\zeta((r'_1, r'_2, \cdots, r'_n)), \\
\ker(\zeta) &= \{(r_1, r_2, \cdots, r_n) \in R^n \mid \zeta((r_1, r_2, \cdots, r_n)) = 0_{n\times n}\}, \\
&= \{(0_R, 0_R, \cdots, 0_R)\},
\end{aligned}$$

and for every $\mathrm{diag}(r_1, r_2, \cdots, r_n) \in D_n(R)$, there exists $(r_1, r_2, \cdots, r_n) \in R^n$ such that $\zeta((r_1, r_2, \cdots, r_n)) = \mathrm{diag}(r_1, r_2, \cdots, r_n)$. Therefore, ζ is a ring isomorphism or $R^n \cong D_n(R)$. Thus, it has been proved that the ring R is a clear ring if and only if $D_n(R)$ is a clear ring. □

Next, the following proposition presents a property of clear rings related to triangular matrix rings denoted by $T_n(R)$.

Proposition 3.14. *Let $(R, +, \cdot)$ be a ring. If $T_n(R)$ is a clear ring, then R is also a clear ring.*

Proof. Let $(R, +, \cdot)$ be a ring. Suppose that $T_n(R)$ is a clear ring. Define a function $\zeta : T_n(R) \longrightarrow R$ by $\zeta([a]_{ij}) = a_{11}$, for each $[a]_{ij} \in T_n(R)$. Let $[a]_{ij}, [b]_{ij} \in T_n(R)$. We will prove that ζ is an epimorphism. Note that:

$$\begin{aligned}
\zeta([a]_{ij} + [b]_{ij}) &= \zeta([a + b]_{ij}) = a_{11} + b_{11} = \zeta([a]_{ij}) + \zeta([b]_{ij}), \\
\zeta([a]_{ij}[b]_{ij}) &= a_{11}b_{11} = \zeta([a]_{ij})\zeta([b]_{ij}),
\end{aligned}$$

and for every $y \in R$, there exists $X = \begin{pmatrix} y & 0_R & \cdots & 0_R \\ 0_R & 0_R & \cdots & 0_R \\ \vdots & \vdots & \ddots & \vdots \\ 0_R & 0_R & \cdots & 0_R \end{pmatrix} \in T_n(R)$ such that $\zeta(X) = y$. Therefore, ζ is a ring epimorphism. Thus, R is a clear ring. □

Next, we investigate a property of clear rings related to Dorroh ideals. A study on Dorroh ideals can be found in [13,14]. Let $(R, +, \cdot)$ be a ring and B an (R, R)-bimodule such that for every $a, b \in B$ and $r \in R$, we have:

$$(ab)r = a(br), \quad (ar)b = a(rb), \quad \text{and} \quad (ra)b = r(ab).$$

Then the set $I(R, B) = R \oplus B$ forms a ring under pointwise addition and multiplication defined by

$$(r, a)(s, b) = (rs, rb + as + ab).$$

Proposition 3.15. *Let $(R, +, \cdot)$ be a ring and B an (R, R)-bimodule. If $I(R, B)$ is a clear ring, then R is also a clear ring.*

Proof. Let $(R, +, \cdot)$ be a ring and B an (R, R)-bimodule. Assume that $I(R, B)$ is a clear ring. Define a function $\zeta : I(R, B) \longrightarrow R$ by $\zeta((r, b)) = r$, for each $(r, b) \in I(R, B)$. Let $(r, b), (r', b') \in I(R, B)$. We will prove that ζ is an epimorphism. Note that:

$$\zeta((r, b) + (r', b')) = \zeta((r + r', b + b')) = r + r' = \zeta((r, b)) + \zeta((r', b')),$$
$$\zeta((r, b)(r', b')) = \zeta((rr', rb' + br' + bb')) = rr' = \zeta((r, b))\zeta((r', b')),$$

and for every $r \in R$, there exists $(r, 0_B) \in I(R, B)$ such that $\zeta((r, 0_B)) = r$. Therefore, ζ is a ring epimorphism. Thus, R is a clear ring. □

However, the converse of Proposition 3.15 does not hold. The following provides a necessary condition for the converse of Proposition 3.15 to be valid.

Proposition 3.16. *Let $(R, +, \cdot)$ be a ring and B an (R, R)-bimodule. If R is a clear ring and for every $x, y \in R$ and $b \in B$ it follows that $byx + byb + xyb = b$, then $I(R, B)$ is a clear ring.*

Proof. Let $(R, +, \cdot)$ be a ring and B an (R, R)-bimodule. Assume that R is a clear ring and for every $x, y \in R$ and $b \in B$ it follows that $byx + byb + xyb = b$. We will prove that $I(R, B)$ is a clear ring. Let $(r, b) \in I(R, B)$. Since R is a clear ring, then $r = u + v$, for some $u \in U(R)$ and $v \in U_{reg}(R)$. Next, since $u \in U(R)$, we have $ut = tu = 1_R$, for some $t \in R$ and $v \in U_{reg}(R)$, we have $vyv = v$, for some $y \in U(R)$. Consider the following decomposition:

$$(r, b) = (u, 0_B) + (v, b). \tag{3.5}$$

It is clear that $(u, 0_B) \in U(I(R, B))$ since:

$$(u, 0_B)(t, 0_B) = (ut, 0_B) = (1_R, 0_B) = (tu, 0_B) = (t, 0_B)(u, 0_B),$$

for some $(t, 0_B) \in I(R, B)$ and $(v, b) \in U_{reg}(I(R, B))$ since there exists $(y, 0_B) \in U(I(R, B))$ such that

$$(v, b)(y, 0_B)(v, b) = (vyv, vyb + byv + byb) = (v, b).$$

Thus, $(r, b) \in Clr(I(R, B))$. □

Next, we investigate a property of clear rings related to Morita context rings. A study on Morita context rings can be found in [15]. Let $(R, +, \cdot)$ and $(S, +, \cdot)$ be rings, B an (R, S)-bimodule, C an (S, R)-bimodule, and let $\varphi : B \otimes_S C \longrightarrow R$ and $\psi : C \otimes_R B \longrightarrow S$ be bimodule homomorphisms. The set:

$$\Lambda_{(\varphi,\psi)} = \left\{ \begin{pmatrix} r & b \\ c & s \end{pmatrix} \mid r \in R, b \in B, c \in C, s \in S \right\}$$

is a ring under matrix addition and multiplication defined by:

$$\begin{pmatrix} r & b \\ c & s \end{pmatrix} \begin{pmatrix} r' & b' \\ c' & s' \end{pmatrix} = \begin{pmatrix} rr' + \varphi(b \otimes c') & rb' + bs' \\ cr' + sc' & ss' + \psi(c \otimes b') \end{pmatrix}.$$

The ring $\Lambda_{(\varphi,\psi)}$ is called the ring over a Morita context (Morita ring).

Proposition 3.17. *Let* $\Lambda_{(0,0)} = \begin{pmatrix} R & B \\ C & S \end{pmatrix}$ *be the Morita ring with two zero pairings. A ring* $\Lambda_{(0,0)}$ *is a clear ring if and only if* R *and* S *are clear rings.*

Proof. Let $\Lambda_{(0,0)} = \begin{pmatrix} R & B \\ C & S \end{pmatrix}$ be the Morita ring with two zero pairings.

($\Rightarrow$) Assume that $\Lambda_{(0,0)}$ is a clear ring. We will prove that R and S are clear rings. Define functions:

$$\begin{aligned} \zeta : \Lambda_{(0,0)} &\longrightarrow R \\ \begin{pmatrix} r & b \\ c & s \end{pmatrix} &\longmapsto r, \text{ for every } \begin{pmatrix} r & b \\ c & s \end{pmatrix} \in \Lambda_{(0,0)} \end{aligned}$$

and

$$\begin{aligned} \eta : \Lambda_{(0,0)} &\longrightarrow S \\ \begin{pmatrix} r & b \\ c & s \end{pmatrix} &\longmapsto s, \text{ for every } \begin{pmatrix} r & b \\ c & s \end{pmatrix} \in \Lambda_{(0,0)}. \end{aligned}$$

We will prove that ζ and η are ring epimorphisms. Let $\begin{pmatrix} r & b \\ c & s \end{pmatrix}, \begin{pmatrix} r' & b' \\ c' & s' \end{pmatrix} \in \Lambda_{(0,0)}$. Note that:

$$\begin{aligned} \zeta\left(\begin{pmatrix} r & b \\ c & s \end{pmatrix} + \begin{pmatrix} r' & b' \\ c' & s' \end{pmatrix}\right) &= \zeta\left(\begin{pmatrix} r+r' & b+b' \\ c+c' & s+s' \end{pmatrix}\right), \\ &= r + r', \\ &= \zeta\left(\begin{pmatrix} r & b \\ c & s \end{pmatrix}\right) + \zeta\left(\begin{pmatrix} r' & b' \\ c' & s' \end{pmatrix}\right), \\ \zeta\left(\begin{pmatrix} r & b \\ c & s \end{pmatrix}\begin{pmatrix} r' & b' \\ c' & s' \end{pmatrix}\right) &= \zeta\left(\begin{pmatrix} rr' & rb'+bs' \\ cr'+sc' & ss' \end{pmatrix}\right), \\ &= rr', \\ &= \zeta\left(\begin{pmatrix} r & b \\ c & s \end{pmatrix}\right)\zeta\left(\begin{pmatrix} r' & b' \\ c' & s' \end{pmatrix}\right), \end{aligned}$$

and for every $r \in R$, there exists $\begin{pmatrix} r & 0_B \\ 0_C & 0_S \end{pmatrix} \in \Lambda_{(0,0)}$ such that $\zeta\left(\begin{pmatrix} r & 0_B \\ 0_C & 0_S \end{pmatrix}\right) = r$. Therefore, ζ is a ring epimorphism. It is easy to show that η is a ring epimorphism. Thus, R and S are clear rings.

($\Leftarrow$) Assume that R and S are clear rings. We will prove that $\Lambda_{(0,0)} = \begin{pmatrix} R & B \\ C & S \end{pmatrix}$ is a clear ring. Let $\begin{pmatrix} r & b \\ c & s \end{pmatrix} \in \Lambda_{(0,0)}$. It means $r \in R$, $b \in B$, $c \in C$, and $s \in S$. Since R and S are clear rings, then $r = u_1 + v_1$ and $s = u_2 + v_2$, for some $u_1 \in U(R)$, $v_1 \in U_{reg}(R)$, $u_2 \in U(S)$, and $v_2 \in U_{reg}(S)$. Based on the following facts:

(a) for an element $u_1 \in U(R)$, there exists $t_1 \in R$ such that $u_1t_1 = t_1u_1 = 1_R$,
(b) for an element $v_1 \in U_{reg}(R)$, there exists $y_1 \in U(R)$ such that $v_1y_1v_1 = v_1$,
(c) for an element $u_2 \in U(S)$, there exists $t_2 \in S$ such that $u_2t_2 = t_2u_2 = 1_S$, and
(d) for an element $v_2 \in U_{reg}(S)$, there exists $y_2 \in U(S)$ such that $v_2y_2v_2 = v_2$.

Next, consider the following decomposition:

$$\begin{pmatrix} r & b \\ c & s \end{pmatrix} = \begin{pmatrix} u_1 & b \\ c & u_2 \end{pmatrix} + \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}. \tag{3.6}$$

Note that:

$$\begin{pmatrix} u_1 & b \\ c & u_2 \end{pmatrix} \begin{pmatrix} t_1 & -t_1bt_2 \\ -t_2ct_1 & t_2 \end{pmatrix} = \begin{pmatrix} 1_R & 0_B \\ 0_C & 1_S \end{pmatrix} = \begin{pmatrix} t_1 & -t_1bt_2 \\ -t_2ct_1 & t_2 \end{pmatrix} \begin{pmatrix} u_1 & b \\ c & u_2 \end{pmatrix}$$

and

$$\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}.$$

As a result, we have $\begin{pmatrix} u_1 & b \\ c & u_2 \end{pmatrix} \in U(\Lambda_{(0,0)})$ and $\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \in U_{reg}(\Lambda_{(0,0)})$. Hence, $\Lambda_{(0,0)}$ is a clear ring. □

Corollary 3.18. *A ring* $T = \begin{pmatrix} R & B \\ 0 & S \end{pmatrix}$ *is a clear ring if and only if* R *and* S *is also a clear ring.*

Let $(R, +, \cdot)$ be a ring and G a group. The set of all formal linear combinations of elements of G with coefficients in R is defined as:

$$R[G] = \left\{ \sum_{g \in G} r_g g \,\middle|\, r_g \in R, \text{ and } r_g \neq 0 \text{ for only finitely many } g \in G \right\}.$$

The set $R[G]$ is a ring under componentwise addition and convolution multiplication and is called the group ring of G over R. Next, we investigate a property of clear rings related to group rings. However, in this study, the discussion is restricted to the case of finite groups, specifically to the group $G = \{e_G, g\}$.

Proposition 3.19. *Let* $(R, +, \cdot)$ *be a ring with* $2 \in U(R)$ *and* $G = \{e_G, g\}$ *be a group with order 2. A ring* $R[G]$ *is a clear ring if and only if* R *is also a clear ring.*

Proof. Let $(R, +, \cdot)$ be a ring with $2 \in U(R)$ and $G = \{e_G, g\}$ be a group with order 2.

($\Rightarrow$) Assume that $R[G]$ is a clear ring. We will prove that R is a clear ring. Define a function $\zeta : R[G] \longrightarrow R$ by $\zeta(re_G + r'g) = r + r'$, for each $re + G + r'g \in R[G]$. We will prove that ζ is a ring epimorphism. Let $r_1e + G + r_1'g, r_2e_G + r_2'g \in R[G]$.

Note that:

$$\begin{aligned}
\zeta((r_1e_G + r_1'g) + (r_2e_G + r_2'g)) &= \zeta((r_1 + r_2)e_G + (r_1' + r_2')g), \\
&= r_1 + r_2 + r_1' + r_2', \\
&= r_1 + r_1' + r_2 + r_2', \\
&= \zeta(r_1e_G + r_1'g) + \zeta(r_2e_G + r_2'g), \\
\zeta((r_1e_G + r_1'g)(r_2e_G + r_2'g)) &= \zeta((r_1r_2 + r_1'r_2')e_G + (r_1r_2' + r_1'r_2)g), \\
&= r_1r_2 + r_1'r_2' + r_1r_2' + r_1'r_2, \\
&= (r_1 + r_1')(r_2 + r_2'), \\
&= \zeta(r_1e_G + r_1'g)\,\zeta(r_2e_G + r_2'g),
\end{aligned}$$

and for every $y \in R$, there exists $ye_G + 0_Rg \in R[G]$ such that $\zeta(ye_G + 0_Rg) = y$. Therefore, ζ is a ring epimorphism. Thus, R is a clear ring.

($\Leftarrow$) Assume that R is a clear ring. We will prove that $R[G]$ is a clear ring. Define a function $\eta : R[G] \longrightarrow R \times R$ by $\eta(re_G + r'g) = (r + r', r - r')$, for each $re_G + r'g \in R[G]$. We will prove that η is a ring isomorphism. Let $r_1e_G + r_1'g, r_2e_G + r_2'g \in R[G]$. Note that:

$$\begin{aligned}
\eta((r_1e_G + r_1'g) + (r_2e_G + r_2'g)) &= \eta((r_1 + r_2)e_G + (r_1' + r_2')g), \\
&= ((r_1 + r_2) + (r_1' + r_2'), (r_1 + r_2) - (r_1' + r_2')), \\
&= (r_1 + r_1' + r_2 + r_2', r_1 - r_1' + r_2 - r_2'), \\
&= (r_1 + r_1', r_1 - r_1') + (r_2 + r_2', r_2 - r_2'), \\
&= \eta(r_1e_G + r_1'g) + \eta(r_2e_G + r_2'g), \\
\eta((r_1e_G + r_1'g)(r_2e_G + r_2'g)) &= \eta((r_1r_2 + r_1'r_2')e_G + (r_1r_2' + r_1'r_2)g), \\
&= ((r_1r_2 + r_1'r_2') + (r_1r_2' + r_1'r_2), (r_1r_2 + r_1'r_2') - (r_1r_2' + r_1'r_2)), \\
&= ((r_1 + r_1')(r_2 + r_2'), (r_1 - r_1')(r_2 - r_2')), \\
&= \eta(r_1e_G + r_1'g)\,\eta(r_2e_G + r_2'g), \\
\ker(\eta) &= \{re_G + r'g \in R[G] \mid \eta(re_G + r'g) = (0_R, 0_R)\}, \\
&= \{re_G + r'g \in R[G] \mid (r + r', r - r') = (0_R, 0_R)\}, \\
&= \{0_{R[G]}\},
\end{aligned}$$

and for every $(r, r') \in R \times R$, there exists $2^{-1}(r + r')e_G + 2^{-1}(r - r')g \in R[G]$ such that:

$$\begin{aligned}
\eta(2^{-1}(r + r')e_G + 2^{-1}(r - r')g) &= (2^{-1}(r + r') + 2^{-1}(r - r'), 2^{-1}(r + r') - 2^{-1}(r - r')), \\
&= (r, r').
\end{aligned}$$

Therefore, η is a ring isomorphism. Since R is a clear ring, by Proposition 2.13, we have $R \times R$ is a clear ring. Moreover, since $R[G] \cong R \times R$, we conclude that $R[G]$ is also a clear ring.

Thus, it is proved that the ring $R[G]$ is a clear ring if and only if R is also a clear ring. □

Next, we investigate a property of clear rings related to the set of all endomorphisms of the module R over R.

Proposition 3.20. *Let $(R, +, \cdot)$ be a ring. A ring R is a clear ring if and only if $End_R(R)$ is also a clear ring.*

Proof. Let $(R, +, \cdot)$ be a ring. Define a function $\zeta : R \longrightarrow \text{End}_R(R)$ by $\zeta(r) = \eta_r$ and $\eta_r(x) = rx$, for each $r, x \in R$. It is clear that ζ is an isomorphism. Then, $R \cong \text{End}_R(R)$. Therefore, we conclude that ring R is a clear ring if and only if $\text{End}_R(R)$ is also a clear ring. □

4. Conclusion

In this paper, we have proven three new properties of clear elements in a ring. The properties state that if a certain condition is satisfied, then an element in a ring is a clear element. We also have proven clear properties in certain special rings, such as the opposite ring, the quotient ring, the corner ring, the Morita ring, and the group ring. The propositions and corollary state that certain special rings are clear under certain conditions.

For future research, the result of this paper can be explored for other algebraic structures, such as modules, algebras, coalgebras, and comodules. Besides that, research about clean ring can also be explored under tensor product. The research could investigate whether the tensor product of two clear rings is necessarily clear and find sufficient conditions such a property holds.

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