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ON INCLUSIVE TOTAL DISTANCE IRREGULARITY STRENGTH OF JOINT PRODUCT GRAPHS

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Abstract. *Graph theory is one of the branches of mathematics that is rapidly developing due to its applications in solving various problems, including electronic networks, communication network models, transportation systems, and carbon reserve networks. The topology of these networks is simply represented using the concept of graphs. Specifically, graph labeling is widely used to address issues such as radio frequency assignment, computer network coding, data transfer optimization, and marketing distribution. Thus, conducting research to develop graph labeling methods is highly significant. Let* $G = (V_G, E_G)$*, be a simple connected graph, and* $\lambda : V_G \cup E_G \to \{1,\ 2, \cdots, k\}$ *be a labeling function on* G*. The inclusive weight of a vertex* $v \in G$ *is defined as the sum of the labels of* v*, all vertices in the* v *neighborhood, and its incident edges. If all vertices in* V_G *have a distinct inclusive weight, then* λ *is called an inclusive distance vertex irregular total* k*-labeling of* G *. The total distance vertex irregularity strength of* G*, denoted by* $\widehat{tdis}\,(G)$*, is the minimum* k *for which such a labeling exists. This paper investigates the inclusive distance vertex irregular total* k*-labeling for certain classes of joint product graphs. Specifically, we determine the inclusive total distance irregularity strength of the joint product of path, cycle, and complete graphs, providing new insights into their structural labeling properties.*

Keywords: Graph, irregular, labeling, joint product, inclusive

1. Introduction

Graph theory, specifically with regard to graph labeling, has witnessed rapid development due to its wide array of applications in solving real-world problems. As highlighted by Fournier [1], graph labeling plays a crucial role in addressing issues in areas such as electronic networks, communication networks, and transportation

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networks. More particularly, researchers such as Bloom and Golomb [2], Basak [3], Arkut et al. [4], Jin and Yeh [5], and Indriati et al. [6], have applied various types of graph labeling to problems like radio frequency assignment, computer network codes, high-speed backbone data transfer systems (such as MPLS), and marketing distribution. These real-world problems can be simplified and modeled using graph theory.

The notion of graph labeling was first introduced by Wallis [7], who defined it as a mapping from the elements of a graph to a set of non-negative integers. Since its introduction, graph labeling has developed into a rich research area, giving rise to numerous labeling schemes motivated by both theoretical and applied considerations. One notable development is distance magic labeling, which was originally proposed by Vilfred [8] under the name Σ-labeling. This concept was later revisited by Miller et al. [9] and formally established under the term distance magic labeling. Subsequent extensions were introduced by Arumugam and Kamatchi [10], as well as Ngurah et al. [11], who proposed the notion of D-distance (anti)magic labelings on shadows of graphs.

Another important direction in graph labeling concerns irregular labeling. This concept was initiated by Chartrand et al. [12] and later significantly advanced by Baĉa et al. [13,14], who introduced the notion of k-total irregular labeling and studied inclusive distance vertex labelings. Integrating distance considerations into irregular labeling, Slamin [15] defined irregular distance vertex labeling.

Building upon these ideas, Wijayanti et al. [16,17] integrated the concepts of total labeling and irregular distance labeling to define a new type of labeling, the distance vertex irregular total k-labeling. Furthermore, they advanced the research by investigating the total labeling of irregular vertex distances in fan graphs, wheel graphs, and corona product graphs. Moreover, in [18], the concept of total labeling of distances-$\{1\}$ was generalized to total labeling of distances-D.

Motivated by these developments, the present paper continues the study of total labeling of distances by exploring the distance vertex irregular total k-labeling on joint product graphs. Specifically, this study investigates the inclusive total distance irregularity strength of joint product graphs, involving the complement of complete graphs with path and cycle graphs.

2. Preliminaries

This section discusses some basic concepts of graph labeling, specifically distance vertex irregular total k-labeling. The terminology and notation used in this paper follows [17].

Let $G = (V, E)$ which we will then denote by G, be a simple and finite graph, with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood of $x \in V(G)$ is $N(x) = \{y \in V(G) \mid xy \in E(G)\}$ and the degree of x is $deg(x) = |N(x)|$. If $deg(x) = \delta \leq deg(v)$ for every $v \in V(G)$ then δ is the minimum degree of G, and if $deg(x) = \Delta \geq deg(v)$ for every $v \in V(G)$ then Δ is the maximum degree of G.

Definition 2.1. [*17*] *A distance vertex irregular total k-labeling of G is a function $\sigma : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, k\}$ such that the weight of every vertex in $V(G)$ is* *distinct. The weight of $x \in V(G)$ under the labeling σ is defined as:*

$$w_\sigma(x) = \sum_{u \in N(x)} (\sigma(u) + \sigma(ux)). \tag{2.1}$$

The total distance vertex irregularity strength of G, denoted by $tdis(G)$, is defined as the smallest value of k for which G has a distance vertex irregular total labeling.

Theorem 2.2. [*18*] *The lower bound of $tdis(G)$ is:*

$$tdis(G) \geq \left\lceil \frac{2\delta + |V(G)| - 1}{2\Delta} \right\rceil. \tag{2.2}$$

Operations on graphs can be used to construct new graphs. Consider simple graphs G and H, where $V(G) = \{x_1, x_2, \cdots, x_m\}$, $V(H) = \{y_1, y_2, \cdots, y_p\}$, $E(G) = \{e_1, e_2, \cdots, e_n\}$ and $E(H) = \{e'_1, e'_2, \cdots, e'_n\}$. The complement of G, denoted by $\bar{G}$, where $V(\bar{G}) = \{x_1, x_2, \cdots, x_m\}$ and $E(\bar{G}) = E(\bar{G}) = \{y_i y_j | y_i, y_j \in V(G) \wedge y_i y_j \notin E(G)\}$. The complement of a complete graph K_n is a graph $\bar{K_n}$ with $V(\bar{K_n}) = V(K_n)$ and $E(\bar{K_n}) = \{\}$. The complement of a complete graph is also known as the zero graph. The joint product of two graphs G and H, denoted by $G + H$, is a graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. Some examples of graphs resulting from the joint product are the fan graph $F_n = K_1 + P_n$, the wheel graph $W_n = K_1 + C_n$, and the complete bipartite graph $K_{m,n} = \bar{K}_m + \bar{K}_n$ [19,20,21].

3. Result and Discussion

The concept of inclusive labeling is based on the definition of closed neighborhoods, which are neighborhoods that include the vertex itself as a neighbor. This concept is further explained through the following definition,

Definition 3.1. *An inclusive distance vertex irregular total k-labeling of G is a function $\sigma : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ such that the weight of every vertex in $V(G)$ is distinct. The weight of $x \in V(G)$ under the labeling σ is defined as:*

$$w_\sigma(x) = \sigma(x) + \sum_{u \in N(x)} (\sigma(u) + \sigma(ux)). \tag{3.1}$$

The inclusive total distance vertex irregularity strength of G, denoted by $\widehat{tdis}(G)$, is defined as the smallest value of k for which G has an inclusive distance vertex irregular total labeling. If graph G has no such labeling, it is said that $\widehat{tdis}(G) = \infty$.

An example of an inclusive distance vertex irregular total k-labeling on the graph C_4 is shown in Figure 1.

The integers inside the blue box represent the vertex weights, while the integers outside the blue box correspond to the vertex and edge labels. In Figure 1, there are two inclusive distance vertex irregular total k-labelings on graph C_4, with the largest label $k = 3$. This value of 3 is the smallest value of the largest label, therefore, $\widehat{tdis}(C_4) = 3$.

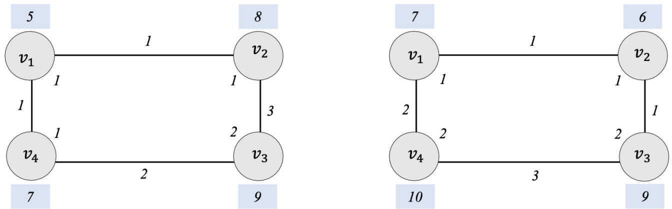

Figure 1. An Inclusive Distance Vertex Irregular Total k-labeling on C_4.

Theorem 3.2 establishes a general lower bound for $\widehat{tdis}(G)$ for all finite simple graphs admitting an inclusive total distance vertex irregularity k-labeling.

Theorem 3.2. *If G be a simple finite graph, then the lower bound of inclusive total distance vertex irregularity strength is,*

$$\widehat{tdis}(G) \geq \left\lceil \frac{2\delta + |V(G)|}{2\Delta + 1} \right\rceil. \tag{3.2}$$

Proof. Let σ be a distance vertex irregular total k-labeling function for G, a finite simple graph with minimum degree δ and maximum degree Δ. Without loss of generality, let $x \in V(G)$ be the vertex with the smallest weight, i.e., $w_\sigma(x) \leq w_\sigma(v)$ for each $v \in V(G)$. The smallest possible value of $w_\sigma(x)$ is $2\delta + 1$, which occurs when $\sigma(x) = 1$, $\sigma(v) = 1$, and $\sigma(xv) = 1$ for each $v \in N(x)$. The labeling is optimal if the weights of all vertices form an arithmetic sequence with common difference 1, namely:

$$2\delta + 1,\ 2\delta + 2,\ \cdots,\ 2\delta + |V(G)|.$$

Without loss of generality, let $y \in V(G)$ be the vertex with weight $2\delta + |V(G)|$. Since the vertex weights form an arithmetic progression, the largest weight must correspond to the vertex of degree Δ. Therefore, the largest label contributing to this weight must satisfy:

$$\widehat{tdis}(G) \geq \left\lceil \frac{2\delta + |V(G)|}{2\Delta + 1} \right\rceil.$$

□

In certain graphs G, the lower bound of $\widehat{tdis}(G)$ formulated in Theorem 3.2 differs significantly from the actual value of $\widehat{tdis}(G)$. This happened, for example, in graphs that have a central vertex, which is a vertex with a very high degree, due to its connection with all other vertices in G. To address this issue, the author provides a new formulation in Theorem 3.3.

Theorem 3.3. *Let G be a simple finite graph with a central vertex v consisting of $n + 1$ vertices: n_1 vertices of degree δ_1, n_2 vertices of degree δ_2, $\cdots$, and n_r vertices of degree δ_r, where $\delta = \delta_1 < \delta_2 < \cdots < \delta_r - 1 < \delta_r = \Delta$. The lower bound of inclusive total distance vertex irregularity strength is:*

$$\widehat{tdis}(G) \geq \max\left\{ \left\lceil \frac{2\delta + n_1 - 1}{2\delta} \right\rceil, \left\lceil \frac{2\delta + n_1 + n_2 - 1}{2\delta_2} \right\rceil, \cdots, \left\lceil \frac{2\delta + n_1 + \cdots + n_{r-1} - 1}{2\delta_{r-1}} \right\rceil, \left\lceil \frac{2\delta + n}{2\Delta} \right\rceil \right\}. \tag{3.3}$$

Proof. Let σ be a distance vertex irregular total k-labeling function for G, a simple graph with a central vertex v of degree Δ, and consisting $n+1$ vertices: n_1 vertices of degree $\delta_1 = \delta$, n_2 vertices of degree $\delta_2, \cdots, n_r$ vertices of degree $\delta_r = \Delta$, where $\delta = \delta_1 < \delta_2 < \cdots < \delta_r = \Delta$, and $n_r = 1$, $|V(G)| = n_1 + n_2 + \cdots + n_r = n + 1$.

Without loss of generality, let $v_{n_i}, i = 1, 2, \cdots, r$ denote the vertices of degree δ_i that have the largest weight among all vertices with the same degree. Since the smallest weight is $2\delta + 1$, and the weights of each vertex must be distinct, then the set of minimum values for $w_\sigma(v_{n_i}), i = 1, 2, \cdots, r$ is given by:

$$\{w_\sigma(v_{n_i}) \mid i = 1, 2, \cdots, r\} = \{2\delta + n_1, 2\delta + n_1 + n_2, \cdots, 2\delta + n_1 + \cdots + n_{r-1}, 2\delta + n\}. \tag{3.4}$$

Furthermore, let $x \in V(G)$ be vertex with the smallest weight, $w_\sigma(x) = 2\delta + 1$, which occurs when $\sigma(x) = 1$, $\sigma(y) = 1$, and $\sigma(xy) = 1$ for each $y \in N(x)$. Since v is adjacent to every vertex, then $v \in N(x)$ and $\sigma(v) = 1$. Moreover, to make the lower bound of $\widehat{tdis}(G)$ approach the actual value of $\widehat{tdis}(G)$ more closely, we must excluded $\sigma(v)$. Then we obtained:

$$\{w_\sigma(v_{n_i}) - 1 \mid i = 1, 2, \cdots, r\} = \{2\delta + n_1 - 1, 2\delta + n_1 + n_2 - 1, \cdots, 2\delta + n_1 + \cdots + n_{r-1} - 1, 2\delta + n - 1\}. \tag{3.5}$$

Since $w_\sigma(v_{n_i}) - 1, i = 1, 2, \ldots, r$ is the result of adding $2\delta_i$ integers, the Inequality 3.3 is proven. □

Some joint product graphs exhibit a graph structure with a central vertex. For instance, the graphs $K_1 + \bar{K_n}$, $K_1 + P_n$, and $K_1 + C_n$, all possess this structure. The distance vertex irregular total k-labeling of these graphs is discussed specifically in the following theorems.

Theorem 3.4. *If $G = K_1 + \bar{K_n}$, then the inclusive total distance vertex irregularity strength of G is,*

$$\widehat{tdis}(G) = \begin{cases} \infty, & \text{for } n = 1, \\ \left\lceil \frac{n+1}{2} \right\rceil, & \text{for } n > 1. \end{cases} \tag{3.6}$$

Proof. Since $G = K_1 + \bar{K_n}$, then $V(G) = \{v_i \mid i = 1, 2, \cdots, n\} \cup \{v\}$, $|V(G)| = n+1$, $deg(v_i) = \delta_1 = \delta = 1$ and $deg(v) = \delta_2 = \Delta = n$. It is clear that $|\{v_i \mid i = 1, 2, \cdots, n\}| = n$ and $|\{v\}| = 1$.

Let σ be a total labeling function of G,

(Case 1) If $n = 1$ then $|V(G)| = 2$, namely $v_1, v_2 \in V(G)$. $N(v_1) = v_2$ and $N(v_2) = v_1$, which result in $w_\sigma(v_1) = \sigma(v_1) + \sigma(v_2) + \sigma(v_1 v_2) = w_\sigma(v_2)$. Since there are two vertices with the same weight, for $n = 1$, the graph G has no distance vertex irregular total k-labeling, and thus $\widehat{tdis}(G) = \infty$.

(Case 2) To prove that $\widehat{tdis}(G) = \left\lceil \frac{n+1}{2} \right\rceil$ for $n > 1$, we show that $\widehat{tdis}(G) \geq \left\lceil \frac{n+1}{2} \right\rceil$ and $\widehat{tdis}(G) \leq \left\lceil \frac{n+1}{2} \right\rceil$. The inequality $\widehat{tdis}(G) \geq \left\lceil \frac{n+1}{2} \right\rceil$ is established by Theorem 3.3. To prove $\widehat{tdis}(G) \leq \left\lceil \frac{n+1}{2} \right\rceil$, we demonstrate that G has an inclusive distance vertex irregular total k-labeling, where the value of k is $\left\lceil \frac{n+1}{2} \right\rceil$.

Let σ be a total labeling function of G, defined as follows.

$$\sigma(v) = 1,$$
$$\sigma(v_i) = 1 + \left\lceil \frac{i-1}{2} \right\rceil, \text{ for } i = 1, 2, \cdots, n,$$
$$\sigma(vv_i) = \left\lceil \frac{i}{2} \right\rceil, \text{ for } i = 1, 2, \cdots, n.$$

We obtained, the weight of all vertices in $V(G)$ are:

$$w_\sigma(v_1) = \sigma(v_1) + \sigma(vv_1) + \sigma(v) = 1 + 1 + 1 = 3,$$
$$w_\sigma(v_2) = \sigma(v_2) + \sigma(vv_2) + \sigma(v) = 2 + 1 + 1 = 4,$$
$$\vdots$$
$$w_\sigma(v_n) = \sigma(v_n) + \sigma(vv_n) + \sigma(v) = \left(1 + \left\lceil \frac{n-1}{2} \right\rceil\right) + \left\lceil \frac{n}{2} \right\rceil + 1.$$

For n odd,

$$1 + \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil \text{ and } \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil.$$

For n even,

$$1 + \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil \text{ and } \left\lceil \frac{n}{2} \right\rceil < \left\lceil \frac{n+1}{2} \right\rceil.$$

Hence, $\{w_\sigma(v_i) \mid 1 = 1, 2, \cdots, n\} = \{3, 4, \cdots, n+2\}$, forms a progressive arithmetic sequence, with the highest label $\lceil \frac{n+1}{2} \rceil$.

This provides evidence that G has an inclusive distance vertex irregular total k-labeling, with $k = \lceil \frac{n+1}{2} \rceil$. □

Next, Theorem 3.5 discusses the labeling of $G = K_1 + P_n$,

Theorem 3.5. *If $G = K_1 + P_n$, then the inclusive total distance vertex irregularity strength of G is:*

$$\widehat{tdis}(G) = \begin{cases} 3, & \text{for } n = 2, \\ 2, & \text{for } n = 3, \\ \lceil \frac{n+3}{6} \rceil, & \text{for } n > 3. \end{cases} \tag{3.7}$$

Proof. Let σ be a total labeling function of G. To prove that $\widehat{tdis}(G) = \lceil \frac{n+3}{6} \rceil$ for $n > 3$, we show that $\widehat{tdis}(G) \geq \lceil \frac{n+3}{6} \rceil$ and $\widehat{tdis}(G) \leq \lceil \frac{n+3}{6} \rceil$.

Since $G = K_1 + P_n$, then $V(G) = \{v_i \mid i = 1, 2, \cdots, n\} \cup \{v\}$, and $|V(G)| = n+1$. Let v_1 and v_n be the vertices at the ends of the path graph P_n, hence $deg(v_1) = deg(v_n) = \delta_1 = \delta = 2$, $deg(v_i) = \delta_2 = 3$ and $deg(v) = \delta_3 = \Delta = n$. It is clear that $|\{v_1, v_n\}| = n_1 = 2$, $|\{v_i \mid i = 2, 3, \cdots, n-1\}| = n_2 = n-2$ and $|\{v\}| = n_3 = 1$.

By Theorem 3.3, we obtain that:

$$\widehat{tdis}(G) \geq \max\left\{ \left\lceil \frac{2.2 + 2 - 1}{2.2} \right\rceil, \left\lceil \frac{2.2 + 2 + n - 2 - 1}{2.3} \right\rceil, \left\lceil \frac{2.2 + n}{2.n} \right\rceil \right\},$$
$$= \max\left\{ \left\lceil \frac{5}{4} \right\rceil, \left\lceil \frac{n+3}{6} \right\rceil, \left\lceil \frac{n+4}{2n} \right\rceil \right\} = \left\lceil \frac{n+3}{6} \right\rceil.$$

Furthermore, to prove $\widehat{tdis}(G) \leq \lceil \frac{n+3}{6} \rceil$, we show that the graph G admits an inclusive distance vertex irregular total k-labeling. In particular, the labeling is constructed with $k = \lceil \frac{n+3}{6} \rceil$ for $n > 3$, while for the exceptional cases $k = 2$ for $n = 3$, and $k = 3$ for $n = 2$.

(Case 1) When $n = 2$, the graph $G = K_1 + P_n$ becomes a cycle graph C_3, where all vertices are interconnected and consequently share the same neighbors in the context of closed neighborhoods. Therefore, in order for the vertex weights to be distinct, the labels of the edges must also be distinct. The description of the distance vertex irregular total k-labeling, with $k = 3$, on the graph $K_1 + P_2$, can be seen in Figure 2. It clarifies the argument that when $n = 2$, then $\text{tdis}(K_1 + P_n) = 3$.

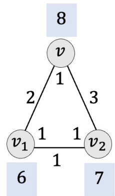

Figure 2. The Inclusive Distance Vertex Irregular Total k-Labeling on $G = K_1 + P_2$.

(Case 2) For $2 < n < 7$, the labeling is given in Figure 3.

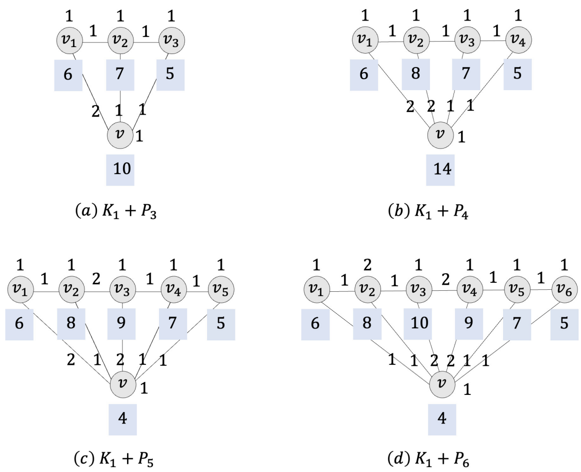

Figure 3. Total Distance Labeling on $G = K_1 + P_n$ for $2 < n < 7$.

For $G = K_1 + P_n, 2 < n < 7$, G has a distance vertex irregular total k-labeling, with $k = \lceil \frac{n+3}{6} \rceil$.

(Case 3) For $n \equiv 1 \pmod 6$, σ is defined as follows.

$$\sigma(v) = 1, \text{for central vertex } v,$$

$$\sigma(v_i) = \begin{cases} \left\lceil \frac{i}{3} \right\rceil, & \text{for } i = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+1}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \cdots, n, \end{cases}$$

$$\sigma(v_i v) = \begin{cases} 2, & \text{for } i = 1, \\ \left\lceil \frac{i-1}{3} \right\rceil + 1, & \text{for } i = 2, \cdots, \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+1}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \\ \left\lceil \frac{n-i+2}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 2, \cdots, n, \end{cases}$$

$$\sigma(v_i v_{i+1}) = \begin{cases} \left\lceil \frac{i+1}{3} \right\rceil, & \text{for } i = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \cdots, n-1. \end{cases}$$

(Case 4) For $n \equiv 2, 3, 4 \pmod 6$, define the labeling σ for vertices v, v_i, and edges $v_i v_{i+1}$ the same as in Case 3. While for edges $v_i v$, define the labeling σ as follows.

$$\sigma(v_i v) = \begin{cases} 2, & \text{for } i = 1, \\ \left\lceil \frac{i-1}{3} \right\rceil + 1, & \text{for } i = 2, \cdots, \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+2}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \cdots, n. \end{cases}$$

(Case 5) For $n \equiv 5 \pmod 6$, the labeling σ of vertices v, and v_i are define the same as in Case 3. While for edges $v_i v$, and $v_i v_{i+1}$, define the labeling σ as follows.

$$\sigma(v_i v) = \begin{cases} 2, & \text{for } i = 1, \\ \left\lceil \frac{i-1}{3} \right\rceil + 1, & \text{for } i = 2, \cdots, \left\lceil \frac{n}{2} \right\rceil - 2, \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{i-1}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{n-i+2}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \cdots, n. \end{cases}$$

$$\sigma(v_i v_{i+1}) = \begin{cases} \left\lceil \frac{i+1}{3} \right\rceil, & \text{for } i = 1, \cdots, \left\lceil \frac{n}{2} \right\rceil - 2, \\ \left\lceil \frac{i+1}{3} \right\rceil + 1, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{n-i}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \cdots, n-1. \end{cases}$$

(Case 6) For $n \equiv 0 \pmod 6$, σ is defined as follows.

$$\sigma(v) = 1, \text{for central vertex } v,$$

$$\sigma(v_i) = \begin{cases} \left\lceil \frac{i}{3} \right\rceil, & \text{for } i = 1, \cdots, \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{n-i+1}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \cdots, n, \end{cases}$$

$$\sigma(v_i v) = \begin{cases} 2, & \text{for } i = 1, \\ \left\lceil \frac{i-1}{3} \right\rceil + 1, & \text{for } i = 2, \cdots, \left\lceil \frac{n}{2} \right\rceil - 2, \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{i-1}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{n-i+2}{3} \right\rceil - 1, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \\ \left\lceil \frac{n-i+2}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 2, \cdots, n, \end{cases}$$

$$\sigma(v_i v_{i+1}) = \begin{cases} \left\lceil \frac{i+1}{3} \right\rceil, & \text{for } i = 1, \cdots, \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{n-i}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \cdots, n-1. \end{cases}$$

From Case 3 to Case 6, it is obtained that the weight of all vertices in $V(G)$

are as follows.

$$\begin{aligned}
w_\sigma(v_n) &= \sigma(v_n)+\sigma(v)+\sigma(v_{n-1})+\sigma(v_nv)+\sigma(v_nv_{n-1}),\\
&= 1+1+1+1+1=5,\\
w_\sigma(v_1) &= \sigma(v_1)+\sigma(v)+\sigma(v_2)+\sigma(v_1v)+\sigma(v_1v_2),\\
&= 1+1+1+2+1=6,\\
w_\sigma(v_{n-1}) &= \sigma(v_{n-1})+\sigma(v)+\sigma(v_{n-2})+\sigma(v_n)+\sigma(v_{n-1}v)+\sigma(v_{n-1}v_{n-2})+\sigma(v_{n-1}v_n),\\
&= 1+1+1+1+1+1+1=7,\\
w_\sigma(v_2) &= \sigma(v_2)+\sigma(v)+\sigma(v_1)+\sigma(v_3)+\sigma(v_2v)+\sigma(v_2v_1)+\sigma(v_2v_3)\\
&= 1+1+1+1+2+1+1=8,\\
&\ \vdots\\
w_\sigma(v_{\lceil\frac{n}{2}\rceil}) &= \sigma(v_{\lceil\frac{n}{2}\rceil-1})+\sigma(v_{\lceil\frac{n}{2}\rceil})+\sigma(v_{\lceil\frac{n}{2}\rceil+1})+\sigma(v)+\sigma(v_{\lceil\frac{n}{2}\rceil-1}v_{\lceil\frac{n}{2}\rceil})\\
&\quad+\sigma(v_{\lceil\frac{n}{2}\rceil}v_{\lceil\frac{n}{2}\rceil+1})+\sigma(v_{\lceil\frac{n}{2}\rceil}v)\\
&= \left\lceil\frac{\lceil\frac{n}{2}\rceil}{3}\right\rceil+\left\lceil\frac{\lceil\frac{n}{2}\rceil-1}{3}\right\rceil+\left\lceil\frac{n-(\lceil\frac{n}{2}\rceil+1)+1}{3}\right\rceil+1+\left\lceil\frac{(\lceil\frac{n}{2}\rceil-1)+1}{3}\right\rceil\\
&\quad+\left\lceil\frac{\lceil\frac{n}{2}\rceil+1}{3}\right\rceil+\left\lceil\frac{\lceil\frac{n}{2}\rceil-1}{3}\right\rceil+1\\
&= 2\left\lceil\frac{\lceil\frac{n}{2}\rceil}{3}\right\rceil+2\left\lceil\frac{\lceil\frac{n}{2}\rceil-1}{3}\right\rceil+\left\lceil\frac{n-(\lceil\frac{n}{2}\rceil+1)+1}{3}\right\rceil+\left\lceil\frac{\lceil\frac{n}{2}\rceil+1}{3}\right\rceil+2.
\end{aligned}$$

For n odd,

$$w_\sigma(v_{\lceil\frac{n}{2}\rceil}) = 2\left\lceil\frac{n+1}{6}\right\rceil+3\left\lceil\frac{n-1}{6}\right\rceil+\left\lceil\frac{n+3}{6}\right\rceil+2.$$

For n even,

$$w_\sigma(v_{\lceil\frac{n}{2}\rceil}) = 3\left\lceil\frac{n}{6}\right\rceil+2\left\lceil\frac{n-2}{6}\right\rceil+\left\lceil\frac{n+2}{6}\right\rceil+2.$$

For various cases, the value of $w_\sigma(v_{\lceil\frac{n}{2}\rceil})$ can be readily obtained as $n+3$, with the largest label corresponds to $\lceil\frac{n+3}{6}\rceil$. Hence, $\{w_\sigma(v_i) \mid i = 1,2,\cdots,n\} = \{5,6,\cdots,n+3\}$, forms a progressive arithmetic sequence, with the highest label $\lceil\frac{n+3}{6}\rceil$. It is proven that K_1+P_n has an inclusive distance vertex irregular total k-labeling, with $k=\lceil\frac{n+3}{6}\rceil$. □

Furthermore, we will discuss the labeling of the graph $G = K_1 + C_n$ through Theorem 3.6.

Theorem 3.6. *If $G = K_1+C_n$, with $n \geq 3$, then the inclusive total distance vertex irregularity strength of G is:*

$$\widehat{tdis}(G) = \begin{cases} 3, & \text{for } n = 3,\\ \lceil\frac{n+5}{6}\rceil, & \text{for } n > 3.\end{cases} \tag{3.8}$$

Proof. Let $G = K_1 + C_n$ and σ be a total labeling function of G. For $n = 3$, we prove that $\widehat{tdis}(G) = 3$ by a direct construction in Case 1. For $n > 3$, we prove that $\widehat{tdis}(G) = \lceil \frac{n+5}{6} \rceil$ by establishing both $\widehat{tdis}(G) \geq \lceil \frac{n+5}{6} \rceil$ and $\widehat{tdis}(G) \leq \lceil \frac{n+5}{6} \rceil$.

Since $G = K_1 + C_n$, we have $V(G) = \{v_i \mid i = 1, 2, \cdots, n\} \cup \{v\}$. Clearly, $|\{v_i \mid i = 1, 2, \cdots, n\}| = n_1 = n$ and $|\{v\}| = n_2 = 1$, and hence $|V(G)| = n_1 + n_2 = n+1$, $deg(v_i) = \delta = 3$ and $deg(v) = \Delta = n$. By Theorem 3.3, it follows that:

$$\widehat{tdis}(G) \geq \max\left\{ \left\lceil \frac{2.3 + n - 1}{2.3} \right\rceil, \left\lceil \frac{2.3 + n + 1 - 1}{2.n} \right\rceil \right\}.$$

Thus,

$$\widehat{tdis}(G) \geq \max\left\{ \left\lceil \frac{n+5}{6} \right\rceil, \left\lceil \frac{n+6}{2n} \right\rceil \right\} = \left\lceil \frac{n+5}{6} \right\rceil.$$

Furthermore, to show that $\widehat{tdis}(G) \leq \lceil \frac{n+5}{6} \rceil$, we construct an inclusive distance vertex irregular total k-labeling of G, where $k = \lceil \frac{n+5}{6} \rceil$.

(Case 1) For $n = 3$, the description of the distance vertex irregular total k-labeling, with $k = 3$, on the graph $K_1 + C_3$, can be seen in Figure 4. It clarifies the argument that when $n = 3$, then $\widehat{\text{tdis}}(K_1 + C_n) = 3$.

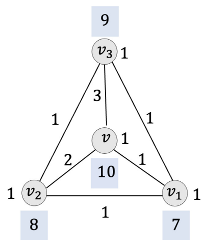

Figure 4. The Inclusive Distance Vertex Irregular Total k-Labeling on $G = K_1 + C_3$.

(Case 2) For $n \geq 3$ and for $n \equiv 0, 1 \pmod 6$, σ is defined as:

$$\sigma(v) = 1, \text{for central vertex } v,$$

$$\sigma(v_i) = \begin{cases} \lceil \frac{i+1}{3} \rceil, & \text{for } i = 1, 2, \cdots, \lceil \frac{n}{2} \rceil, \\ \lceil \frac{n-i+3}{3} \rceil, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \cdots, n, \end{cases}$$

$$\sigma(v_i v) = \begin{cases} \lceil \frac{i+2}{3} \rceil, & \text{for } i = 1, 2, \cdots, \lceil \frac{n}{2} \rceil, \\ \lceil \frac{n-i+3}{3} \rceil, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \cdots, n, \end{cases}$$

$$\sigma(v_i v_{i+1}) = \begin{cases} \lceil \frac{i+2}{3} \rceil, & \text{for } i = 1, 2, \cdots, \lceil \frac{n}{2} \rceil, \\ \lceil \frac{n-i+3}{3} \rceil, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \cdots, n-1, \end{cases}$$

$$\sigma(v_n v_1) = 1,$$

(Case 3) For $n \equiv 2 \pmod 6$, σ is defined identically to Case 2, for vertices v and v_i, and edges $v_i v_{i+1}$ and $v_n v_1$.

For $v_i v$, σ is defined as follows.

$$\sigma(v_i v) = \begin{cases} \lceil \frac{i+2}{3} \rceil, & \text{for } i = 1, 2, \cdots, \lceil \frac{n}{2} \rceil + 1, \\ \lceil \frac{n-i+3}{3} \rceil, & \text{for } i = \lceil \frac{n}{2} \rceil + 2, \cdots, n, \end{cases}$$

(Case 4) For $n \equiv 3 \pmod 6$, σ is defined identically to Case 2, for vertices v and v_i, and edges v_iv and v_nv_1.

For v_iv_{i+1}, σ is defined as follows.

$$\sigma(v_iv_{i+1}) = \begin{cases} \left\lceil \frac{i+1}{3} \right\rceil, & \text{for } i = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{n-i+1}{3} \right\rceil + 1, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+1}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \cdots, n-1, \end{cases}$$

(Case 5) For $n \equiv 4 \pmod 6$, σ is defined identically to Case 2, for vertices v and v_i, and edge v_nv_1.

For v_iv and v_iv_{i+1}, σ is defined as follows.

$$\sigma(v_iv) = \begin{cases} \left\lceil \frac{i+2}{3} \right\rceil, & \text{for } i = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{i+2}{3} \right\rceil - 1, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+3}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \cdots, n, \end{cases}$$

$$\sigma(v_iv_{i+1}) = \begin{cases} \left\lceil \frac{i+1}{3} \right\rceil, & \text{for } i = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{i+1}{3} \right\rceil + 1, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+1}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \cdots, n-1, \end{cases}$$

(Case 6) For $n \equiv 5 \pmod 6$, σ is defined identically to Case 2, for vertices v and v_i, and edges v_iv_{i+1} and v_nv_1.

For v_iv, σ IS defined as follows.

$$\sigma(v_iv) = \begin{cases} \left\lceil \frac{i+2}{3} \right\rceil, & \text{for } i = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{n-i+3}{3} \right\rceil - 1, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \\ \left\lceil \frac{n-i+3}{3} \right\rceil, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 2, \cdots, n. \end{cases}$$

Based on Definition 3.1, the weight of each vertex $v_i \in V(G)$ was determined to be:

$$w_\sigma(v_i) = \sigma(v_i) + \sigma(v_{i-1}) + \sigma(v_{i+1}) + \sigma(vv_i) + \sigma(v_iv_{i-1}) + \sigma(v_iv_{i+1}).$$

Hence, the calculation of weight for all vertices in $V(G)$ are:

$$\begin{aligned} w_\sigma(v_1) &= \sigma(v_1) + \sigma(v_n) + \sigma(v_2) + \sigma(v) + \sigma(vv_1) + \sigma(v_1v_n) + \sigma(v_1v_2) = 7, \\ w_\sigma(v_n) &= \sigma(v_1) + \sigma(v_n) + \sigma(v_{n-1}) + \sigma(v) + \sigma(vv_n) + \sigma(v_1v_n) + \sigma(v_nv_{n-1}) = 8, \\ &\vdots \\ w_\sigma(v_{\lceil \frac{n}{2} \rceil}) &= \sigma(v_{\lceil \frac{n}{2} \rceil - 1}) + \sigma(v_{\lceil \frac{n}{2} \rceil}) + \sigma(v_{\lceil \frac{n}{2} \rceil + 1}) + \sigma(v) + \sigma(v_{\lceil \frac{n}{2} \rceil - 1}v_{\lceil \frac{n}{2} \rceil}) \\ &\quad + \sigma(v_{\lceil \frac{n}{2} \rceil}v_{\lceil \frac{n}{2} \rceil + 1}) + \sigma(v_{\lceil \frac{n}{2} \rceil}v) = n + 5. \end{aligned}$$

The weights for all vertices $v_1, v_2, \cdots, v_n$ form an arithmetic progression starting from 7 and progressing to the highest weights $w_\sigma(v_{\lceil \frac{n}{2} \rceil}) = n + 5$ and the highest label being $\left\lceil \frac{n+5}{6} \right\rceil$. This complete all prove that $K_1 + C_n$ has an inclusive distance vertex irregular total k-labeling, where $k = \left\lceil \frac{n+5}{6} \right\rceil$. □

4. Conclusion

In this paper, we have investigated the inclusive total distance irregularity strength of several classes of joint product graphs. The first contribution of this work is the establishment of a new lower bound for the inclusive total distance irregularity strength, denoted by $\widehat{tdis}(G)$ which improves the general estimation for graphs with specific structural properties, especially those containing a central vertex. Theorem 3.2 provided a universal lower bound for arbitrary finite simple graphs, while Theorem 3.3 refined this result by giving a sharper bound for graphs with degree partitions around a central vertex. These results not only strengthen the theoretical foundation of inclusive irregular labeling but also show that the lower bounds can be made closer to the actual value of $\widehat{tdis}(G)$.

The second contribution lies in the explicit determination of $\widehat{tdis}(G)$ for specific joint product graphs. For $G = K_1 + K_n$, we proved that the value of $\widehat{tdis}(G)$ is infinite when $n = 1$, and $\lceil \frac{n+1}{2} \rceil$ when $n > 1$. For the joint product $G = K_1 + P_n$, we established that $\widehat{tdis}(G) = 3$ for $n = 2$, while for $n > 2$, it is given by $\lceil \frac{n+3}{6} \rceil$. Similarly, for the graph $G = K_1 + C_n$, we obtained that $\widehat{tdis}(G) = 3$ when $n = 3$, and for larger n, the strength is $\lceil \frac{n+5}{6} \rceil$. These explicit results confirm that the inclusive total distance irregularity strength in joint product graphs strongly depends on the underlying structure of the base graph, particularly the degree distribution and the presence of a dominating central vertex.

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