

INCLUSIVE DISTANCE ANTIMAGIC LABELING OF SHADOW GRAPH OF COMPLETE AND CIRCULANT GRAPHS

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Abstract. Consider a graph $G = (V, E)$ with order n . Suppose that we have a bijection $f : V(G) \rightarrow \{1, 2, \dots, n\}$. A graph G is said to admit an inclusive distance antimagic labeling if every pair of distinct vertices has different weights, with a vertex weight is defined by $w(v) = \sum_{u \in N(v)} f(u) + f(v)$. A graph G is said to have an (a, d) -inclusive distance antimagic labeling if, after labeling its vertices, the calculated vertex weights create an arithmetic progression where the first term is a and the common difference between consecutive terms is d . This paper investigates the inclusive distance antimagic labeling of the shadow graph of the complete and circulant graph.

Keywords: (a, d) -Inclusive distance antimagic labeling, Inclusive distance antimagic labeling, Shadow graph

1. Introduction

We follow the notation and terminologies in [1,2,3,4,5,6]. Let G be a simple, finite, and undirected graph. The study of distance magic labeling on graphs has appeared under various terminologies, such as sigma labeling introduced by Vilfred in 1994 [7], 1-vertex-magic vertex labeling investigated by Miller, Rodger and Simanjuntak in 2003 [8], and distance magic labeling proposed by Sugeng et al. in 2009 [9]. Subsequently, the notion of distance antimagic labeling emerged as a natural variation of distance magic labeling. The (a, d) -distance antimagic labeling is introduced by Arumugam and Kamatchi [3], which arises naturally from distance magic labeling.

A distance magic labeling of a graph G of order n is a bijection $f : V(G) \rightarrow \{1, 2, \dots, n\}$ with the property that there exists a positive integer k such that $\sum_{y \in N(x)} f(y) = k$ for every $x \in V(G)$. The constant k is called the magic constant

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of the labeling f . Let $N(x) = \{y \in V(G) : xy \in E(G)\}$ be the set of vertex neighbors $x \in V(G)$. The summation $\sum_{y \in N(x)} f(y)$ refers to the weight of the vertex x , namely $w(x)$ [9].

A distance antimagic labeling (DAL) of a graph G is a bijection $f : V(G) \rightarrow \{1, 2, \dots, n\}$, such that any two distinct vertices have different weight, namely $w(x) \neq w(y)$ for every $x, y \in V(G)$, with $w(x) = \sum_{y \in N(x)} f(y)$ [2]. If we add the summation of its own label, we form a new weight, namely $w(x) = \sum_{y \in N(x)} f(y) + f(x)$. In this case, if a graph's vertices are labeled such that all the resulting vertex weights are unique (i.e., no two vertices have the same weight), the graph is said to admit an inclusive distance antimagic labeling (IDAL) [4]. Furthermore, if the resulting values form an arithmetic progression with the first term a and the common difference d , namely $\{a, a + d, \dots, a + (n - 1)d\}$, where a and d are integers and $d \geq 0$, then G admits an (a, d) -IDAL.

Consider a graph G that has no isolated vertices. The shadow graph of G , namely $D_2(G)$, is obtained from two copies of G by joining each vertex in the second copy to all neighbors of its corresponding vertex in the first copy. The first and second components are denoted respectively by G and G' , where $V(G) = \{u_i : 1 \leq i \leq |V(G)|\}$ and $V(G') = \{v_i : 1 \leq i \leq |V(G')|\}$. In $D_2(G)$, the distance between each vertex u and its shadow v is two, and both share the same neighborhood, namely $N(u) = N(v)$ [6].

2. Known Result

Distance antimagic labeling (DAL) has been widely studied in various classes of graphs. In particular, several known results have been established for complete and circulant graphs. Simanjuntak and Wijaya [10] studied DAL of complete graph and showed that it admits an (a, d) -DAL. Patel and Vasava [11] studied DAL of circulant graph and showed that it admits an (a, d) -DAL.

As a natural extension of DAL, IDAL and (a, d) -IDAL have also attracted research interest. Several research has been conducted related to IDAL and (a, d) -IDAL in some classes of graphs. In [4], Dafik et al studied IDAL of path, cycle, star, double star, broom, and wheel graph. They also show that there are some graphs that do not admit IDAL, namely complete, friendship, and joint graph of $P_2 + H$, $(P_2 \cup mK_1) + H$, and $K_n + H$. In [6], Ngurah et al studied IDAL of shadow graph of cycle and complete bipartite graph. In [12], Ngurah and Inayah studied IDAL of 2-regular graph.

The following results are used related to our main theorem.

Conjecture 2.1. [4] *Every r -regular graphs except complete graph K_n is inclusive distance antimagic.*

Lemma 2.2. [4] *If G is a connected graphs, with the two adjacent vertices $u, v \in G$ have the same neighborhood $N(u) - \{v\} = N(v) - \{u\}$, then G does not admit inclusive distance antimagic.*

Corollary 2.3. [6] *For every integer $m \geq 1$ and $n \geq 3$, the graph $mD_2(C_n)$ is $(4nm + 3, 1)$ - $\{0, 1\}$ -DAL.*

- (1) For every integer $m \geq 1$, the graph $mD_2(C_3)$ is $(1, 2)$ - $\{2\}$ -DAL.
- (2) For every integer $m \geq 1$, the graph $mD_2(C_4)$ is $(8m + 2, 1)$ - $\{2\}$ -DAL.
- (3) For every integer $m \geq 1$ and $n \geq 5$, the graph $mD_2(C_n)$ is $(4mn + 3, 1)$ - $\{2\}$ -DAL.

Corollary 2.4. [6] The graph $mD_2(K_{n,n})$ is $(n(4mn + 1) + 1, 1)$ - $\{0, 1\}$ -DAL and $(n(4mn - 4m + 1), 1)$ - $\{2\}$ -DAL for every integer $m, n \geq 1$.

Up to now, the study of IDAL of shadow graph is only done for two class graphs, namely, the shadow graph of cycle and the complete bipartite graph. Motivated by these results, and as mentioned in the conjecture above, we conduct a related study on regular graphs. We study IDAL for shadow graph of complete graph and circulant graph.

3. Result and Discussion

In [4], it is proved that the complete graph does not admit an IDAL, since all vertices are pairwise adjacent. Then, the weight of each vertex is equal to the sum of all vertex labels, which results in identical weight for all vertices. However, for a shadow graph of complete graph $D_2(K_n)$, we arrive at the following theorems.

Theorem 3.1. For every integer $n \geq 2$, the graph $D_2(K_n)$ admits IDAL.

Proof. Let $D_2(K_n)$ be a shadow graph of complete graph with $V(D_2(K_n)) = \{u_i, v_i \mid 1 \leq i \leq n\}$ and $E(D_2(K_n)) = \{u_i u_j, u_i v_j, v_i v_j, v_i u_j \mid 1 \leq i < j \leq n\}$.

Define a labeling $f : V(D_2(K_n)) \rightarrow \{1, 2, \dots, 2n\}$ as:

$$\begin{aligned} f(u_i) &= i, \text{ and} \\ f(v_i) &= n + i, \end{aligned}$$

where $1 \leq i \leq n$.

If $A = \{f(u_i) \mid 1 \leq i \leq n\} = \{1, 2, \dots, n\}$ and $B = \{f(v_i) \mid 1 \leq i \leq n\} = \{n+1, n+2, \dots, 2n\}$, then $A \cap B = \emptyset$. Consider the domain function $\{1, 2, \dots, 2n\}$, then each element in the domain appears exactly once in the union of the sets $A \cup B = \{1, 2, \dots, 2n\} = V(D_2(K_n))$. Since $|A \cup B|$ equals to the range of the function f , then f is a bijection function.

Given the labeling f , we have the following weight of vertices, for $1 \leq i \leq n$,

$$\begin{aligned} w(u_i) &= 2n^2 - i, \text{ and} \\ w(v_i) &= 2n^2 + n - i, \end{aligned}$$

If $C = \{w(u_i) \mid 1 \leq i \leq n\} = \{2n^2 - n, 2n^2 - n + 1, \dots, 2n^2 - 1\}$ and $D = \{2n^2, 2n^2 + 1, \dots, 2n^2 + n - 1\}$, it is clear that $w(u) \neq w(v)$ for all $v \in V(D_2(K_n))$. Thus, $D_2(K_n)$ admits an IDAL. □

The shadow graph of complete graph automatically admits an (a, d) -IDAL with $a = 2n^2 - n$ and $d = 1$.

Corollary 3.2. For every integer $n \geq 2$, graph $D_2(K_n)$ admits $(2n^2 - n, 1)$ -IDAL.

Proof. By Theorem 3.1., the elements of the set $C \cup D$ are ordered integers from $2n^2 - n$ to $2n^2 + n - 1$ and form an arithmetic progression with $a = 2n^2 - n$ and $d = 1$. Such that $D_2(K_n)$ admits $(2n^2 - n, 1)$ -IDAL. \square

As an illustration, we present a labeling for the shadow graph of a complete graph $D_2(K_6)$ as in Figure 1.

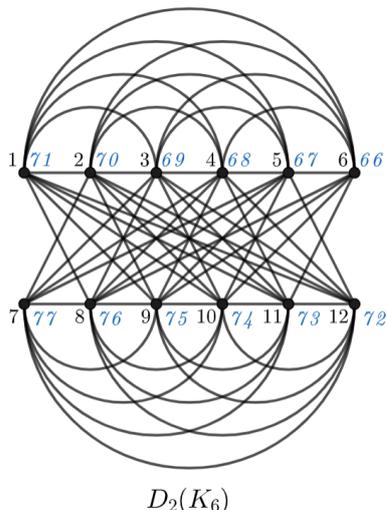


Figure 1. Illustration of IDAL of $D_2(K_6)$.

In the next theorem, we consider another regular graph that admits an IDAL, namely the shadow graph of circulant graph $D_2(C_n(1,2))$. A circulant graph $C_n(m_1, m_2, \dots, m_k)$ is defined as a graph whose set of vertex is $V = \{v_0, v_1, \dots, v_{n-1}\}$, with edges connecting each vertex v_i to v_{i+m_j} for $i = 0, 1, \dots, n-1$ and $j = 1, 2, \dots, k$, where the sum $i + m_j$ is taken modulo n [5]. Observe that Theorem 3.1 can be applied directly for $n = 3, 4, 5$, since $D_2(C_3(1)) \approx D_2(K_3)$, $D_2(C_4(1,2)) \approx D_2(K_4)$, and $D_2(C_5(1,2)) \approx D_2(K_5)$, where the notation \approx indicates that the graphs are isomorphic. Consequently, the labeling method and the corresponding weights remain the same.

Theorem 3.3. For every integer $n \geq 6$, graph $D_2(C_n(1,2))$ admits IDAL.

Proof. Let $D_2(C_n(1,2))$ be a shadow graph of circulant graph with $V(D_2(C_n(1,2))) = \{u_i, v_i \mid 1 \leq i \leq n\}$ and $E(D_2(C_n(1,2))) = \{u_i u_{i+1}, u_i v_{i+1}, v_i v_{i+1}, v_i u_{i+1} \mid 1 \leq i \leq n-1\} \cup \{u_i u_{i+2}, u_i v_{i+2}, v_i v_{i+2}, v_i u_{i+2} \mid 1 \leq i \leq n-2\} \cup \{u_1 u_n, u_1 u_{n-1}, u_1 v_n, u_1 v_{n-1}, v_1 v_n, v_1 v_{n-1}, v_1 u_n, v_1 u_{n-1}\} \cup \{u_2 u_n, u_2 v_n, v_2 v_n, v_2 u_n\}$.

Define a labeling $f : V(D_2(C_n(1,2))) \rightarrow \{1, 2, \dots, 2n\}$ with:

$$\begin{aligned} f(u_i) &= i, \text{ and} \\ f(v_i) &= 2n + 1 - i, \end{aligned}$$

where $1 \leq i \leq n$.

If $A = \{(f(u_i) \mid 1 \leq i \leq n)\} = \{1, 2, \dots, n\}$ and $B = \{f(v_i) \mid 1 \leq i \leq n\} = \{n+1, n+2, \dots, 2n\}$, then $A \cap B = \emptyset$. Consider the domain function $\{1, 2, \dots, 2n\}$, then each element in the domain appears exactly once in the union of the sets $A \cup B = \{1, 2, \dots, 2n\} = V(D_2(C_n(1, 2)))$. Since $|A \cup B|$ is equal to the range of the function f , then f is a bijection function.

Given the labeling f , we can obtain the following weights:

$$w(u_i) = \begin{cases} 8n + 5, & i = 1, \\ 8n + 6, & i = 2, \\ 8n + i + 4, & 3 \leq i \leq n - 2, \\ 9n + 3, & i = n - 1, \\ 9n + 4, & i = n, \end{cases}$$

$$w(v_i) = \begin{cases} 10n + 4, & i = 1, \\ 10n + 3, & i = 2, \\ 10n - i + 5, & 3 \leq i \leq n - 2, \\ 9n + 6, & i = n - 1, \\ 9n + 5, & i = n. \end{cases}$$

If $C = \{w(u_i) \mid 1 \leq i \leq n\} = \{8n + 5, 8n + 6, \dots, 9n + 2, 9n + 3, 9n + 4\}$ and $D = \{9n + 5, 9n + 6, \dots, 10n + 2, 10n + 3, 10n + 4\}$, then it is clear that $w(u) \neq w(v)$ for all $u, v \in V(D_2(C_n(1, 2)))$. Thus, $D_2(C_n(1, 2))$ admits IDAL. \square

The shadow graph of circulant graph also automatically admits an (a, d) -IDAL with $a = 8n + 5$ and $d = 1$.

Corollary 3.4. *Graph $D_2(C_n(1, 2))$ admits $(8n + 5, 1)$ -IDAL.*

Proof. By Theorem 3.3., the elements of the set $C \cup D$ are ordered integers from $8n + 5$ to $10n + 4$ and forming arithmetic progression with $a = 8n + 5$ and $d = 1$. Therefore, $D_2(C_n(1, 2))$ admits $(8n + 5, 1)$ -IDAL. \square

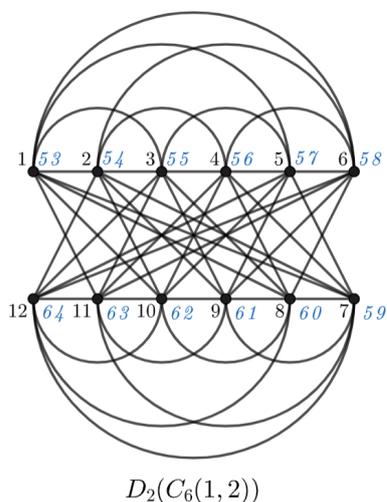
As an illustration, we present a labeling for the shadow graph of a circulant graph $D_2(C_6(1, 2))$ as in Figure 2.

4. Conclusion

In this paper, we study and prove the existance of IDAL and (a, d) -IDAL of shadow graph of complete graph and circulant graph. For future work, one may explore other classes of regular graphs that satisfy such labelings.

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Figure 2. Illustration of IDAL of $D_2(C_6(1, 2))$.

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