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THE PARTITION DIMENSION OF ROSE GRAPHS AND ITS BARBELL

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Abstract. *The rose graph, denoted by $R(C_n)$, $n \geq 3$, constructed by a cycle graph C_n with n isolated vertices that connect every two vertices in the cycle graph with one isolated vertex. The barbell rose graph, denoted by $B_{R(C_n)}$ is a simple graph formed by connecting two rose graphs $R(C_n)$ by edges v_1, v_1' as a bridge. In this paper, we determined the partition dimension of the rose graph and its barbell.*

Keywords: Partition dimension, Barbell rose graph, Rose graph

1. Introduction

Let $G = (V, E)$ be a connected graph. For a subset $S \subset V(G)$ and a vertex $v \in V(G)$, the distance between v and S is defined as $d(v, S) = \min\{d(v, x), x \in S\}$ with $d(v, x)$ is the distance from vertex v to x. Suppose $V(G)$ is partitioned into t mutually exclusive sets $\{S_1, S_2, \cdots, S_t\}$. For t-ordered partitions $\mathcal{A} = \{S_1, S_2, \cdots, S_t\}$ of $V(G)$ and a vertex $v \in V(G)$, the representation of v with respect to $\mathcal{A}$, is defined as $k(v \mid \mathcal{A}) = (d(v, S_1), d(v, S_2), \cdots, d(v, S_t))$. A partition $\mathcal{A}$ is called the resolving partition of $V(G)$ if $k(u \mid \mathcal{A}) \neq k(v \mid \mathcal{A})$ has two distinct vertices $u, v \in V(G)$. The smallest t value (minimum cardinality) such that graph G has a resolving partition with t partition class is the partition dimension of graph G. The partition dimension of graph G is denoted by $pd(G)$ [1].

Asmiati in [2] successfully determined the partition dimension of the amalgamation of star graphs. Amrullah et al. in [3] successfully determined the partition dimension of subdivision graph on the star and in 2020, Amrullah successfully determined the partition dimension for a subdivision of a homogeneous firecracker [4]. Other studies on similar topics such as [5], [6], [7], and [8]. Latest research on this topic can be found in [9], [10], [11], [12], [13], and [14].

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In this paper, we determine the partition dimension of the rose graph and the partition dimension of the barbell rose graph. The rose graph, denoted by $R(C_n)$, $n \geq 3$ is a connected graph constructed by a cycle graph C_n with vertices $v_1, v_2, \cdots, v_n$ and n isolated vertices $w_1, w_2, \cdots, w_n$, by connecting every two vertices v_s, v_{s+1} with w_s, for $s = 1, 2, \cdots, n$ where $v_{n+1} = v_1$ [15]. The barbell rose graph, denoted by $B_{R(C_n)}$ is a simple graph formed by connecting two rose graphs $R(C_n)$ by edges $v_1, v_1^{'}$ as a bridge [9]. Theorem 1.1 gives the partition dimension of cycle C_n for $n \geq 3$.

Theorem 1.1. [1] *The partition dimension of the cycle graph C_n for $n \geq 3$ is 3.*

2. Result and Discussion

In this section, we determine the partition dimension of the rose graph and the barbell rose graph.

Theorem 2.1. *The partition dimension of the rose graph $pd(R(C_n))$ is 3 for $n \geq 3$.*

Proof. Let $R(C_n)$ with $n \geq 3$ be the rose graph with the set of vertex $V(R(C_n)) = \{v_s \mid s = 1, 2, \cdots, n\} \cup \{w_s \mid s = 1, 2, \cdots, n\}$, and the set of edges $E(R(C_n)) = \{v_s v_{s+1} \mid s = 1, 2, \cdots, n\} \cup \{v_s w_s \mid s = 1, 2, \cdots, n\} \cup \{v_{s+1} w_s \mid s = 1, 2, \cdots, n\}$ with $v_{n+1} = v_1$.

We begin by determining the lower bound of the partition dimension of the rose graph $R(C_n)$, for $n \geq 3$. Since the rose graph $R(C_n)$ contains the cycle graph C_n, it follows from Theorem 1.1 that:

$$pd(R(C_n)) \geq 3, \quad \text{for } n \geq 3. \tag{2.1}$$

To find the upper bound of the partition dimension of $R(C_n)$, we examine the following three cases.

(Case 1) For $n \geq 3$, with $n \equiv 0$ $(mod\ 3)$.

Let $\mathcal{A} = \{S_1, S_2, S_3\}$ be a partition of the set of vertex $V(R(C_n))$. The partition classes of the graph $R(Cn)$ are defined as follows.

$$S_1 = \{v_s \mid 1 \leq s \leq \frac{n}{3}\} \cup \{w_s \mid 1 \leq s \leq \frac{n}{3}\},$$
$$S_2 = \{v_s \mid \frac{n+3}{3} \leq s \leq \frac{2n}{3}\} \cup \{w_s \mid \frac{n+3}{3} \leq s \leq \frac{2n}{3}\},$$
$$S_3 = \{v_s \mid \frac{2n+3}{3} \leq s \leq n\} \cup \{w_s \mid \frac{2n+3}{3} \leq s \leq n\}.$$

The representation of all vertices in the rose graph $R(C_n)$ with respect to $\mathcal{A}$ are defined as follows:

$$r(v_s|\mathcal{A}) = (0, \frac{n+3}{3} - s, s), \text{ for } 1 \leq s \leq \frac{n}{3},$$
$$r(v_s|\mathcal{A}) = (s - \frac{n}{3}, 0, \frac{2n+3}{3} - s), \text{ for } \frac{n+3}{3} \leq s \leq \frac{2n}{3},$$
$$r(v_s|\mathcal{A}) = (n - s + 1, s - \frac{2n}{3}, 0), \text{ for } \frac{2n+3}{3} \leq s \leq n,$$

$$r(w_s|\mathcal{A}) = (0, \frac{n+3}{3} - s, s+1), \text{ for } 1 \le s \le \frac{n}{3},$$
$$r(w_s|\mathcal{A}) = (s - \frac{n+3}{3}, 0, \frac{2n+3}{3} - s), \text{ for } \frac{n+3}{3} \le s \le \frac{2n}{3},$$
$$r(w_s|\mathcal{A}) = (n - s + 1, s - \frac{2n+3}{3}, 0), \text{ for } \frac{2n+3}{3} \le s \le n.$$

Since each vertex in the rose graph $R(C_n)$ for $n \ge 3$ with $n \equiv 0$ $(mod\ 3)$ has a distinct representation with respect to $\mathcal{A}$, it follows that $\mathcal{A}$ is a resolving partition of the rose graph $R(C_n)$. Therefore,

$$pd(R(C_n)) \le 3. \tag{2.2}$$

(Case 2) For $n \ge 3$, with $n \equiv 1$ $(mod\ 3)$.
Given $\mathcal{A} = \{S_1, S_2, S_3\}$, the partition classes of the graph $R(C_n)$ are defined as follows:

$$S_1 = \{v_s \mid 1 \le s \le \frac{n+2}{3}\} \cup \{w_s \mid 1 \le s \le \frac{n-1}{3}\},$$
$$S_2 = \{v_s \mid \frac{n+5}{3} \le s \le \frac{2n+1}{3}\} \cup \{w_s \mid \frac{n+2}{3} \le s \le \frac{2n+1}{3}\},$$
$$S_3 = \{v_s \mid \frac{2n+4}{3} \le s \le n\} \cup \{w_s \mid \frac{2n+4}{3} \le s \le n\}.$$

So, the representations of all the vertices of the rose graph $R(C_n)$ with respect to $\mathcal{A}$ are defined as follows:

$$r(v_s|\mathcal{A}) = (0, \frac{n+5}{3} - s, s), \text{ for } 1 \le s \le \frac{n+2}{3},$$
$$r(v_s|\mathcal{A}) = (s - \frac{n+2}{3}, 0, \frac{2n+4}{3} - s), \text{ for } \frac{n+5}{3} \le s \le \frac{2n+1}{3},$$
$$r(v_s|\mathcal{A}) = (n - s + 1, s - \frac{2n+1}{3}, 0), \text{ for } \frac{2n+4}{3} \le s \le n,$$
$$r(w_s|\mathcal{A}) = (0, \frac{n+5}{3} - s, s+1), \text{ for } 1 \le s \le \frac{n-1}{3},$$
$$r(w_s|\mathcal{A}) = (s - \frac{n-1}{3}, 0, \frac{2n+4}{3} - s), \text{ for } \frac{n+2}{3} \le s \le \frac{2n+1}{3},$$
$$r(w_s|\mathcal{A}) = (n - s + 1, s - \frac{2n-2}{3}, 0), \text{ for } \frac{2n+4}{3} \le s \le n.$$

Since each vertex in the rose graph $R(C_n)$ for $n \ge 3$, with $n \equiv 1$ $(mod\ 3)$ has a distinct representation with respect to $\mathcal{A}$, it follows that $\mathcal{A}$ is a resolving partition of the rose graph $R(C_n)$. Therefore,

$$pd(R(C_n)) \le 3. \tag{2.3}$$

(Case 3) For $n \ge 3$, with $n \equiv 2$ $(mod\ 3)$.
Given $\mathcal{A} = \{S_1, S_2, S_3\}$, the partition classes of the graph $R(C_n)$ are defined as follows:

$$S_1 = \{v_s \mid 1 \le s \le \frac{n+1}{3}\} \cup \{w_s \mid 1 \le s \le \frac{n+1}{3}\},$$
$$S_2 = \{v_s \mid \frac{n+4}{3} \le s \le \frac{2n+2}{3}\} \cup \{w_s \mid \frac{n+4}{3} \le s \le \frac{2n-1}{3}\},$$
$$S_3 = \{v_s \mid \frac{2n+5}{3} \le s \le n\} \cup \{w_s \mid \frac{2n+2}{3} \le s \le n\}.$$

So, the representations of all the vertices of the rose graph $R(C_n)$ with respect to $\mathcal{A}$ are defined as follows:

$$\begin{aligned}
r(v_s|\mathcal{A}) &= (0, \frac{n+4}{3} - s, s), \text{ for } 1 \le s \le \frac{n+1}{3},\\
r(v_s|\mathcal{A}) &= (s - \frac{n+1}{3}, 0, \frac{2n+5}{3} - s), \text{ for } \frac{n+4}{3} \le s \le \frac{2n+2}{3},\\
r(v_s|\mathcal{A}) &= (n - s + 1, s - \frac{2n+2}{3}, 0), \text{ for } \frac{2n+5}{3} \le s \le n,\\
r(w_s|\mathcal{A}) &= (0, \frac{n+4}{3} - s, s+1), \text{ for } 1 \le s \le \frac{n+1}{3},\\
r(w_s|\mathcal{A}) &= (s - \frac{n-2}{3}, 0, \frac{2n+5}{3} - s), \text{ for } \frac{n+4}{3} \le s \le \frac{2n-1}{3},\\
r(w_s|\mathcal{A}) &= (n - s + 1, s - \frac{2n-1}{3}, 0), \text{ for } \frac{2n+2}{3} \le s \le n.
\end{aligned}$$

Since each vertex in the rose graph $R(C_n)$ for $n \ge 3$, with $n \equiv 2$ (mod 3) has a distinct representation with respect to $\mathcal{A}$, it follows that $\mathcal{A}$ is a resolving partition of the rose graph $R(C_n)$. Therefore,

$$pd(R(C_n)) \le 3. \tag{2.4}$$

The following Figure 1 shows a minimum resolving partition of graph $R(C_5)$.

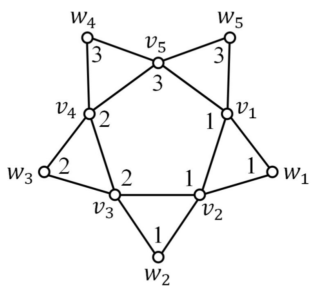

Figure 1. A minimum resolving partition of $R(C_5)$.

From Eq. (2.2) - Eq. (2.4), it follows that the upper bound of $R(C_n)$ for $n \ge 3$ is:

$$pd(R(C_n)) \le 3. \tag{2.5}$$

Therefore, from Eq. (2.1) and Eq. (2.5), the partition dimension of the rose graph $R(C_n)$ is 3 for $n \ge 3$. □

Theorem 2.2. *The partition dimension of the barbell rose graph* $pd(B_{R(C_n)})$ *is 4, for* $n \ge 3$.

Proof. Let $B_{R(C_n)}$ with $n \ge 3$ is a barbell rose graph with the set of vertices $V(B_{R(C_n)}) = \{v_s \mid s = 1, 2, \cdots, n\} \cup \{w_s \mid s = 1, 2, \cdots, n\} \cup \{v_s' \mid s = 1, 2, \cdots, n\} \cup \{w_s' \mid s = 1, 2, \cdots, n\}$, and the set of edges $E(B_{R(C_n)}) = \{v_s v_{s+1} \mid s = 1, 2, \cdots, n\}$

$\cup \{v_s'w_s \mid s = 1, 2, \cdots, n\} \cup \{v_{s+1}w_s \mid s = 1, 2, \cdots, n\} \cup \{v_1v_1'\} \cup \{v_s'v_{s+1}' \mid s = 1, 2, \cdots, n\} \cup \{v_s'w_s' \mid s = 1, 2, \cdots, n\} \cup \{v_{s+1}'w_s' \mid s = 1, 2, \cdots, n\}$ with $v_{n+1} = v_1$.

First, we will determine the lower bound of the partition dimensions of the barbell rose graph $B_{R(C_n)}$ for $n \geq 3$. Since the barbell rose graph $B_{R(C_n)}$ contains the rose graph $R(C_n)$, it follows by Theorem 2.1 that:

$$pd(B_{R(C_n)}) \geq 3, \quad \text{for } n \geq 3. \tag{2.6}$$

Suppose that the vertices of the barbell rose graph $B_{R(C_n)}$ are partitioned into three partitions, namely $\mathcal{A} = \{S_1, S_2, S_3\}$. As a result, some vertices share the same representation, as there are at least two vertices that have the same distances to the other vertices in the barbell rose graph $B_{R(C_n)}$. This contradicts the assumption. Therefore, at least four partition classes are required to determine the partition dimension of the barbell rose graph $B_{R(C_n)}$ for $n \geq 3$. Thus,

$$pd(B_{R(C_n)}) \geq 4, \quad \text{for } n \geq 3. \tag{2.7}$$

In the following, we determine the upper bound of the partition dimensions of the barbell rose graph $B_{R(C_n)}$. Consider the following three cases to determine the upper bound on the partition dimensions of the barbell rose graph $B_{R(C_n)}$.

(Case 1) For $n \geq 3$, with $n \equiv 0 \ (mod\ 3)$.
Consider the following three subcases.

(Subcase 1) For $n = 3$.
Given $\mathcal{A} = \{S_1, S_2, S_3, S_4\}$, the partition classes of the graph $B_{R(C_n)}$ are defined as follows.

$$\begin{aligned}
S_1 &= \{v_1\} \cup \{w_n\} \cup \{v_1'\} \cup \{w_n'\},\\
S_2 &= \{v_{n-1}\} \cup \{w_1\} \cup \{v_{n-1}'\} \cup \{w_1'\},\\
S_3 &= \{v_n\} \cup \{w_{n-1}\},\\
S_4 &= \{v_n'\} \cup \{w_{n-1}'\}.
\end{aligned}$$

Therefore, the representations of all the vertices of the barbell rose graph $B_{R(C_n)}$ with respect to $\mathcal{A}$ are defined as follows.

$$\begin{aligned}
r(v_s|\mathcal{A}) &= (0, \frac{n+9}{6} - s, \frac{n-3}{6} + s, \frac{n+3}{6} + s), \text{ for } s = 1,\\
r(v_s|\mathcal{A}) &= (s - \frac{n+3}{6}, 0, \frac{n+3}{2} - s, \frac{n+3}{6} + s), \text{ for } s = n-1,\\
r(v_s|\mathcal{A}) &= (\frac{5n+9}{6} - s, s - \frac{n+1}{2}, 0, \frac{n+9}{6} + s), \text{ for } s = n,\\
r(w_s|\mathcal{A}) &= (s - \frac{n-3}{6}, 0, \frac{n+3}{2} - s, \frac{n+9}{6} + s), \text{ for } s = 1,\\
r(w_s|\mathcal{A}) &= (\frac{5n+9}{6} - s, s - \frac{n-1}{2}, 0, \frac{7n+15}{6} - s), \text{ for } s = n-1,\\
r(w_s|\mathcal{A}) &= (0, \frac{7n+9}{6} - s, s - \frac{5n-3}{6}, \frac{7n+15}{6} - s), \text{ for } s = n,
\end{aligned}$$

$$r(v_s^{'}|\mathcal{A}) = (0, \frac{n+9}{6} - s, \frac{n+3}{6} + s, \frac{n-3}{6} + s), \text{ for } s = 1,$$
$$r(v_s^{'}|\mathcal{A}) = (s - \frac{n+3}{6}, 0, \frac{n+3}{6} + s, \frac{n+3}{2} - s), \text{ for } s = n-1,$$
$$r(v_s^{'}|\mathcal{A}) = (\frac{5n+9}{6} - s, s - \frac{n+1}{2}, \frac{n+9}{6} + s, 0), \text{ for } s = n,$$
$$r(w_s^{'}|\mathcal{A}) = (s - \frac{n-3}{6}, 0, \frac{n+9}{6} + s, \frac{n+3}{2} - s), \text{ for } s = 1,$$
$$r(w_s^{'}|\mathcal{A}) = (\frac{5n+9}{6} - s, s - \frac{n-1}{2}, \frac{7n+15}{6} - s, 0), \text{ for } s = n-1,$$
$$r(w_s^{'}|\mathcal{A}) = (0, \frac{7n+9}{6} - s, \frac{7n+15}{6} - s, s - \frac{5n-3}{6}), \text{ for } s = n.$$

(Subcase 2) For $n > 3$ odd, with $n \equiv 0$ $(mod\ 3)$.
Given $\mathcal{A} = \{S_1, S_2, S_3, S_4\}$, the partition classes of the graph $B_{R(C_n)}$ are defined as follows.

$$\begin{aligned}
S_1 = &\{ v_s \mid 1 \le s \le \tfrac{n+3}{6} \} \cup \{ v_s \mid \tfrac{5n+9}{6} \le s \le n\} \\
&\cup \{ w_s \mid 1 \le s \le \tfrac{n-3}{6} \} \cup \{ w_s \mid \tfrac{5n+3}{6} \le s \le n\} \\
&\cup \{ v_s^{'} \mid 1 \le s \le \tfrac{n+3}{6} \} \cup \{ v_s^{'} \mid \tfrac{5n+9}{6} \le s \le n\}, \\
&\cup \{ w_s^{'} \mid 1 \le s \le \tfrac{n-3}{6}\} \cup \{ w_s^{'} \mid \tfrac{5n+3}{6} \le s \le n\}, \\
S_2 = &\{ v_s \mid \tfrac{n+9}{6} \le s \le \tfrac{n+1}{2}\} \cup \{ w_s \mid \tfrac{n+3}{6} \le s \le \tfrac{n-1}{2}\} \\
&\cup \{ v_s^{'} \mid \tfrac{n+9}{6} \le s \le \tfrac{n+1}{2}\} \cup \{ w_s^{'} \mid \tfrac{n+3}{6} \le s \le \tfrac{n-1}{2}\}, \\
S_3 = &\{ v_s \mid \tfrac{n+3}{2} \le s \le \tfrac{5n+3}{6}\} \cup \{ w_s \mid \tfrac{n+1}{2} \le s \le \tfrac{5n-3}{6}\}, \\
S_4 = &\{ v_s^{'} \mid \tfrac{n+3}{2} \le s \le \tfrac{5n+3}{6}\} \cup \{ w_s^{'} \mid \tfrac{n+1}{2} \le s \le \tfrac{5n-3}{6}\}.
\end{aligned}$$

Therefore, the representations of all the vertices of the barbell rose graph $B_{R(C_n)}$ with respect to $\mathcal{A}$ are defined as follows.

$$r(v_s|\mathcal{A}) = (0, \frac{n+9}{6} - s, \frac{n-3}{6} + s, \frac{n+3}{6} + s), \text{ for } 1 \le s \le \frac{n+3}{6},$$
$$r(v_s|\mathcal{A}) = (s - \frac{n+3}{6}, 0, \frac{n+3}{2} - s, \frac{n+3}{6} + s), \text{ for } \frac{n+9}{6} \le s \le \frac{n+1}{2},$$
$$r(v_s|\mathcal{A}) = (\frac{5n+9}{6} - s, s - \frac{n+1}{2}, 0, \frac{7n+15}{6} - s), \text{ for } \frac{n+3}{2} \le s \le \frac{5n+3}{6},$$
$$r(v_s|\mathcal{A}) = (0, \frac{7n+9}{6} - s, s - \frac{5n+3}{6}, \frac{7n+15}{6} - s), \text{ for } \frac{5n+9}{6} \le s \le n,$$
$$r(w_s|\mathcal{A}) = (0, \frac{n+9}{6} - s, \frac{n+3}{6} + s, \frac{n+9}{6} + s), \text{ for } 1 \le s \le \frac{n-3}{6},$$
$$r(w_s|\mathcal{A}) = (s - \frac{n-3}{6}, 0, \frac{n+3}{2} - s, \frac{n+9}{6} + s), \text{ for } \frac{n+3}{6} \le s \le \frac{n-1}{2},$$
$$r(w_s|\mathcal{A}) = (\frac{5n+9}{6} - s, s - \frac{n-1}{2}, 0, \frac{7n+15}{6} - s), \text{ for } \frac{n+1}{2} \le s \le \frac{5n-3}{6},$$
$$r(w_s|\mathcal{A}) = (0, \frac{7n+9}{6} - s, s - \frac{5n-3}{6}, \frac{7n+15}{6} - s), \text{ for } \frac{5n+3}{6} \le s \le n,$$

$$r(v_s^{'}|\mathcal{A}) = (0, \frac{n+9}{6} - s, \frac{n+3}{6} + s, \frac{n-3}{6} + s), \text{ for } 1 \leq s \leq \frac{n+3}{6},$$
$$r(v_s^{'}|\mathcal{A}) = (s - \frac{n+3}{6}, 0, \frac{n+3}{6} + s, \frac{n+3}{2} - s), \text{ for } \frac{n+9}{6} \leq s \leq \frac{n+1}{2},$$
$$r(v_s^{'}|\mathcal{A}) = (\frac{5n+9}{6} - s, s - \frac{n+1}{2}, \frac{7n+15}{6} - s, 0), \text{ for } \frac{n+3}{2} \leq s \leq \frac{5n+3}{6},$$
$$r(v_s^{'}|\mathcal{A}) = (0, \frac{7n+9}{6} - s, \frac{7n+15}{6} - s, s - \frac{5n+3}{6}), \text{ for } \frac{5n+9}{6} \leq s \leq n,$$
$$r(w_s^{'}|\mathcal{A}) = (0, \frac{n+9}{6} - s, \frac{n+9}{6} + s, \frac{n+3}{6} + s), \text{ for } 1 \leq s \leq \frac{n-3}{6},$$
$$r(w_s^{'}|\mathcal{A}) = (s - \frac{n-3}{6}, 0, \frac{n+9}{6} + s, \frac{n+3}{2} - s), \text{ for } \frac{n+3}{6} \leq s \leq \frac{n-1}{2},$$
$$r(w_s^{'}|\mathcal{A}) = (\frac{5n+9}{6} - s, s - \frac{n-1}{2}, \frac{7n+15}{6} - s, 0), \text{ for } \frac{n+1}{2} \leq s \leq \frac{5n-3}{6},$$
$$r(w_s^{'}|\mathcal{A}) = (0, \frac{7n+9}{6} - s, \frac{7n+15}{6} - s, s - \frac{5n-3}{6}), \text{ for } \frac{5n+3}{6} \leq s \leq n.$$

(Subcase 3) For $n > 3$ even, with $n \equiv 0 \ (mod\ 3)$.
Given $\mathcal{A} = \{S_1, S_2, S_3, S_4\}$, the partition classes of the graph $B_{R(C_n)}$ are defined as follows.

$$\begin{aligned} S_1 = &\{v_s \mid 1 \leq s \leq \tfrac{n}{6}\} \cup \{v_s \mid \tfrac{5n+6}{6} \leq s \leq n\} \\ &\cup \{w_s \mid 1 \leq s \leq \tfrac{n}{6}\} \cup \{w_s \mid \tfrac{5n+6}{6} \leq s \leq n\} \\ &\cup \{v_s^{'} \mid 1 \leq s \leq \tfrac{n}{6}\} \cup \{v_s^{'} \mid \tfrac{5n+6}{6} \leq s \leq n\} \\ &\cup \{w_s^{'} \mid 1 \leq s \leq \tfrac{n}{6}\} \cup \{w_s^{'} \mid \tfrac{5n+6}{6} \leq s \leq n\}, \\ S_2 = &\{v_s \mid \tfrac{n+6}{6} \leq s \leq \tfrac{n}{2}\} \cup \{w_s \mid \tfrac{n+6}{6} \leq s \leq \tfrac{n}{2}\} \\ &\cup \{v_s^{'} \mid \tfrac{n+6}{6} \leq s \leq \tfrac{n}{2}\} \cup \{w_s^{'} \mid \tfrac{n+6}{6} \leq s \leq \tfrac{n}{2}\}, \\ S_3 = &\{v_s \mid \tfrac{n+2}{2} \leq s \leq \tfrac{5n}{6}\} \cup \{w_s \mid \tfrac{n+2}{2} \leq s \leq \tfrac{5n}{6}\}, \\ S_4 = &\{v_s^{'} \mid \tfrac{n+2}{2} \leq s \leq \tfrac{5n}{6}\} \cup \{w_s^{'} \mid \tfrac{n+2}{2} \leq s \leq \tfrac{5n}{6}\}. \end{aligned}$$

Therefore, the representations of all the vertices of the barbell rose graph $B_{R(C_n)}$ with respect to $\mathcal{A}$ are defined as follows.

$$r(v_s|\mathcal{A}) = (0, \frac{n+6}{6} - s, \frac{n}{6} + s, \frac{n+6}{6} + s), \text{ for } 1 \leq s \leq \frac{n}{6},$$
$$r(v_s|\mathcal{A}) = (s - \frac{n}{6}, 0, \frac{n+2}{2} - s, \frac{n+6}{6} + s), \text{ for } \frac{n+6}{6} \leq s \leq \frac{n}{2},$$
$$r(v_s|\mathcal{A}) = (\frac{5n+6}{6} - s, s - \frac{n}{2}, 0, \frac{7n+18}{6} - s), \text{ for } \frac{n+2}{2} \leq s \leq \frac{5n}{6},$$
$$r(v_s|\mathcal{A}) = (0, \frac{7n+6}{6} - s, s - \frac{5n}{6}, \frac{7n+18}{6} - s), \text{ for } \frac{5n+6}{6} \leq s \leq n,$$
$$r(w_s|\mathcal{A}) = (0, \frac{n+6}{6} - s, \frac{n+6}{6} + s, \frac{n+12}{6} + s), \text{ for } 1 \leq s \leq \frac{n}{6},$$
$$r(w_s|\mathcal{A}) = (s - \frac{n+6}{6}, 0, \frac{n+2}{2} - s, \frac{n+12}{6} + s), \text{ for } \frac{n+6}{6} \leq s \leq \frac{n}{2},$$
$$r(w_s|\mathcal{A}) = (\frac{5n+6}{6} - s, s - \frac{n-2}{2}, 0, \frac{7n+18}{6} - s), \text{ for } \frac{n+2}{2} \leq s \leq \frac{5n}{6},$$

$$
\begin{aligned}
r(w_s|\mathcal{A}) &= (0, \frac{7n+6}{6} - s, s - \frac{5n-6}{6}, \frac{7n+18}{6} - s), \text{ for } \frac{5n+6}{6} \le s \le n,\\
r(v_s^{'}|\mathcal{A}) &= (0, \frac{n+6}{6} - s, \frac{n+6}{6} + s, \frac{n}{6} + s), \text{ for } 1 \le s \le \frac{n}{6},\\
r(v_s^{'}|\mathcal{A}) &= (s - \frac{n}{6}, 0, \frac{n+6}{6} + s, \frac{n+2}{2} - s), \text{ for } \frac{n+6}{6} \le s \le \frac{n}{2},\\
r(v_s^{'}|\mathcal{A}) &= (\frac{5n+6}{6} - s, s - \frac{n}{2}, \frac{7n+18}{6} - s, 0, \text{ for } \frac{n+2}{2} \le s \le \frac{5n}{6},\\
r(v_s^{'}|\mathcal{A}) &= (0, \frac{7n+6}{6} - s, \frac{7n+18}{6} - s, s - \frac{5n}{6}), \text{ for } \frac{5n+6}{6} \le s \le n,\\
r(w_s^{'}|\mathcal{A}) &= (0, \frac{n+6}{6} - s, \frac{n+12}{6} + s, \frac{n+6}{6} + s), \text{ for } 1 \le s \le \frac{n}{6},\\
r(w_s^{'}|\mathcal{A}) &= (s - \frac{n+6}{6}, 0, \frac{n+12}{6} + s, \frac{n+2}{2} - s), \text{ for } \frac{n+6}{6} \le s \le \frac{n}{2},\\
r(w_s^{'}|\mathcal{A}) &= (\frac{5n+6}{6} - s, s - \frac{n-2}{2}, \frac{7n+18}{6} - s, 0), \text{ for } \frac{n+2}{2} \le s \le \frac{5n}{6},\\
r(w_s^{'}|\mathcal{A}) &= (0, \frac{7n+6}{6} - s, \frac{7n+18}{6} - s, s - \frac{5n-6}{6}), \text{ for } \frac{5n+6}{6} \le s \le n.
\end{aligned}
$$

Since each vertex in the barbell rose graph $B_{R(C_n)}$ for $n \geq 3$, with $n \equiv 0$ $(mod$ $3)$ has a distinct representation with respect to $\mathcal{A}$, it follows that $\mathcal{A}$ is a resolving partition of the barbell rose graph $B_{R(C_n)}$. Therefore,

$$pd(B_{R(C_n)}) \leq 4. \tag{2.8}$$

(Case 2) For $n \geq 4$, with $n \equiv 1$ $(mod\ 3)$.
Consider the following three subcases.

(Subcase 1) For $n = 4$.
Given $\mathcal{A} = \{S_1, S_2, S_3, S_4\}$, the partition classes of the graph $B_{R(C_n)}$ are defined as follows.

$$
\begin{aligned}
S_1 &= \{\, v_1 \,\} \cup \{\, v_n \,\} \cup \{\, w_n \,\} \cup \{\, v_1^{'} \,\} \cup \{\, v_n^{'} \,\} \cup \{\, w_n^{'} \,\},\\
S_2 &= \{\, v_{n-2} \,\} \cup \{\, w_s \mid 1 \le s \le n-2 \,\} \cup \{\, v_{n-2}^{'} \,\} \cup \{\, w_s^{'} \mid 1 \le s \le n-2 \,\},\\
S_3 &= \{\, v_{n-1} \,\} \cup \{\, w_{n-1} \,\},\\
S_4 &= \{\, v_{n-1}^{'} \,\} \cup \{\, w_{n-1}^{'} \,\}.
\end{aligned}
$$

Therefore, the representations of all the vertices of the barbell rose graph $B_{R(C_n)}$ with respect to $\mathcal{A}$ are defined as follows.

$$
\begin{aligned}
r(v_s|\mathcal{A}) &= (0, \frac{n+8}{6} - s, \frac{n+2}{6} + s, \frac{n+8}{6} + s), \text{ for } s = 1,\\
r(v_s|\mathcal{A}) &= (s - \frac{n+2}{6}, 0, \frac{n+2}{2} - s, \frac{n+8}{6} + s), \text{ for } s = n-2,\\
r(v_s|\mathcal{A}) &= (\frac{5n+4}{6} - s, s - \frac{n}{2}, 0, \frac{7n+6}{6} - s), \text{ for } s = n-1,\\
r(v_s|\mathcal{A}) &= (0, \frac{7n+8}{6} - s, s - \frac{5n-2}{6}, s - \frac{7n+20}{6}), \text{ for } s = n,\\
r(w_s|\mathcal{A}) &= (s - \frac{n-4}{6}, 0, \frac{n+2}{2} - s, \frac{n+14}{6} + s), \text{ for } 1 \le s \le n-2,
\end{aligned}
$$

$$r(w_s|\mathcal{A}) = (\frac{5n+4}{6} - s, s - \frac{n-2}{2}, 0, \frac{7n+20}{6} - s),\ \text{for } s = n-1,$$
$$r(w_s|\mathcal{A}) = (0, \frac{7n+8}{6} - s, s - \frac{5n-8}{6}, \frac{7n+20}{6} - s),\ \text{for } s = n,$$
$$r(v_s^{'}|\mathcal{A}) = (0, \frac{n+8}{6} - s, \frac{n+8}{6} + s, \frac{n+2}{6} + s),\ \text{for } s = 1,$$
$$r(v_s^{'}|\mathcal{A}) = (s - \frac{n+2}{6}, 0, \frac{n+8}{6} + s, \frac{n+2}{2} - s),\ \text{for } s = n-2,$$
$$r(v_s^{'}|\mathcal{A}) = (\frac{5n+4}{6} - s, s - \frac{n}{2}, \frac{7n+6}{6} - s, 0),\ \text{for } s = n-1,$$
$$r(v_s^{'}|\mathcal{A}) = (0, \frac{7n+8}{6} - s, s - \frac{7n+20}{6}, s - \frac{5n-2}{6}),\ \text{for } s = n,$$
$$r(w_s^{'}|\mathcal{A}) = (s - \frac{n-4}{6}, 0, \frac{n+14}{6} + s, \frac{n+2}{2} - s),\ \text{for } 1 \le s \le n-2,$$
$$r(w_s^{'}|\mathcal{A}) = (\frac{5n+4}{6} - s, s - \frac{n-2}{2}, \frac{7n+20}{6} - s, 0),\ \text{for } s = n-1,$$
$$r(w_s^{'}|\mathcal{A}) = (0, \frac{7n+8}{6} - s, \frac{7n+20}{6} - s, s - \frac{5n-8}{6}),\ \text{for } s = n.$$

(Subcase 2) For $n > 4$ odd, with $n \equiv 1\ (mod\ 3)$.
Given $\mathcal{A} = \{S_1, S_2, S_3, S_4\}$, the partition classes of the graph $B_{R(C_n)}$ are defined as follows.

$$\begin{aligned}
S_1 = &\{\, v_s \mid 1 \le s \le \tfrac{n-1}{6} \,\} \cup \{\, v_s \mid \tfrac{5n+7}{6} \le s \le n \,\}\\
&\cup \{\, w_s \mid 1 \le s \le \tfrac{n-1}{6} \,\} \cup \{\, w_s \mid \tfrac{5n+1}{6} \le s \le n \,\}\\
&\cup \{\, v_s^{'} \mid 1 \le s \le \tfrac{n-1}{6} \,\} \cup \{\, v_s^{'} \mid \tfrac{5n+7}{6} \le s \le n \,\}\\
&\cup \{\, w_s^{'} \mid 1 \le s \le \tfrac{n-1}{6} \,\} \cup \{\, w_s^{'} \mid \tfrac{5n+1}{6} \le s \le n \,\},\\
S_2 = &\{\, v_s \mid \tfrac{n+5}{6} \le s \le \tfrac{n+1}{2} \,\} \cup \{\, w_s \mid \tfrac{n+5}{6} \le s \le \tfrac{n-1}{2} \,\}\\
&\cup \{\, v_s^{'} \mid \tfrac{n+5}{6} \le s \le \tfrac{n+1}{2} \,\} \cup \{\, w_s^{'} \mid \tfrac{n+5}{6} \le s \le \tfrac{n-1}{2} \,\},\\
S_3 = &\{\, v_s \mid \tfrac{n+3}{2} \le s \le \tfrac{5n+1}{6} \,\} \cup \{\, w_s \mid \tfrac{n+1}{2} \le s \le \tfrac{5n-5}{6} \,\},\\
S_4 = &\{\, v_s^{'} \mid \tfrac{n+3}{2} \le s \le \tfrac{5n+1}{6} \,\} \cup \{\, w_s^{'} \mid \tfrac{n+1}{2} \le s \le \tfrac{5n-5}{6} \,\}.
\end{aligned}$$

Therefore, the representations of all the vertices of the barbell rose graph $B_{R(C_n)}$ with respect to $\mathcal{A}$ are defined as follows.

$$r(v_s|\mathcal{A}) = (0, \frac{n+5}{6} - s, \frac{n-1}{6} + s, \frac{n+5}{6} + s),\ \text{for } 1 \le s \le \frac{n-1}{6},$$
$$r(v_s|\mathcal{A}) = (s - \frac{n-1}{6}, 0, \frac{n+3}{2} - s, \frac{n+5}{6} + s),\ \text{for } \frac{n+5}{6} \le s \le \frac{n+1}{2},$$
$$r(v_s|\mathcal{A}) = (\frac{5n+7}{6} - s, s - \frac{n+1}{2}, 0, \frac{7n+17}{6} - s),\ \text{for } \frac{n+3}{2} \le s \le \frac{5n+1}{6},$$
$$r(v_s|\mathcal{A}) = (0, \frac{7n+5}{6} - s, s - \frac{5n+1}{6}, \frac{7n+17}{6} - s),\ \text{for } \frac{5n+7}{6} \le s \le n,$$
$$r(w_s|\mathcal{A}) = (0, \frac{n+5}{6} - s, \frac{n+5}{6} + s, \frac{n+11}{6} + s),\ \text{for } 1 \le s \le \frac{n-1}{6},$$
$$r(w_s|\mathcal{A}) = (s - \frac{n-7}{6}, 0, \frac{n+3}{2} - s, \frac{n+11}{6} + s),\ \text{for } \frac{n+5}{6} \le s \le \frac{n-1}{2},$$

$$r(w_s|\mathcal{A}) = (\frac{5n+7}{6} - s, s - \frac{n-1}{2}, 0, \frac{7n+17}{6} - s), \text{ for } \frac{n+1}{2} \leq s \leq \frac{5n-5}{6},$$
$$r(w_s|\mathcal{A}) = (0, \frac{7n+5}{6} - s, s - \frac{5n-5}{6}, \frac{7n+17}{6} - s), \text{ for } \frac{5n+1}{6} \leq s \leq n,$$
$$r(v_s^{'}|\mathcal{A}) = (0, \frac{n+5}{6} - s, \frac{n+5}{6} + s, \frac{n-1}{6} + s), \text{ for } 1 \leq s \leq \frac{n-1}{6},$$
$$r(v_s^{'}|\mathcal{A}) = (s - \frac{n-1}{6}, 0, \frac{n+5}{6} + s, \frac{n+3}{2} - s), \text{ for } \frac{n+5}{6} \leq s \leq \frac{n+1}{2},$$
$$r(v_s^{'}|\mathcal{A}) = (\frac{5n+7}{6} - s, s - \frac{n+1}{2}, \frac{7n+17}{6} - s, 0), \text{ for } \frac{n+3}{2} \leq s \leq \frac{5n+1}{6},$$
$$r(v_s^{'}|\mathcal{A}) = (0, \frac{7n+5}{6} - s, \frac{7n+17}{6} - s, s - \frac{5n+1}{6}), \text{ for } \frac{5n+7}{6} \leq s \leq n,$$
$$r(w_s^{'}|\mathcal{A}) = (0, \frac{n+5}{6} - s, \frac{n+11}{6} + s, \frac{n+5}{6} + s), \text{ for } 1 \leq s \leq \frac{n-1}{6},$$
$$r(w_s^{'}|\mathcal{A}) = (s - \frac{n-7}{6}, 0, \frac{n+11}{6} + s, \frac{n+3}{2} - s), \text{ for } \frac{n+5}{6} \leq s \leq \frac{n-1}{2},$$
$$r(w_s^{'}|\mathcal{A}) = (\frac{5n+7}{6} - s, s - \frac{n-1}{2}, \frac{7n+17}{6} - s, 0), \text{ for } \frac{n+1}{2} \leq s \leq \frac{5n-5}{6},$$
$$r(w_s^{'}|\mathcal{A}) = (0, \frac{7n+5}{6} - s, \frac{7n+17}{6} - s, s - \frac{5n-5}{6}), \text{ for } \frac{5n+1}{6} \leq s \leq n.$$

(Subcase 3) For $n > 4$ even, with $n \equiv 1$ $(mod\ 3)$.
Given $\mathcal{A} = \{S_1, S_2, S_3, S_4\}$, the partition classes of the graph $B_{R(C_n)}$ are defined as follows.

$$\begin{aligned}
S_1 = & \{ v_s \mid 1 \leq s \leq \tfrac{n+2}{6} \} \cup \{ v_s \mid \tfrac{5n+4}{6} \leq s \leq n \} \\
& \cup \{ w_s \mid 1 \leq s \leq \tfrac{n-4}{6} \} \cup \{ w_s \mid \tfrac{5n+4}{6} \leq s \leq n \} \\
& \cup \{ v_s^{'} \mid 1 \leq s \leq \tfrac{n+2}{6} \} \cup \{ v_s^{'} \mid \tfrac{5n+4}{6} \leq s \leq n \} \\
& \cup \{ w_s^{'} \mid 1 \leq s \leq \tfrac{n-4}{6} \} \cup \{ w_s^{'} \mid \tfrac{5n+4}{6} \leq s \leq n \}, \\
S_2 = & \{ v_s \mid \tfrac{n+8}{6} \leq s \leq \tfrac{n}{2} \} \cup \{ w_s \mid \tfrac{n+2}{6} \leq s \leq \tfrac{n}{2} \} \\
& \cup \{ v_s^{'} \mid \tfrac{n+8}{6} \leq s \leq \tfrac{n}{2} \} \cup \{ w_s^{'} \mid \tfrac{n+2}{6} \leq s \leq \tfrac{n}{2} \}, \\
S_3 = & \{ v_s \mid \tfrac{n+2}{2} \leq s \leq \tfrac{5n-2}{6} \} \cup \{ w_s \mid \tfrac{n+2}{2} \leq s \leq \tfrac{5n-2}{6} \}, \\
S_4 = & \{ v_s^{'} \mid \tfrac{n+2}{2} \leq s \leq \tfrac{5n-2}{6} \} \cup \{ w_s^{'} \mid \tfrac{n+2}{2} \leq s \leq \tfrac{5n-2}{6} \}.
\end{aligned}$$

Therefore, the representations of all the vertices of the barbell rose graph $B_{R(C_n)}$ with respect to $\mathcal{A}$ are defined as follows.

$$r(v_s|\mathcal{A}) = (0, \frac{n+8}{6} - s, \frac{n+2}{6} + s, \frac{n+8}{6} + s), \text{ for } 1 \leq s \leq \frac{n+2}{6},$$
$$r(v_s|\mathcal{A}) = (s - \frac{n+2}{6}, 0, \frac{n+2}{2} - s, \frac{n+8}{6} + s), \text{ for } \frac{n+8}{6} \leq s \leq \frac{n}{2},$$
$$r(v_s|\mathcal{A}) = (\frac{5n+4}{6} - s, s - \frac{n}{2}, 0, \frac{7n+6}{6} - s), \text{ for } \frac{n+2}{2} \leq s \leq \frac{5n-2}{6},$$
$$r(v_s|\mathcal{A}) = (0, \frac{7n+8}{6} - s, s - \frac{5n-2}{6}, \frac{7n+20}{6} - s), \text{ for } \frac{5n+4}{6} \leq s \leq n,$$
$$r(w_s|\mathcal{A}) = (0, \frac{n+8}{6} - s, \frac{n+8}{6} + s, \frac{n+14}{6} + s), \text{ for } 1 \leq s \leq \frac{n-4}{6},$$

$$r(w_s|\mathcal{A}) = (s - \frac{n-4}{6}, 0, \frac{n+2}{2} - s, \frac{n+14}{6} + s), \text{ for } \frac{n+2}{6} \le s \le \frac{n}{2},$$
$$r(w_s|\mathcal{A}) = (\frac{5n+4}{6} - s, s - \frac{n-2}{2}, 0, \frac{7n+20}{6} - s), \text{ for } \frac{n+2}{2} \le s \le \frac{5n-2}{6},$$
$$r(w_s|\mathcal{A}) = (0, \frac{7n+8}{6} - s, s - \frac{5n-8}{6}, \frac{7n+20}{6} - s), \text{ for } \frac{5n+4}{6} \le s \le n,$$
$$r(v_s^{'}|\mathcal{A}) = (0, \frac{n+8}{6} - s, \frac{n+8}{6} + s, \frac{n+2}{6} + s), \text{ for } 1 \le s \le \frac{n+2}{6},$$
$$r(v_s^{'}|\mathcal{A}) = (s - \frac{n+2}{6}, 0, \frac{n+8}{6} + s, \frac{n+2}{2} - s), \text{ for } \frac{n+8}{6} \le s \le \frac{n}{2},$$
$$r(v_s^{'}|\mathcal{A}) = (\frac{5n+4}{6} - s, s - \frac{n}{2}, \frac{7n+6}{6} - s, 0), \text{ for } \frac{n+2}{2} \le s \le \frac{5n-2}{6},$$
$$r(v_s^{'}|\mathcal{A}) = (0, \frac{7n+8}{6} - s, \frac{7n+20}{6} - s, s - \frac{5n-2}{6}), \text{ for } \frac{5n+4}{6} \le s \le n,$$
$$r(w_s^{'}|\mathcal{A}) = (0, \frac{n+8}{6} - s, \frac{n+14}{6} + s, \frac{n+8}{6} + s), \text{ for } 1 \le s \le \frac{n-4}{6},$$
$$r(w_s^{'}|\mathcal{A}) = (s - \frac{n-4}{6}, 0, \frac{n+14}{6} + s, \frac{n+2}{2} - s), \text{ for } \frac{n+2}{6} \le s \le \frac{n}{2},$$
$$r(w_s^{'}|\mathcal{A}) = (\frac{5n+4}{6} - s, s - \frac{n-2}{2}, \frac{7n+20}{6} - s, 0), \text{ for } \frac{n+2}{2} \le s \le \frac{5n-2}{6},$$
$$r(w_s^{'}|\mathcal{A}) = (0, \frac{7n+8}{6} - s, \frac{7n+20}{6} - s, s - \frac{5n-8}{6}), \text{ for } \frac{5n+4}{6} \le s \le n.$$

Since each vertex in the barbell rose graph $B_{R(C_n)}$ for $n \ge 3$, with $n \equiv 1$ (*mod* 3) has a distinct representation with respect to $\mathcal{A}$, it follows that $\mathcal{A}$ is a resolving partition of the barbell rose graph $B_{R(C_n)}$. Therefore,

$$pd(B_{R(C_n)}) \le 4. \tag{2.9}$$

(Case 3) For $n \ge 5$, with $n \equiv 2$ (*mod* 3).
Consider the following two subcases.

(Subcase 1) For $n \ge 5$ odd, with $n \equiv 2$ (*mod* 3).
Given $\mathcal{A} = \{S_1, S_2, S_3, S_4\}$, the partition classes of the graph $B_{R(C_n)}$ are defined as follows.

$$\begin{aligned} S_1 = &\{ v_s \mid 1 \le s \le \tfrac{n+1}{6} \} \cup \{ v_s \mid \tfrac{5n+5}{6} \le s \le n \} \\ &\cup \{ w_s \mid 1 \le s \le \tfrac{n+1}{6} \} \cup \{ w_s \mid \tfrac{5n+5}{6} \le s \le n \} \\ &\cup \{ v_s^{'} \mid 1 \le s \le \tfrac{n+1}{6} \} \cup \{ v_s^{'} \mid \tfrac{5n+6}{6} \le s \le n \} \\ &\cup \{ w_s^{'} \mid 1 \le s \le \tfrac{n+1}{6} \} \cup \{ w_s^{'} \mid \tfrac{5n+6}{6} \le s \le n \}, \\ S_2 = &\{ v_s \mid \tfrac{n+7}{6} \le s \le \tfrac{n+1}{2} \} \cup \{ w_s \mid \tfrac{n+7}{6} \le s \le \tfrac{n-1}{2} \} \\ &\cup \{ v_s^{'} \mid \tfrac{n+7}{6} \le s \le \tfrac{n+1}{2} \} \cup \{ w_s^{'} \mid \tfrac{n+7}{6} \le s \le \tfrac{n-1}{2} \}, \\ S_3 = &\{ v_s \mid \tfrac{n+3}{2} \le s \le \tfrac{5n-1}{6} \} \cup \{ w_s \mid \tfrac{n+1}{2} \le s \le \tfrac{5n-1}{6} \}, \\ S_4 = &\{ v_s^{'} \mid \tfrac{n+3}{2} \le s \le \tfrac{5n-1}{6} \} \cup \{ w_s^{'} \mid \tfrac{n+1}{2} \le s \le \tfrac{5n-1}{6} \}. \end{aligned}$$

Therefore, the representations of all the vertices of the barbell rose graph $B_{R(C_n)}$ with respect to $\mathcal{A}$ are defined as follows.

$$
\begin{aligned}
r(v_s|\mathcal{A}) &= (0, \frac{n+7}{6} - s, \frac{n+1}{6} + s, \frac{n+7}{6} + s), \text{ for } 1 \le s \le \frac{n+1}{6},\\
r(v_s|\mathcal{A}) &= (s - \frac{n+1}{6}, 0, \frac{n+3}{2} - s, \frac{n+7}{6} + s), \text{ for } \frac{n+7}{6} \le s \le \frac{n+1}{2},\\
r(v_s|\mathcal{A}) &= (\frac{5n+5}{6} - s, s - \frac{n+1}{2}, 0, \frac{7n+19}{6} - s), \text{ for } \frac{n+3}{2} \le s \le \frac{5n-1}{6},\\
r(v_s|\mathcal{A}) &= (0, s - \frac{n+1}{2}, s - \frac{5n-1}{6}, \frac{7n+19}{6} - s), \text{ for } \frac{5n+5}{6} \le s \le n,\\
r(w_s|\mathcal{A}) &= (0, \frac{n+7}{6} - s, \frac{n+7}{6} + s, \frac{n+13}{6} + s), \text{ for } 1 \le s \le \frac{n+1}{6},\\
r(w_s|\mathcal{A}) &= (s - \frac{n-5}{6}, 0, \frac{n+3}{2} - s, \frac{n+13}{6} + s), \text{ for } \frac{n+7}{6} \le s \le \frac{n-1}{2},\\
r(w_s|\mathcal{A}) &= (\frac{5n+5}{6} - s, s - \frac{n-1}{2}, 0, \frac{7n+19}{6} - s), \text{ for } \frac{n+1}{2} \le s \le \frac{5n-1}{6},\\
r(w_s|\mathcal{A}) &= (0, \frac{7n+7}{6} - s, s - \frac{5n-7}{6}, \frac{7n+19}{6} - s), \text{ for } \frac{5n+5}{6} \le s \le n,\\
r(v_s^{'}|\mathcal{A}) &= (0, \frac{n+7}{6} - s, \frac{n+7}{6} + s, \frac{n+1}{6} + s), \text{ for } 1 \le s \le \frac{n+1}{6},\\
r(v_s^{'}|\mathcal{A}) &= (s - \frac{n+1}{6}, 0, \frac{n+7}{6} + s, \frac{n+3}{2} - s), \text{ for } \frac{n+7}{6} \le s \le \frac{n+1}{2},\\
r(v_s^{'}|\mathcal{A}) &= (\frac{5n+5}{6} - s, s - \frac{n+1}{2}, \frac{7n+19}{6} - s, 0), \text{ for } \frac{n+3}{2} \le s \le \frac{5n-1}{6},\\
r(v_s^{'}|\mathcal{A}) &= (0, s - \frac{n+1}{2}, \frac{7n+19}{6} - s, s - \frac{5n-1}{6}), \text{ for } \frac{5n+5}{6} \le s \le n,\\
r(w_s^{'}|\mathcal{A}) &= (0, \frac{n+7}{6} - s, \frac{n+13}{6} + s, \frac{n+7}{6} + s), \text{ for } 1 \le s \le \frac{n+1}{6},\\
r(w_s^{'}|\mathcal{A}) &= (s - \frac{n-5}{6}, 0, \frac{n+13}{6} + s, \frac{n+3}{2} - s), \text{ for } \frac{n+7}{6} \le s \le \frac{n-1}{2},\\
r(w_s^{'}|\mathcal{A}) &= (\frac{5n+5}{6} - s, s - \frac{n-1}{2}, \frac{7n+19}{6} - s, 0), \text{ for } \frac{n+1}{2} \le s \le \frac{5n-1}{6},\\
r(w_s^{'}|\mathcal{A}) &= (0, \frac{7n+7}{6} - s, \frac{7n+19}{6} - s, s - \frac{5n-7}{6}), \text{ for } \frac{5n+5}{6} \le s \le n.
\end{aligned}
$$

(Subcase 2) For $n > 5$ even, with $n \equiv 2$ $(mod\ 3)$.
Given $\mathcal{A} = \{S_1, S_2, S_3, S_4\}$, the partition classes of the graph $B_{R(C_n)}$ are defined as follows.

$$
\begin{aligned}
S_1 = &\{ v_s \mid 1 \le s \le \tfrac{n+4}{6} \} \cup \{ v_s \mid \tfrac{5n+8}{6} \le s \le n \}\\
&\cup \{ w_s \mid 1 \le s \le \tfrac{n-2}{6} \} \cup \{ w_s \mid \tfrac{5n+2}{6} \le s \le n \}\\
&\cup \{ v_s^{'} \mid 1 \le s \le \tfrac{n+4}{6} \} \cup \{ v_s^{'} \mid \tfrac{5n+8}{6} \le s \le n \}\\
&\cup \{ w_s^{'} \mid 1 \le s \le \tfrac{n-2}{6} \} \cup \{ w_s^{'} \mid \tfrac{5n+2}{6} \le s \le n \},\\
S_2 = &\{ v_s \mid \tfrac{n+10}{6} \le s \le \tfrac{n}{2} \} \cup \{ w_s \mid \tfrac{n+4}{6} \le s \le \tfrac{n}{2} \}\\
&\cup \{ v_s^{'} \mid \tfrac{n+10}{6} \le s \le \tfrac{n}{2} \} \cup \{ w_s^{'} \mid \tfrac{n+4}{6} \le s \le \tfrac{n}{2} \},\\
S_3 = &\{ v_s \mid \tfrac{n+2}{2} \le s \le \tfrac{5n+2}{6} \} \cup \{ w_s \mid \tfrac{n+2}{2} \le s \le \tfrac{5n-4}{6} \},\\
S_4 = &\{ v_s^{'} \mid \tfrac{n+2}{2} \le s \le \tfrac{5n+2}{6} \} \cup \{ w_s^{'} \mid \tfrac{n+2}{2} \le s \le \tfrac{5n-4}{6} \}.
\end{aligned}
$$

Therefore, the representations of all the vertices of the barbell rose graph $B_{R(C_n)}$ with respect to $\mathcal{A}$ are defined as follows.

$$
\begin{aligned}
r(v_s|\mathcal{A}) &= (0, \frac{n+10}{6} - s, \frac{n-2}{6} + s, \frac{n+4}{6} + s), \text{ for } 1 \leq s \leq \frac{n+4}{6},\\
r(v_s|\mathcal{A}) &= (s - \frac{n+4}{6}, 0, \frac{n+2}{2} - s, \frac{n+4}{6} + s), \text{ for } \frac{n+10}{6} \leq s \leq \frac{n}{2},\\
r(v_s|\mathcal{A}) &= (\frac{5n+8}{6} - s, s - \frac{n}{2}, 0, \frac{7n+16}{6} - s), \text{ for } \frac{n+2}{2} \leq s \leq \frac{5n+2}{6},\\
r(v_s|\mathcal{A}) &= (0, \frac{7n+10}{6} - s, s - \frac{5n+2}{6}, \frac{7n+16}{6} - s), \text{ for } \frac{5n+8}{6} \leq s \leq n,\\
r(w_s|\mathcal{A}) &= (0, \frac{n+10}{6} - s, \frac{n+4}{6} + s, \frac{n+10}{6} + s), \text{ for } 1 \leq s \leq \frac{n-2}{6},\\
r(w_s|\mathcal{A}) &= (s - \frac{n-2}{6}, 0, \frac{n+2}{2} - s, \frac{n+10}{6} + s), \text{ for } \frac{n+4}{6} \leq s \leq \frac{n}{2},\\
r(w_s|\mathcal{A}) &= (\frac{5n+8}{6} - s, s - \frac{n-2}{2}, 0, \frac{7n+16}{6} - s), \text{ for } \frac{n+2}{2} \leq s \leq \frac{5n-4}{6},\\
r(w_s|\mathcal{A}) &= (0, \frac{7n+10}{6} - s, s - \frac{5n-4}{6}, \frac{7n+16}{6} - s), \text{ for } \frac{5n+2}{6} \leq s \leq n,\\
r(v_s^{'}|\mathcal{A}) &= (0, \frac{n+10}{6} - s, \frac{n+4}{6} + s, \frac{n-2}{6} + s), \text{ for } 1 \leq s \leq \frac{n+4}{6},\\
r(v_s^{'}|\mathcal{A}) &= (s - \frac{n+4}{6}, 0, \frac{n+4}{6} + s, \frac{n+2}{2} - s), \text{ for } \frac{n+10}{6} \leq s \leq \frac{n}{2},\\
r(v_s^{'}|\mathcal{A}) &= (\frac{5n+8}{6} - s, s - \frac{n}{2}, \frac{7n+16}{6} - s, 0), \text{ for } \frac{n+2}{2} \leq s \leq \frac{5n+2}{6},\\
r(v_s^{'}|\mathcal{A}) &= (0, \frac{7n+10}{6} - s, \frac{7n+16}{6} - s, s - \frac{5n+2}{6}), \text{ for } \frac{5n+8}{6} \leq s \leq n,\\
r(w_s^{'}|\mathcal{A}) &= (0, \frac{n+10}{6} - s, \frac{n+10}{6} + s, \frac{n+4}{6} + s), \text{ for } 1 \leq s \leq \frac{n-2}{6},\\
r(w_s^{'}|\mathcal{A}) &= (s - \frac{n-2}{6}, 0, \frac{n+10}{6} + s, \frac{n+2}{2} - s), \text{ for } \frac{n+4}{6} \leq s \leq \frac{n}{2},\\
r(w_s^{'}|\mathcal{A}) &= (\frac{5n+8}{6} - s, s - \frac{n-2}{2}, \frac{7n+16}{6} - s, 0), \text{ for } \frac{n+2}{2} \leq s \leq \frac{5n-4}{6},\\
r(w_s^{'}|\mathcal{A}) &= (0, \frac{7n+10}{6} - s, \frac{7n+16}{6} - s, s - \frac{5n-4}{6}), \text{ for } \frac{5n+2}{6} \leq s \leq n.
\end{aligned}
$$

Since each vertex in the barbell rose graph $B_{R(C_n)}$ for $n \geq 3$, with $n \equiv 2$ (mod 3) has a distinct representation with respect to $\mathcal{A}$, it follows that $\mathcal{A}$ is a resolving partition of the barbell rose graph $B_{R(C_n)}$. Therefore,

$$pd(B_{R(C_n)}) \leq 4. \tag{2.10}$$

From Eq. (2.8) – Eq. (2.10), it follows that the upper bound of $B_{R(C_n)}$ for $n \geq 3$ is:

$$pd(B_{R(C_n)}) \leq 4. \tag{2.11}$$

Therefore, from Eq. (2.7) and Eq. (2.11), the partition dimension of the barbell rose graph $B_{R(C_n)}$ is $pd(B_{R(C_n)}) = 4$ for $n \geq 3$. □

The following Figure 2 shows a minimal resolving partition of graph $B_{R(C_5)}$.

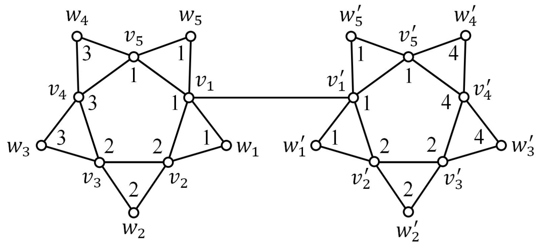

Figure 2. A minimum resolving partition of $B_{R(C_5)}$.

3. Conclusion

The partition dimension of the rose graph, $pd(R(C_n))$ is 3 for $n \geq 3$ and the partition dimension of the barbell rose graph, $pd(B_{R(C_n)})$ is 4 for $n \geq 3$.

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