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OPTIMAL BONUS-MALUS PREMIUM WITH THE CLAIM FREQUENCY DISTRIBUTION IS NEGATIVE BINOMIAL AND THE CLAIM SEVERITY DISTRIBUTION IS TRUNCATED WEIBULL

IKHSAN MAULIDI[a,*], IZZAH MAULIDIA[a], RADHIAH[a], VINA APRILIANI[b]
[a] *Department of Mathematics, Universitas Syiah Kuala, Banda Aceh, Indonesia,*
[b] *Department of Mathematics Education, Universitas Islam Negeri Ar-Raniry, Banda Aceh, Indonesia.*
email : ikhsanmaulidi@usk.ac.id

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Abstract. *The optimal bonus-malus system is a system for determining the amount of premium in the next period based on the frequency and severity of claims filed by policyholders in the previous period. In this article, we study the insurance premium formula using the optimal bonus-malus system with the claim frequency has a negative binomial distribution and the claim severity has a truncated Weibull distribution. The method to derive the bonus-malus formula uses a Bayes solution with a quadratic loss function. The formula obtained has been also applied to the data of motor vehicle insurance to calculate the risk premium that must be paid by the policyholder. From the calculation results, the insurance premiums that use the optimal bonus-malus system with a truncated Weibull distribution are more profitable for both the company and the insurer than the Weibull distribution.*

Keywords: Negative Binomial, optimal Bonus-Malus system, risk premium, truncated Weibull.

1. Introduction

Insurance is an agreement mechanism offered by an insurance company to insurance users (policyholders) with the aim of minimizing the risk of loss. The insurance company promises to pay compensation to the policyholder with a certain amount, if the policyholder filed a claim (a loss that must be paid by the company) within the timeframe set by both parties. On the other hand, policyholders are required to pay a premium (monthly fee) to the insurance company in exchange for guaranteed financial protection provided by the insurance company. Motor vehicle insurance is

*Corresponding author

a branch of general insurance. One system that is often used in determining the premium of motor vehicle insurance is the bonus-malus system.

The bonus-malus system (BMS) is a system used to determine the risk premium price based on the claim history of each policyholder. Bonus will be given to policyholders in the form of reduced premium prices if there are no claims filed in the previous year, while malus will be given to policyholders in the form of increased premium prices if there have been claims in the previous year [1,2]. The bonus-malus system is said to be optimal if it is financially balanced for the insurance company and fair for the policyholders. The bonus-malus system used in this study is the optimal bonus-malus system in which the premium is determined based on the frequency and severity of claims filed by policyholders.

The BMS has been extensively studied in actuarial science, particularly in the context of determining motor vehicle insurance premiums. Previous research has investigated various combinations of claim frequency and severity distributions. For instance, the Pareto distribution for claim severity was analyzed in [3]. The Weibull distribution for severity has also been adopted in BMS models, as demonstrated in [4], with results highlighting its flexibility in modeling heterogeneous risks. Recent work in [5] introduced a dependency model between claim frequency and severity using bivariate random effects, addressing the limitations of traditional systems that focus solely on frequency.

However, in practical insurance applications, claim severity is often subject to policy limits and deductibles, resulting in truncated distributions. Ignoring truncation can lead to biased premium estimates and mispricing of risk, particularly for high-severity claims. Incorporating truncated severity models ensures that the Bonus-Malus System accurately reflects contractual conditions and improves fairness and financial stability from an actuarial perspective.

Integrating claim severity into the Bonus-Malus System (BMS) is becoming increasingly important to address the 'hunger for bonus' phenomenon, where policyholders avoid filing minor claims to maintain their premium discounts [6,7]. Traditional BMS models primarily focus on claim frequency, which can unfairly penalize policyholders who report low-severity claims [6]. An effective system should account for both claim frequency and severity [7]. Further research [8] highlights the need for optimized premium structures that accurately reflect claim risks. However, most studies have yet to incorporate truncated severity distributions—a key modern actuarial tool for limiting extreme losses by capping maximum claim payouts [9].

The truncated Weibull distribution has gained prominence in insurance and risk management due to its ability to mitigate risk by imposing an upper limit on claim amounts, thereby reducing variability and enhancing portfolio stability [10,11]. Comparative studies, such as that in [12], highlight its superiority over traditional models like the Pareto distribution, particularly for heavy-tailed and heterogeneous claim data, as it offers more equitable premiums and better aligns with real-world loss patterns. Further supporting this, [13] and [14] emphasize the efficacy of truncated and mixed distributions in handling complex datasets, such as motor insurance, where diverse risk profiles necessitate more precise pricing and risk assessment strategies. Collectively, these findings underscore the truncated Weibull distribution's robustness in improving actuarial fairness, predictive accuracy, and decision-making in insurance modeling.

This study extends prior frameworks by combining a negative binomial distribution for claim frequency with a truncated Weibull distribution for claim severity. This approach enables more realistic risk modeling: negative binomial overdispersion accommodates frequency heterogeneity, while severity truncation mitigates corporate exposure to large claims. Similar findings were demonstratedin [15], who utilized a geometric distribution to model claim frequency. A Bayesian methodology with a quadratic loss function is employed to derive optimal premium formulas, building on the frameworks of [16] and [5]. Simulation results with real-world data demonstrate that this system not only enhances premium relativity fairness but also safeguards profitability.

2. Basic Consepts

2.1. *Negative Binomial Distribution*

A random variable X is said to have a negative binomial distribution with parameters α and τ if it has a probability density function:

$$P(X) = \binom{x+\alpha-1}{x}\left(\frac{1}{1+\tau}\right)^{x}\left(\frac{\tau}{1+\tau}\right)^{\alpha}, \quad x = 0, 1, 2, \cdots, \alpha > 0, \beta > 0.$$

The negative binomial distribution has an expected value of $E(X) = \frac{\alpha}{\tau}$ and a variance of $Var(X) = \frac{\alpha}{\tau}\left(1 + \frac{1}{\tau}\right)$.

The negative binomial distribution is one of the mixed distributions. The negative binomial distribution is a mixed distribution of the Poisson distribution and the Gamma distribution, [17].

2.2. *Truncated Weibull Distribution*

A continuous random variable X is said to have a Weibull distribution with parameters c and γ, if it has a probability density function:

$$f_X(x; c, \gamma) = \frac{\gamma}{c}\left(\frac{x}{c}\right)^{\gamma-1} e^{-\left(\frac{x}{c}\right)^{\gamma}}, x \geq 0,$$

and has a distribution function:

$$F(x) = 1 - e^{-cx^{\gamma}}, \quad x \geq 0.$$

If a random variable with a limit b is said to have a truncated Weibull distribution, then its probability distribution function is:

$$F(x) = \begin{cases} 1 - e^{-\left(\frac{x}{c^{-2}}\right)^{k}}, & 0 < x < b, \\ 1, & x \geq b. \end{cases}$$

The Weibull distribution serves as a foundational model from which many generalized distributions are derived. Its inherent flexibility facilitates extensions such as the exponentiated Weibull, beta-Weibull, and Weibull-Gamma distributions [18,19].

2.3. Bayes Solution

Suppose $\hat{\theta}$ is an estimator of the parameter θ under a quadratic loss function:

$$l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2,$$

where the expected loss (risk) is given by $E[L(\hat{\theta}, \theta)]$. The estimator $\widehat{\theta}$ is called a Bayesian solution if it minimizes the posterior expected loss:

$$E\left[L(\hat{\theta}, \theta) \mid Y = y\right] = \int_{-\infty}^{\infty} (\hat{\theta} - \theta)^2 \, k(\theta \mid y) \, d\theta. \tag{2.1}$$

Here, $k(\theta \mid y)$ is the posterior density of θ given the observed data $Y = y$. The minimizer of this expression is the posterior mean:

$$\widehat{\theta} = E[\theta \mid y],$$

which serves as the optimal Bayesian estimator under quadratic loss [20].

2.4. Modified Bessel Functions

Let $B_v(x)$ be a modified Bessel function. Then for every $x > 0$, $B_v(x)$ satisfies the following two properties:

$$\begin{aligned} B_v(x) &= B_{-v}(x), \\ B_{v+1}(x) &= \frac{2v}{x} B_v(x) + B_{v-1}(x), \end{aligned} \tag{2.2}$$

where $B_v(x) = \int_0^{\infty} e^{-\frac{x}{2}\left(y + \frac{1}{y}\right)} y^{v-1} \, dy$ [21].

2.5. Bonus-Malus System

The bonus-malus system is an insurance pricing mechanism that adjusts premiums based on a policyholder's claim history, ensuring fairness by differentiating between high-risk and low-risk individuals. Under this system, all policyholders initially pay the same base premium, but their subsequent payments are modified annually according to their claims record from the previous period. If a policyholder files a claim, a malus (penalty) is applied, increasing their premium for the next year as a reflection of higher perceived risk; conversely, those who do not file a claim receive a bonus (discount), reducing their premium as a reward for risk-free behavior. This dynamic adjustment aligns premiums more closely with individual risk profiles, making the system equitable by charging higher premiums to those with poor claim histories and lower premiums to those with favorable records [22].

The optimal bonus-malus system is a system in insurance that focuses on the classification of premium levels which is influenced by the claim frequency and the claim severity filed by policyholders each year. In the bonus-malus system, at the beginning of each period, policyholders pay the same amount of premium according to each category/level. If there is an extension for the next period, premium payments will change based on experience during the previous period.

In the optimal bonus-malus system, the formula for determining the risk premium is the multiplication of the expected value of the claim frequency and the expected value of the claim severity. Suppose K is a random variable of the claim frequency distribution under condition Λ, and X is a random variable of the claim severity distribution under condition Θ. If claim frequency and claim severity are assumed to be conditionally independent given their risk factors, then the formula of the optimal bonus-malus premium is given in [23] by:

$$\text{Premium}_{t+1} = E(\Lambda|K) \cdot E(\Theta|X). \tag{2.3}$$

The flowchart for the premium determination process using the Bonus–Malus method is presented in Figure 1.

3. Result and Discussion

3.1. *Bonus-Malus System Model*

3.1.1. *Claim Frequency Component*

Suppose K_i denotes the number of claims reported by a given policyholder during period i, for $i = 1, 2, \cdots, t$. Conditional on the risk parameter $\Lambda = \lambda$, the random variables $K_1, K_2, \cdots, K_t$ are assumed to be independent and identically distributed according to a Poisson distribution with mean λ. Let

$$K = \sum_{i=1}^{t} K_i$$

denote the total number of claims observed over t periods.

The unknown claim frequency parameter Λ is treated as a random variable. We assume that Λ follows a Gamma prior distribution with shape parameter $\alpha > 0$ and rate parameter $\tau > 0$. This choice is motivated by the conjugacy between the Poisson likelihood and the Gamma prior.

Under a squared error loss function

$$L(\lambda, \hat{\lambda}_{t+1}) = (\hat{\lambda}_{t+1} - \lambda)^2,$$

the Bayes estimator of the claim frequency for period $t+1$ is given by the posterior mean

$$\hat{\lambda}_{t+1} = E(\Lambda \mid K).$$

Using the Poisson likelihood and the Gamma prior, the posterior distribution of $\Lambda \mid K$ is again a Gamma distribution with updated parameters $\alpha + K$ and $\tau + t$. Consequently, the Bayes estimator takes the closed-form expression:

$$\hat{\lambda}_{t+1} = E(\Lambda \mid K) = \frac{K + \alpha}{t + \tau}. \tag{3.1}$$

The detailed derivation of the likelihood function, posterior distribution, and the computation of the posterior expectation is provided in Appendix A.

3.1.2. *Claim Severity Component*

Suppose X_i denotes the claim severity of a given policyholder in period i, for $i = 1, 2, \cdots, t$. The random variables X_i are assumed to be independent and identically distributed conditional on an unknown severity parameter $\Theta = \theta$.

For a given policyholder, K claims are observed. Among them, n claims satisfy $0 < X_i < b$, where $b > 0$ is the policy limit, while the remaining $K - n$ claims are censored at b. Let

$$N = \sum_{i=1}^{n} X_i$$

denote the aggregate severity of uncensored claims. Then, the total observed claim severity is

$$\sum_{i=1}^{K} X_i = N + b(K - n).$$

The unknown severity parameter Θ is treated as a random variable. Conditional on $\Theta = \theta$, the uncensored claim severities follow an exponential distribution, leading to a truncated Weibull-type likelihood structure. The prior distribution of Θ is assumed to follow a Lévy distribution with scale parameter $c > 0$.

Under the squared error loss function:

$$L(\theta, \hat{\theta}_{t+1}) = (\hat{\theta}_{t+1} - \theta)^2,$$

the Bayes estimator of the claim severity parameter for period $t + 1$ is given by the posterior mean:

$$\hat{\theta}_{t+1} = E(\Theta \mid X).$$

Using the resulting posterior distribution, the Bayes estimator admits the closed-form expression:

$$\hat{\theta}_{t+1} = \left(\frac{2\sqrt{N + b(K - n)}}{c}\right) \frac{B_{K-\frac{3}{2}}\left(c\sqrt{N + b(K - n)}\right)}{B_{K-\frac{1}{2}}\left(c\sqrt{N + b(K - n)}\right)}, \tag{3.2}$$

where $B_\nu(\cdot)$ denotes the modified Bessel function of the second kind. The full derivation of the likelihood, posterior distribution, and posterior expectation is provided in Appendix B.

3.1.3. *Formula of Optimal Bonus-Malus Premium*

The optimal Bonus-Malus premium formula with the claim frequency distribution following a Negative Binomial and the claim severity distribution following a Truncated Weibull in period $t + 1$ is given by the value of $N + b(K - n)$. When the claim severity is not equal to zero or $N + b(K - n) \neq 0$, from Eq. (3.1), Eq. (3.2), and Eq. (2.3):

$$\text{Premium}_{t+1} = E(\Lambda|K) \cdot E(\Theta|X) = \frac{K + \alpha}{t + \tau} \left(\frac{2\sqrt{N + b(K - n))}}{c}\right) \left(\frac{B_{K-\frac{3}{2}}(c\sqrt{N + b(K - n)}))}{B_{K-\frac{1}{2}}(c\sqrt{N + b(K - n)}))}\right), \tag{3.3}$$

where:

K : Total claim frequency,

b : Maximum limit of claim severity,

N : Claim severity below the limit b,

n : Claim frequency below the limit b,

α, τ : Parameter of the negative binomial distribution,

c : Parameters of the truncated Weibull distribution.

When the claim severity is equal to zero ($N + b(K - n) = 0$) or there is no claim filed ($K = 0$), then the calculation of the risk premium in Eq. (3.3) is not defined, so it needs to be analyzed again. The following is the result of risk premium analysis for the claim severity $N + b(K - n) = 0$ (see Appendix C):

$$\text{Premium}_{t+1}|_{N+b(K-n)=0} = \frac{\alpha}{t+\tau}\left(\frac{2}{c^2}\right), \tag{3.4}$$

where:

α, τ : Parameter of the negative binomial distribution,

c : Parameters of the truncated Weibull distribution.

3.2. *Application of Premium Value Calculation*

The data used in this study consist of the distribution of claim frequency and claim severity. The claim frequency data (n_k) were obtained from the motor vehicle insurance portfolio in Singapore, as reported by [24]. These data cover the period from 1993 to 2001 and are presented in Table 1. Table 1 shows the number of policyholders who filed k claims, denoted by n_k. The claim severity data used in this study were sourced from the research conducted by [14]. Table 2 provides the distribution of claim severity associated with the claims filed by policyholders, while Table 3 presents the claim severity and frequency after applying a maximum deductible of IDR 2,300,000. When the claim severity exceeds this limit, the claim frequency recorded is 18 claims.

The claim severity was not only assumed but statistically tested to follow a Weibull distribution. Goodness-of-fit diagnostics (Kolmogorov-Smirnov and Anderson-Darling tests) do not reject the Weibull model at the 5% level . Therefore, we retain the Weibull specification with parameters $c^{-2} = 371.642$ and $c = 0.052$ for subsequent BMS modelling.The estimated value of parameter c will be used in constructing the optimal Bonus-Malus System model. The maximum coverage limit provided by the insurance company is $b =$ IDR 2,300,000.

Table 1. Portfolio on the distribution of the claim frequency resulting from observations

k	0	1	2	3	4	5	> 5	Total
n_k	178,080	19,224	1,859	177	11	1	0	199,352

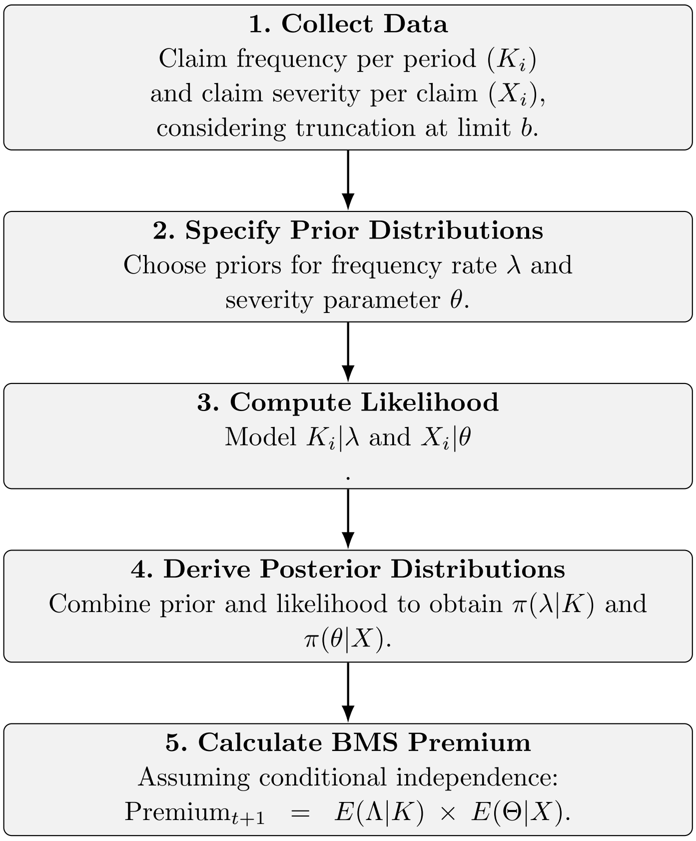

Figure. 1. Steps for calculating premiums using BMS

Table 2. The claim severity and the claim severity

The claim severity (in thousands)	n_k
[0,50)	52
(50,100)	14
(100,200)	22
(200,400)	23
(400,800)	22
(800,1600)	23
(1600,3200)	19
(3200,6400)	6
(6400,12800)	1
(12800,25600)	3
Total	185

Table 3. The claim severity and the claim frequency with $b = 2,300,000$ (IDR)

The claim severity (in thousands)	n_k
[0,50)	52
(50,130)	20
(130,250)	27
(250,410)	13
(410,610)	12
(610,850)	11
(850,1130)	9
(1130,1450)	7
(1450,1810)	10
(1810,2300)	6
(2300,∞)	18
Total	185

The data intervals in Table 2 and Table 3 are distributed exponentially. The claim severity x_i is assumed to be the claim severity below the maximum limit b which is IDR 1,000,000. The value of $\alpha = 1.29$ and $\tau = 10.9$ are obtained from parameter estimation. The estimated value of c which is the truncated Weibull distribution parameter is 0.052. The results of determining the premium are calculated using the R software and are shown in Table 4.

Table 4. Risk premium price (IDR) with a maximum total claim severity $b = 2,300,000$ (IDR)

t	Claim Frequency (K) (in thousands)				
	0	1	2	3	4
Claim Frequency Less than $b(n)$					
0	87	0	0	0	0
1	80	355	234	633	458
2	74	327	216	584	422
3	67	304	200	542	392
4	64	283	187	506	366
5	60	266	175	474	343
6	56	250	165	446	322
7	53	236	156	421	304

Description:
If the claim frequency less than $b(n) = 0$, then there are no claim severity that are smaller than the maximum limit.
If the claim frequency less than $b(n) = 1$, there is one filing for a claim severity that is smaller than the maximum limit.

If the claim frequency less than $b(n) = 2$, there are two filings for claim severity that are smaller than the maximum limit, and so on.

The risk premium prices in Table 4 are read in thousands. The description below the table follows the claim frequency less than $b(n)$. When the claim frequency is less than $b(n) = 0$, it means that there are no claims that are less than the maximum limit, and all claims filed are greater than the maximum limit. Likewise with claim frequency that less than $b(n) = 1$, $b(n) = 2$, $b(n) = 3$, and $b(n) = 4$. Table 4 shows that at the beginning of the month, the policyholder pays an initial risk premium of IDR 87,000. If in the first month, the policyholder does not file a claim, the risk premium payment is reduced to IDR 80,000. Likewise, for the following months, there will be a reduction (bonus) if there are no claims filed by the policyholder. When in the first month the policyholder files a claim once, he is required to pay a premium of IDR 355,000 if the claim filed is more than the maximum limit (IDR 2,300,000). Meanwhile, if the claim filed is below the maximum limit, the premium that must be paid is IDR 234,000.

Furthermore in the second month, if the policyholder files one claim, then the claim history will be seen in the first month. If filing a claim in the first month is greater than the maximum limit and filing a claim in the second month is below the maximum limit (in this case, there is one claim that is less than the maximum limit), then the policyholder must pay a premium of IDR 422,000 (can be seen in the column for the second month, the claim frequency is two, and the claim frequency is less than $b(n) = 1$).

Another example is that a policyholder files a claim four times in the first month, where the claim severity filed is all greater than the maximum limit (IDR 2,300,000), so no claim is filed below the maximum limit. Therefore, the policyholder must pay a premium of IDR 1,040,000 (can be seen in the column for the first month, the claim frequency is four, and the claim frequency is less than $b(n) = 0$). However, if the policyholder files a claim four times with all claims severity less than IDR 2,300,000, then the policyholder will only pay a premium of IDR 157,000 (can be seen in the column for the first month, the claim frequency is four, and the claim frequency is less than $b(n) = 4$).

Based on the example and the description above, it can be seen that the more claims and the bigger the claim severity filed, the higher the premium to be paid, and vice versa, the fewer claims and the smaller the claim severity filed, the smaller the premium to be paid. This is very beneficial for the insurance company and the insurance users. The insurance company itself can prepare the claim severity according to its ability. Motor vehicle insurance users can consider filing a claim severity. The claim severity filed will greatly affect the cost of premiums that must be paid for the company.

3.3. *Comparison of Insurance Premium Benefits*

In this section, we compare the benefits of an optimal bonus-malus system with a truncated Weibull distribution and an optimal bonus-malus system with a Weibull distribution. The total claim severity used is IDR 8,000,000 and IDR 10,000,000.

According to the calculation model of optimal bonus-malus premium with claim frequency is negative binomial distribution and claim severity is Weibull distribution are as follows:

$$P(t,n) = \left(\frac{n+\alpha}{t+\tau}\right) \frac{2\sqrt{M}}{c} \left(\frac{B_{K-\frac{3}{2}}(c\sqrt{M})}{B_{K-\frac{1}{2}}(c\sqrt{M})}\right).$$

For $K = 0$, the risk premium is

$$P(0,0) = \left(\frac{\alpha}{\tau}\right) \frac{2}{c^2}.$$

Table 5. Price of risk premium (IDR) with a maximum total claim severity of IDR 8,000,000

Month (t)	Claim Frequency (K)					
	0	1	2	3	4	5
0	87,000	0	0	0	0	0
1	80,000	662,000	783,000	844,000	871,000	877,000
2	74,000	611,000	722,000	779,000	803,000	809,000
3	67,000	567,000	670,000	723,000	745,000	750,000
4	64,000	529,000	625,000	675,000	695,000	700,000
5	60,000	495,000	586,000	632,000	652,000	656,000
6	56,000	466,000	551,000	595,000	613,000	617,000
7	53,000	440,000	520,000	562,000	579,000	582,000

Table 6. Price of risk premium (IDR) with a maximum total claim severity of IDR 10,000,000

Month (t)	Claim Frequency (K)					
	0	1	2	3	4	5
0	87,000	0	0	0	0	0
1	80,000	740,000	892,000	979,000	1,025,000	1,045,000
2	74,000	683,000	823,000	904,000	946,000	964,000
3	67,000	634,000	764,000	839,000	878,000	895,000
4	64,000	591,000	712,000	782,000	819,000	834,000
5	60,000	554,000	667,000	733,000	767,000	782,000
6	56,000	521,000	628,000	690,000	722,000	736,000
7	53,000	492,000	593,000	651,000	681,000	695,000

Table 5 with a maximum total claim severity of IDR 8,000,000 shows that the premium that must be paid at the beginning of the month is IDR 87,000 and if there are no claims, the premium price is reduced to IDR 80,000. In this case, there is a bonus (premium price cut). Likewise, if in the first month, the policyholder files one claim with a claim severity of IDR 8,000,000, then the policyholder must pay

IDR 662,000 as can be seen in Table 5. Then in the second month, the policyholder files one claim with a claim severity of IDR 2,000,000, the claim frequency for two months will be two claims and the total claim severity will be IDR 10,000,000. Based on this, policyholders are required to pay a premium of IDR 823,000 which can be seen in Table 6. Both tables show that the risk premium paid by policyholders varies.

Table 4 is the risk premium with truncated Weibull distribution, while Table 5 and Table 6 are the risk premium with Weibull distribution. Based on these three tables, it can be seen that when there are no claims ($K = 0$), there is an equal in the results of motor vehicle insurance risk premiums. However, when the claim is not the same as 0, the difference between the three tables is clear. The difference can be seen when the claim frequency is less than $b(n) = 0$. When the claim frequency is less than $b(n) = 0$, Table 4 is not limited by its maximum limit.

Suppose a policyholder files one claim in the first month with a claim severity of IDR 5,000,000, more than the maximum limit b, then the policyholder must pay IDR 355,000 based on the truncated Weibull distribution (can be seen in Table 4, the column for the first month, the claim frequency is one, and the claim frequency is less than $b(n) = 0$). On the other hand, the policyholder must pay IDR 662,000 based on the Weibull distribution (can be seen in Table 5, the column for the first month, the claim frequency is one). Based on this example, it can be seen that the Weibull distribution is deducted at a smaller price premium than the Weibull distribution. A much cheaper premium price will greatly assist policyholders in paying premiums. Therefore, the truncated Weibull distribution is far more profitable than the Weibull distribution on the policyholder's side.

4. Conclusion

In this article, an insurance premium formula has been given using the optimal bonus-malus system with the claim frequency has a negative binomial distribution and the claim severity has a truncated Weibull distribution. From the results of data calculations, the more claims and the higher the claim severity filed, the greater the premium to be paid. In addition, insurance premiums using the bonus-malus system with a truncated Weibull distribution are more profitable than the Weibull distribution. The insurance premium with a truncated Weibull distribution can benefit both parties, both the company and the insurance user.

5. Acknowledgment

We would like to express our sincere gratitude to the reviewers for their insightful comments and constructive suggestions, which have significantly enhanced the quality of this paper.

Appendix A. Proof of (3.1)

A.1 Likelihood Function

Conditional on $\Lambda = \lambda$, each claim count K_i follows a Poisson distribution with parameter λ, that is:

$$P(K_i = k_i \mid \lambda) = \frac{e^{-\lambda}\lambda^{k_i}}{k_i!}, \quad k_i = 0, 1, 2, \cdots$$

Assuming conditional independence, the joint likelihood function of $K_1, K_2, \cdots, K_t$ given λ is:

$$\begin{aligned} p(K_1, \cdots, K_t \mid \lambda) &= \prod_{i=1}^{t} \frac{e^{-\lambda}\lambda^{k_i}}{k_i!}, \\ &= \frac{e^{-t\lambda}\lambda^{\sum_{i=1}^{t} k_i}}{\prod_{i=1}^{t} k_i!}. \end{aligned}$$

A.2 Prior Distribution

The prior distribution of the claim frequency parameter Λ is assumed to be Gamma with shape parameter $\alpha > 0$ and rate parameter $\tau > 0$, with density:

$$u_\Lambda(\lambda) = \frac{\tau^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\tau\lambda}, \quad \lambda > 0.$$

A.3 Posterior Distribution

By Bayes' theorem, the posterior density of Λ given the observed data $K_1 = k_1, \cdots, K_t = k_t$ is:

$$u_{\Lambda|K}(\lambda \mid k_1, \cdots, k_t) = \frac{p(k_1, \cdots, k_t \mid \lambda)\, u_\Lambda(\lambda)}{\int_0^\infty p(k_1, \cdots, k_t \mid \lambda)\, u_\Lambda(\lambda)\, d\lambda}.$$

Substituting the likelihood and the prior yields:

$$u_{\Lambda|K}(\lambda \mid k) \propto e^{-(t+\tau)\lambda}\lambda^{\sum_{i=1}^{t} k_i + \alpha - 1}.$$

Hence, the posterior distribution is a Gamma distribution with updated parameters:

$$\Lambda \mid K \sim \text{Gamma}\left(\alpha + \sum_{i=1}^{t} k_i,\ t + \tau\right),$$

with density:

$$u_{\Lambda|K}(\lambda \mid k) = \frac{(t+\tau)^{\alpha + \sum_{i=1}^{t} k_i}}{\Gamma\left(\alpha + \sum_{i=1}^{t} k_i\right)}\lambda^{\alpha + \sum_{i=1}^{t} k_i - 1}e^{-(t+\tau)\lambda}.$$

The posterior mean of Λ is given by:

$$\begin{aligned}E(\Lambda \mid K) &= \int_0^\infty \lambda\, u_{\Lambda|K}(\lambda \mid k)\, d\lambda \\ &= \frac{\Gamma\Big(\alpha + \sum_{i=1}^t k_i + 1\Big)}{\Gamma\Big(\alpha + \sum_{i=1}^t k_i\Big)} \cdot \frac{1}{t+\tau} \\ &= \frac{\alpha + \sum_{i=1}^t k_i}{t+\tau}.\end{aligned}$$

This posterior expectation coincides with the Bayes estimator under the squared error loss function.

Appendix B. Proof of (3.2)

B.1 Likelihood Function

For a given policyholder, K claims are observed. Among them, n claims satisfy $0 < X_i < b$, while the remaining $K - n$ claims are censored at the policy limit b. Let

$$N = \sum_{i=1}^n X_i.$$

Conditional on $\Theta = \theta$, the joint likelihood function is:

$$\begin{aligned}f(X_1, \cdots, X_K \mid \theta) &= \prod_{i=1}^n \theta e^{-\theta x_i} \prod_{i=1}^{K-n} e^{-\theta b} \\ &= \theta^n \exp\big[-\theta\big(N + b(K-n)\big)\big] .\end{aligned}$$

B.2 Prior Distribution

The prior distribution of the severity parameter Θ is assumed to follow a Lévy distribution with scale parameter $c > 0$, whose density is

$$\pi(\theta) = \frac{c}{2\sqrt{\pi}} \theta^{-\frac{3}{2}} \exp\left(-\frac{c^2}{4\theta}\right), \quad \theta > 0.$$

B.3 Posterior Distribution

By Bayes' theorem, the posterior density of Θ given the observed claim severities is:

$$\pi(\theta \mid X) \propto \theta^{n-\frac{3}{2}} \exp\left[-\left(\theta\big(N + b(K-n)\big) + \frac{c^2}{4\theta}\right)\right].$$

The normalizing constant of this distribution involves the modified Bessel function of the second kind, yielding:

$$\pi(\theta \mid X) = \frac{\left(\dfrac{c}{2\sqrt{N + b(K-n)}}\right)^{K-\frac{1}{2}}}{2\, B_{K-\frac{1}{2}}\Big(c\sqrt{N + b(K-n)}\Big)} \theta^{K-\frac{3}{2}} \exp\left[-\left(\theta\big(N + b(K-n)\big) + \frac{c^2}{4\theta}\right)\right].$$

B.4 Posterior Expectation

The expected value of the posterior distribution is:

$$
\begin{aligned}
E(\Theta|X) &= \int_0^\infty \theta\,\pi(\theta \mid X)\,d\theta \\
&= \int_0^\infty \frac{1}{\theta}\left(\frac{c}{2\sqrt{N+b(K-n)}}\right)^{-\left(K-\frac{1}{2}\right)} \theta^{K-\frac{3}{2}} e^{-\left(\frac{c^2}{4\theta}+\theta(N+b(K-n))\right)} \frac{d\theta}{2B_{K-\frac{1}{2}}(c\sqrt{N+b(K-n)}))} \\
&= \int_0^\infty \left(\frac{2\sqrt{N+b(K-n)}}{c}\right)^{K-\frac{5}{2}} \theta^{K-\frac{5}{2}} e^{-\left(\frac{c^2}{4\theta}+\theta(N+b(K-n))\right)} d\theta \cdot \frac{1}{2B_{K-\frac{1}{2}}(c\sqrt{N+b(K-n)}))}
\end{aligned}
$$

By assuming $\alpha = 2(N+b(K-n)), \beta = \frac{c^2}{2}$, and $v = K-\frac{1}{2}$, then we have:

$$
E(\Theta|X) = \int_0^\infty \left(\frac{\alpha}{\beta}\right)^{\frac{v}{2}} \theta^{v-2} e^{-\frac{1}{2}\left(\frac{\beta}{\theta}+\alpha\theta\right)} d\theta \cdot \frac{1}{2B_v(\sqrt{\alpha\beta})} = \int_0^\infty \left(\sqrt{\frac{\alpha}{\beta\theta}}\right)^{v} \theta^{-2} e^{-\frac{\sqrt{\alpha\beta}}{2}\left(\theta+\frac{1}{\theta}\right)} \frac{d\theta}{2B_v(\sqrt{\alpha\beta})}
$$

Suppose $y = \sqrt{\frac{\alpha}{\beta\theta}}, \theta = \sqrt{\frac{\beta}{\alpha}}y \to d\theta = \sqrt{\frac{\beta}{\alpha}}dy$, then we have:

$$
\begin{aligned}
E(\Theta|X) &= \int_0^\infty y^v \left(\sqrt{\frac{\beta}{\alpha}}y\right)^{-2} e^{-\frac{\sqrt{\alpha\beta}}{2}\left(y+\frac{1}{y}\right)} \sqrt{\frac{\beta}{\alpha}}dy \cdot \frac{1}{2B_v(\sqrt{\alpha\beta})} \\
&= \sqrt{\frac{\alpha}{\beta}}\left(\int_0^\infty y^{v-2} e^{-\frac{\sqrt{\alpha\beta}}{2}\left(y+\frac{1}{y}\right)} dy \cdot \frac{1}{2B_v(\sqrt{\alpha\beta})}\right) = \sqrt{\frac{\alpha}{\beta}}\left(\frac{B_{v-1}(\sqrt{\alpha\beta})}{B_v(\sqrt{\alpha\beta})}\right)
\end{aligned}
$$

By substituting $\alpha = 2(N+b(K-n)), \beta = \frac{c^2}{2}$, and $v = K-\frac{1}{2}$, we get:

$$
E(\Theta|X) = \left(\frac{2\sqrt{N+b(K-n)}}{c}\right)\left(\frac{B_{K-\frac{3}{2}}(c\sqrt{N+b(K-n)}))}{B_{K-\frac{1}{2}}(c\sqrt{N+b(K-n)}))}\right).
$$

Appendix C. Proof of (3.4)

Let the ratio of two Bessel functions be:

$$
Q_k\left(c\sqrt{N+b(K-n)}\right) = \frac{B_{\nu-1}\left(c\sqrt{N+b(K-n)}\right)}{B_\nu\left(c\sqrt{N+b(K-n)}\right)} = \frac{B_{K-\frac{3}{2}}\left(c\sqrt{N+b(K-n)}\right)}{B_{K-\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right)}. \tag{C.1}
$$

By using (2.2), the recursive function for Q_k is:

$$
B_{K+\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right) = \frac{2\left(K-\frac{1}{2}\right)}{c\sqrt{N+b(K-n)}} B_{K-\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right) + B_{K-\frac{3}{2}}\left(c\sqrt{N+b(K-n)}\right).
$$

Then the ratioL

$$
\frac{B_{K+\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right)}{B_{K-\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right)} = \frac{\frac{2\left(K-\frac{1}{2}\right)}{c\sqrt{N+b(K-n)}} B_{K-\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right) + B_{K-\frac{3}{2}}\left(c\sqrt{N+b(K-n)}\right)}{B_{K-\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right)}. \tag{C.2}
$$

From Eq. (C.1) we have:

$$\frac{1}{Q_{k+1}\left(c\sqrt{N+b(K-n)}\right)} = \frac{2\left(K-\frac{1}{2}\right)}{c\sqrt{N+b(K-n)}} + \frac{B_{K-\frac{3}{2}}\left(c\sqrt{N+b(K-n)}\right)}{B_{K-\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right)}$$
$$= \frac{2\left(K-\frac{1}{2}\right)}{c\sqrt{N+b(K-n)}} + Q_k\left(c\sqrt{N+b(K-n)}\right). \quad \text{(C.3)}$$

When no claims are filed ($k=0$) or when $N+b(K-n)=0$, then for $K=1$ we have:

$$Q_1\left(c\sqrt{N+b(K-n)}\right) = \frac{B_{1-\frac{3}{2}}\left(c\sqrt{N+b(K-n)}\right)}{B_{1-\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right)} = 1.$$

If we subtitute this Q_1 and for $K=0$ from Eq. (C.3), we get:

$$Q_0\left(c\sqrt{N+b(K-n)}\right) = \frac{c\sqrt{N+b(K-n)}+1}{c\sqrt{N+b(K-n)}}.$$

The premium rate when the magnitude of loss $N+b(K-n)=0$ is:

$$\text{Premi}_{t+1|N+b(K-n)=0} = \frac{K+\alpha}{t+\tau}\cdot\left(\frac{2\sqrt{N+b(K-n)}}{c}\right)\left(\frac{B_{K-\frac{3}{2}}\left(c\sqrt{N+b(K-n)}\right)}{B_{K-\frac{1}{2}}\left(c\sqrt{N+b(K-n)}\right)}\right)$$
$$= \frac{\alpha}{t+\tau}\cdot\left(\frac{2}{c^2}\right).$$

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