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ALGEBRAIC LAWS AND PROPERTIES OF PICTURE FUZZY SETS

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Abstract. *This paper investigates the fundamental algebraic laws and properties that hold in the framework of Picture Fuzzy Sets (PFS). Picture fuzzy sets extend classical fuzzy and intuitionistic fuzzy sets by incorporating an additional degree of neutrality, providing a more refined representation of uncertainty. This study examine the validity of standard algebraic laws such as commutativity, associativity, distributivity, and idempotency under picture fuzzy operations. The result show that several classical properties-such as the transitivity of the subset relation, as well as the commutative, associative, distributive, De Morgan's, idempotent, dominance, and zero/one complement laws—are preserved. However, some other properties are not fully maintained; for instance, the union of subsets is not necessarily equal to their superset, and the complement and absorption laws do not hold strictly. These findings provide an important contribution to a deeper understanding of the algebraic behavior of PFS and forms a theoretical basis for their further applications in fuzzy decision-making and information processing.*

Keywords: Algebraic Laws, Set Theoretic, Picture Fuzzy Set

1. Introduction

The concept of uncertainty was first introduced by Lotfi A. Zadeh in 1965 through the theory of fuzzy sets [1]. A fuzzy set employs a membership degree that allows an element to belong to a set at varying levels rather than in a binary manner (0 or 1). This approach provides greater flexibility in representing and managing uncertainty and has been widely applied in diverse fields such as artificial intelligence [2], decision-making [3], and data analysis [4].

In 1986, K. Atanassov extended Zadeh's concept by introducing the *Intuitionistic Fuzzy Set* (IFS) [5], which provides a more comprehensive frame-

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work than the classical fuzzy set. An IFS incorporates both membership and non-membership degrees, thereby enabling the representation of uncertainty that cannot be fully captured by a single membership degree. The IFS theory has found extensive applications in decision-making, including environmental management [6], medical diagnosis [7,8,9], and education, such as school placement and student specialization [10,11,12].

Further advancements in fuzzy set theory were made by Bui Cong Cuong and Vladik Kreinovich in 2013, who introduced the *Picture Fuzzy Set* (PFS) [13]. PFS extends the intuitionistic fuzzy framework by incorporating an additional degree neutrality or indeterminacy alongside positive and negative membership. Each element in a PFS is thus characterized by three parameters: positive membership, neutral (or hesitation) membership, and negative membership. This structure enables a more detailed representation of uncertainty in contexts where belief, hesitation, and rejection coexist[13].

The development of PFS theory has opened new perspectives for both theoretical and practical research in fuzzy algebra. From a theoretical standpoint, PFS has inspired the generalization of various algebraic structures, including picture fuzzy subgroups [14], kernel [15], subspace [16], subring [17], ideal [18,19]. However, due to the distinct definitions of the empty set and the universal set in the context of PFS [20], not all classical algebraic laws remain valid. The novelty of this research lies in its specific focus on algebraic laws. While previous studies in [13] have provided definitions and some properties for basic PFS operations, and [20] defines the concepts of one (the universal set) and zero (the empty set) in PFS, no research has comprehensively addressed the validity of classical algebraic laws (such as the commutative, associative, De Morgan's, absorption laws, among others) within the PFS framework by incorporating these specific definitions for the empty set and the universal set in the context of PFS. This observation motivates further investigation into the algebraic properties that hold within the framework of picture fuzzy sets.

2. Relation and Operation on Picture Fuzzy Set

Before exploring the concept of the Picture Fuzzy Set (PFS), it is necessary to first review the Fuzzy Set (FS) and the Intuitionistic Fuzzy Set (IFS), which serve as the theoretical basis for its formulation. The formal definition is presented below.

Definition 2.1. *[1] Let X be a non-empty set. Then the Fuzzy Set (FS) A on X is defined as:*

$$A = \{(x, \mu_A(x)) | x \in X\},$$

where $\mu_A : X \to [0, 1]$. is called the membership function of A, which assigns to each element $x \in X$ a degree of membership between 0 and 1.

Example 2.2. Let $P = \{p, q, r, s, t\}$. Then the fuzzy set on P is:

$$A = \{(p, 0.1), (q, 0.5), (r, 0), (s, 0.7), (t, 1)\}.$$

Definition 2.3. *[5] Let X be a non-empty set. Intuitionistic Fuzzy Set (IFS) A on X is defined as:*

$$A = \{(x, \mu_A(x), \nu_A)|x \in X\},$$

where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ satisfy:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X.$$

The value $\mu_A(x)$ represents the degree of membership of x in A, and $\nu_A(x)$ represents the degree of non-membership of x in A. For an IFS A on X, the hesitation degree (or intuitionistic fuzzy index) of x in A is defined as:

$$\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x)).$$

Clearly, $0 \leq \pi_A(x) \leq 1$ for all $x \in X$.

Example 2.4. Let $P = \{p, q, r, s, t\}$. Then the Intuitionistic Fuzzy Set on P is:

$$A = \{(p, 0.1, 0.4), (q, 0.5, 0.3), (r, 0, 0.7), (s, 0.7, 0.1), (t, 1, 0)\}.$$

Definition 2.5 ([13]). *Let X be a non-empty set. A Picture Fuzzy Set (PFS) A on X is denoted by:*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A(x) \in [0, 1]$ is called the positive membership degree of x in A, $\eta_A(x) \in [0, 1]$ is called the neutral membership degree of x in A, and $\nu_A(x) \in [0, 1]$ is called the negative membership degree of x in A. These functions satisfy the condition:

$$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X.$$

The degree of refusal (or rejection) of x in A is defined as:

$$1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)).$$

Example 2.6. Let $P = \{p, q, r, s, t\}$. Then the Picture Fuzzy Set on P is:

$$A = \{(p, 0.1, 0.2, 0.4), (q, 0.5, 0.1, 0.3), (r, 0, 0, 3, 0.7), (s, 0.7, 0.1, 0.1), (t, 1, 0, 0)\}.$$

The collection of all Picture Fuzzy Sets on X is denoted by $PFS(X)$.

Similar to the classical set, a Picture Fuzzy Set also possesses the notions of an empty set and a universal set, which are defined in accordance with the membership degrees specified in the PFS framework.

Definition 2.7 ([20]). *Let X be a non-empty set. Then,*

$$\hat{0} = \{(x, 0, 0, 1) \mid x \in X\}, \quad \textit{and} \quad \hat{1} = \{(x, 1, 0, 0) \mid x \in X\}.$$

Here, $\hat{0}$ is called the empty set and $\hat{1}$ is called the universal set in $PFS(X)$.

The formal definitions of relations and operations on Picture Fuzzy Sets are given in the following definitions.

Definition 2.8 ([21]). *Let $A, B \in PFS(X)$, then:*

(1) $A \subseteq B$ iff $\forall x \in X, \mu_A(x) \leq \mu_B(x)$, $\eta_A(x) \leq \eta_A(x)$ and $\nu_A(x) \geq \nu_B(x)$.

(2) $A = B$ *iff* $\forall x \in X, \mu_A(x) = \mu_B(x),\ \eta_A(x) = \eta_A(x)$ *and* $\nu_A(x) = \nu_B(x)$.
(3) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\},\ \min\{\nu_A(x), \nu_B(x)\})|x \in X\}$.
(4) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\},\ \max\{\nu_A(x), \nu_B(x)\})|x \in X\}$.
(5) $A^c = \{(x, \nu_A(x), \eta_A(x), \mu_A(x))|x \in X\}$.
(6) $A \times B = \{((x, y), \mu_{A\times B}(x, y), \eta_{A\times B}(x, y), \nu_{A\times B}(x, y)|\ x \in A, y \in B\}$, *where* $\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$, $\eta_{A\times B}(x, y) = \min\{\eta_A(x), \eta_B(y)\}$ *and* $\nu_{A\times B}(x, y) = \max\{\nu_A(x), \nu_B(y)\}$.

Example 2.9. Let $P = \{p, q, r, s, t\}$. The Picture Fuzzy Set on P is defined as:

$$A = \{(p, 0.1, 0.2, 0.4), (q, 0.5, 0.1, 0.3), (r, 0, 0.3, 0.7)\}, \text{ and}$$
$$B = \{(p, 0.3, 0.1, 0.5), (q, 0.2, 0.1, 0.3), (r, 0.3, 0, 0.7)\}.$$

Then:

(1) $A \not\subseteq B$ because $\exists q \in P$ then $\mu_A(q) = 0.5 > 0.2 = \mu_B(q)$.
(2) $B \not\subseteq A$ because $\exists p \in P$ then $\mu_B(p) = 0.3 > 0.1 = \mu_A(p)$.
(3) $A \neq B$ because $A \not\subseteq B$ and $B \not\subseteq A$.
(4) $A \cup B = \{(p, 0.3, 0.1, 0.4), (q, 0.5, 0.1, 0.3), (r, 0.3, 0, 0.7)\}$
(5) $A \cap B = \{(p, 0.1, 0.1, 0.5), (q, 0.2, 0.1, 0.3), (r, 0, 0, 0.7)\}$
(6) $A^c = \{(p, 0.4, 0.2, 0.1), (q, 0.3, 0.1, 0.5), (r, 0.7, 0.3, 0)\}$.
(7) $A \times B = \{((p, p), 0.1, 0.1, 0.5), ((p, q), 0.1, 0.1, 0.4), ((p, r), (0.1, 0, 0.7)),$
$((q, p), 0.3, 0.1, 0.5), ((q, q), 0.2, 0.1, 0.3), ((q, r), 0.3, 0, 0.7),$
$((r, p), 0, 0.1, 0.7), ((r, q), 0, 0.1, 0.7), ((r, r), 0, 0, 0.7)\}$.

3. Algebraic Laws on Picture Fuzzy Sets

In this section, the algebraic set laws satisfied by the concepts defined in the Picture Fuzzy Set (PFS) are discussed. It is first demonstrated that the subset relation in PFS is characterized by the transitive property, and that the union and intersection operations are governed by the commutative, associative, distributive, and De Morgan's laws.

Theorem 3.1. *[13] Let* $A, B, C \in PFS(X)$*, then:*

(1) *If* $A \subseteq B$ *and* $B \subseteq C$ *then* $A \subseteq C$.
(2) $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$.
(3) $(A \cup B) \cup C = A \cup (B \cup C)$ *and* $(A \cap B) \cap C = A \cap (B \cap C)$.
(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and* $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$.
(5) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$.
(6) $(A \cup B)^c = A^c \cap B^c$ *and* $(A \cap B)^c = A^c \cup B^c$.

Proof. Let $A, B, C \in PFS(X)$.

(1) Given $A \subseteq B$ and $B \subseteq C$, then based on Definition 2.8, it follows that for all $x \in X$,

$$\mu_A(x) \leq \mu_B(x), \quad \eta_A(x) \leq \eta_B(x), \quad \text{and} \quad \nu_A(x) \geq \nu_B(x),$$

and:

$$\mu_B(x) \leq \mu_C(x), \quad \eta_B(x) \leq \eta_C(x), \quad \text{and} \quad \nu_B(x) \geq \nu_C(x).$$

Therefore, by the transitive property of inequalities in real numbers, it can be concluded that:

$$\mu_A(x) \leq \mu_C(x), \quad \eta_A(x) \leq \eta_C(x), \quad \text{and} \quad \nu_A(x) \geq \nu_C(x).$$

Hence, it is shown that $A \subseteq C$.

(2) It is observed that:

$$\begin{aligned} A \cup B &= \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}) \mid x \in X\}, \\ &= \{(x, \max\{\mu_B(x), \mu_A(x)\}, \min\{\eta_B(x), \eta_A(x)\}, \min\{\nu_B(x), \nu_A(x)\}) \mid x \in X\}, \\ &= B \cup A, \end{aligned}$$

and:

$$\begin{aligned} A \cap B &= \{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}) \mid x \in X\}, \\ &= \{(x, \min\{\mu_B(x), \mu_A(x)\}, \min\{\eta_B(x), \eta_A(x)\}, \max\{\nu_B(x), \nu_A(x)\}) \mid x \in X\}, \\ &= B \cap A. \end{aligned}$$

Hence, it is proved that the union and intersection operations are commutative.

(3) Based on Definition 2.8, it follows that:

$$\begin{aligned} (A \cup B) \cup C &= (\{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \\ &\quad \min\{\nu_A(x), \nu_B(x)\}) | x \in X\}) \cup (\{(x, \mu_C(x), \eta_C(x), \nu_C(x)) | x \in X\}), \\ &= (\{(x, \max\{(\mu_A(x), \mu_B(x)), \mu_C(x)\}, \min\{(\eta_A(x), \eta_B(x)), \eta_C(x)\}, \\ &\quad \min\{(\nu_A(x), \nu_B(x)), \nu_C(x)\}) | x \in X\}), \\ &= (\{(x, \max\{\mu_A(x), (\mu_B(x), \mu_C(x))\}, \min\{\eta_A(x), (\eta_B(x), \eta_C(x))\}, \\ &\quad \min\{\nu_A(x), (\nu_B(x), \nu_C(x))\}) | x \in X\}), \\ &= (\{(x, \mu_A(x), \eta_A(x), \nu_A(x)) | x \in X\}) \cup (\{(x, \max\{\mu_B(x), \mu_C(x)\}, \\ &\quad \min\{\eta_B(x), \eta_C(x)\}, \min\{\nu_B(x), \nu_C(x)\}) | x \in X\}) \\ &= A \cup (B \cup C), \end{aligned}$$

and

$$\begin{aligned} (A \cap B) \cap C &= (\{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \\ &\quad \max\{\nu_A(x), \nu_B(x)\}) | x \in X\}) \cap (\{(x, \mu_C(x), \eta_C(x), \nu_C(x)) | x \in X\}), \\ &= (\{(x, \min\{(\mu_A(x), \mu_B(x)), \mu_C(x)\}, \min\{(\eta_A(x), \eta_B(x)), \eta_C(x)\}, \\ &\quad \max\{(\nu_A(x), \nu_B(x)), \nu_C(x)\}) | x \in X\}), \\ &= (\{(x, \min\{\mu_A(x), (\mu_B(x), \mu_C(x))\}, \min\{\eta_A(x), (\eta_B(x), \eta_C(x))\}, \\ &\quad \max\{\nu_A(x), (\nu_B(x), \nu_C(x))\}) | x \in X\}), \\ &= (\{(x, \mu_A(x), \eta_A(x), \nu_A(x)) | x \in X\}) \cap (\{(x, \min\{\mu_B(x), \mu_C(x)\}, \\ &\quad \min\{\eta_B(x), \eta_C(x)\}, \max\{\nu_B(x), \nu_C(x)\}) | x \in X\}) \\ &= A \cap (B \cap C). \end{aligned}$$

Hence, it is proved that the union and intersection operations are associative.

(4) Based on Definition 2.8, it follows that:

$$\begin{aligned}A\cup(B\cap C) &= (\{(x,\mu_A(x),\eta_A(x),\nu_A(x))|x\in X\})\cup(\{(x,\min\{\mu_B(x),\mu_C(x)\},\\ &\quad \min\{\eta_B(x),\eta_C(x)\},\max\{\nu_A(x),\nu_B(x)\})|x\in X\}),\\ &= \{(x,\max\{\mu_A(x),\min(\mu_B(x),\mu_C(x))\},\\ &\quad \min\{\eta_A(x),\min(\eta_B(x),\eta_C(x))\},\\ &\quad \min\{\nu_A(x),\max(\nu_B(x),\nu_C(x))|x\in X\}),\\ &= \{(x,\min\{\max(\mu_A(x),\mu_B(x)),\max(\mu_A(x),\mu_C(x))\},\\ &\quad \min\{\min(\eta_A(x),\eta_B(x)),\min(\eta_A(x),\eta_C(x))\},\\ &\quad \max\{\min(\nu_A(x),\nu_B(x)),\min(\nu_A(x),\nu_C(x)))|x\in X\}),\\ &= \{(x,\max\{\mu_A(x),\mu_B(x)\},\min\{\eta_A(x),\eta_B(x)\},\\ &\quad \min\{\nu_A(x),\nu_B(x)\})|x\in X\}\cap\{(x,\max\{\mu_A(x),\mu_C(x)\},\\ &\quad \min\{\eta_A(x),\eta_C(x)\},\min\{\nu_A(x),\nu_C(x)\})|x\in X\}\\ &= (A\cup B)\cap(A\cup C).\end{aligned}$$

Because the union operation is commutative, it is implied that:

$$\begin{aligned}(B\cap C)\cup A &= A\cup(B\cap C),\\ &= (A\cup B)\cap(A\cup C),\\ &= (B\cup A)\cap(C\cup A).\end{aligned}$$

(5) Based on Definition 2.8, it follows that:

$$\begin{aligned}A\cap(B\cup C) &= (\{(x,\mu_A(x),\eta_A(x),\nu_A(x))|x\in X\})\cap(\{(x,\max\{\mu_B(x),\mu_C(x)\},\\ &\quad \min\{\eta_B(x),\eta_C(x)\},\min\{\nu_A(x),\nu_B(x)\})|x\in X\}),\\ &= \{(x,\min\{\mu_A(x),\max(\mu_B(x),\mu_C(x))\},\\ &\quad \min\{\eta_A(x),\min(\eta_B(x),\eta_C(x))\},\\ &\quad \max\{\nu_A(x),\min(\nu_B(x),\nu_C(x))|x\in X\}),\\ &= \{(x,\max\{\min(\mu_A(x),\mu_B(x)),\min(\mu_A(x),\mu_C(x))\},\\ &\quad \min\{\min(\eta_A(x),\eta_B(x)),\min(\eta_A(x),\eta_C(x))\},\\ &\quad \min\{\max(\nu_A(x),\nu_B(x)),\max(\nu_A(x),\nu_C(x)))|x\in X\}),\\ &= \{(x,\min\{\mu_A(x),\mu_B(x)\},\min\{\eta_A(x),\eta_B(x)\},\\ &\quad \max\{\nu_A(x),\nu_B(x)\})|x\in X\}\cup\{(x,\min\{\mu_A(x),\mu_C(x)\},\\ &\quad \min\{\eta_A(x),\eta_C(x)\},\max\{\nu_A(x),\nu_C(x)\})|x\in X\}\\ &= (A\cap B)\cup(A\cap C).\end{aligned}$$

Because the intersection operation is commutative, it is implied that:

$$\begin{aligned}(B\cup C)\cap A &= A\cap(B\cup C),\\ &= (A\cap B)\cup(A\cap C),\\ &= (B\cap A)\cup(C\cap A).\end{aligned}$$

It has been proven that the union and intersection operations are distributive with respect to both the right-hand and left-hand sides.

(6) De Morgan's Law: $(A \cup B)^c = A^c \cap B^c$ dan $(A \cap B)^c = A^c \cup B^c$.
Based on Definition 2.8, it follows that:

$$A^c = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) | x \in X\}$$

and

$$B^c = \{(x, \nu_B(x), \eta_B(x), \mu_B(x)) | x \in X\},$$

then,

$$A^c \cap B^c = \{(x, \min\{\nu_A(x), \nu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \max\{\mu_A(x), \mu_B(x)\}) | x \in X\},$$

and

$$A^c \cup B^c = \{(x, \max\{\nu_A(x), \nu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \min\{\mu_A(x), \mu_B(x)\}) | x \in X\}.$$

Based on Definition 2.8, it follows that:

$$A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}) | x \in X\},$$

then,

$$\begin{aligned} (A \cup B)^c &= \{(x, \min\{\nu_A(x), \nu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \max\{\mu_A(x), \mu_B(x)\}) | x \in X\} \\ &= A^c \cap B^c. \end{aligned}$$

Based on Definition 2.8, it follows that:

$$A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}) | x \in X\},$$

then:

$$\begin{aligned} (A \cap B)^c &= \{(x, \max\{\nu_A(x), \nu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \min\{\mu_A(x), \mu_B(x)\}) | x \in X\}, \\ &= A^c \cup B^c. \end{aligned}$$

It has been proven that De Morgan's law holds true. □

Although PFS satisfies several properties found in classical sets, there are also properties of classical sets that are not satisfied under the PFS definitions. The classical-set properties that are not satisfied in PFS are as follows.

(1) If $A \subseteq B$ then $A \cup B = B$.
For example, let $X = \{p, q, r\}$ and define the PFS:

$$A = \{(p, 0.3, 0.2, 0.4), (q, 0.5, 0.2, 0.3), (r, 0, 0.3, 0.7)\}$$

and:

$$B = \{(p, 0.3, 0.3, 0.2), (q, 0.6, 0.2, 0.1), (r, 0.4, 0.4, 0.2)\}.$$

For every $x \in X$ it holds that $\mu_A(x) \leq \mu_B(x)$, $\eta_A(x) \leq \eta_B(x)$ and $\nu_A(x) \geq \nu_B(x)$. Consequently $A \subseteq B$. However, $(p, 0.3, 0.2, 0.2) \in (A \cup B)$ while $(p, 0.3, 0.2, 0.2) \notin B$.

(2) $A \subseteq \hat{1}, \forall A \in PFS(X)$.
For example, let $X = \{p, q, r\}$ and define:

$$A = \{(p, 0.3, 0.2, 0.4), (q, 0.5, 0.2, 0.3), (r, 0, 0.3, 0.7)\}, \quad A \in PFS(X).$$

Nevertheless, $A \nsubseteq \hat{1}$ because $\eta_A(p) = 0.2 \geq 0 = \eta_{\hat{1}}(p)$.

Although the union property above is not satisfied, the intersection property is satisfied. The property of the empty set in PFS is also satisfied. The proof is given below.

Theorem 3.2. *Let X be a nonempty set. Then the following hold.*

(1) If $A \subseteq B$ then $A \cap B = A$ for all $A, B \in PFS(X)$.
(2) If $\hat{0}$ denotes the empty set in $PFS(X)$ then $\hat{0} \subseteq A$ for all $A \in PFS(X)$.

Proof.

(1) Assume $A \subseteq B$. Then for all $x \in X$ it holds that $\mu_A(x) \leq \mu_B(x)$, $\eta_A(x) \leq \eta_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, hence:

$$\min\{\mu_A(x), \mu_B(x)\} = \mu_A(x), \quad \min\{\eta_A(x), \eta_B(x)\} = \eta_A(x),$$

and

$$\max\{\nu_A(x), \nu_B(x)\} = \nu_A(x), \qquad \forall x \in X.$$

Therefore,

$$\begin{aligned} A \cap B &= \{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}) \mid x \in X\}, \\ &= \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\} \\ &= A. \end{aligned}$$

(2) Let $A \in PFS(X)$ be arbitrary and write:

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}.$$

The empty set in $PFS(X)$ is given by $\hat{0} = \{(x, 0, 0, 1) \mid x \in X\}$. Hence for each $x \in X$ it holds that $\mu_{\hat{0}}(x) = 0 \leq \mu_A(x)$, $\eta_{\hat{0}}(x) = 0 \leq \eta_A(x)$ and $\nu_{\hat{0}}(x) = 1 \geq \nu_A(x)$. By the definition of subset in PFS, it follows that $\hat{0} \subseteq A$ for all $A \in PFS(X)$. □

Next, it will be shown that the complement, union, and intersection operations in PFS satisfy the idempotent property.

Theorem 3.3. *Let $A \in PFS(X)$. Then the following properties hold:*

(1) $(A^c)^c = A$,
(2) $A \cup A = A$,
(3) $A \cap A = A$.

Proof. Let

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}.$$

(1) Based on the definition of the complement in PFS, we have:

$$A^c = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) \mid x \in X\} = B,$$

hence,

$$B^c = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\} = A.$$

Therefore, it is proven that $(A^c)^c = A$.

(2) Based on the definition of union in PFS, we obtain:

$$\begin{aligned} A \cup A &= \{(x, \max\{\mu_A(x), \mu_A(x)\}, \min\{\eta_A(x), \eta_A(x)\}, \min\{\nu_A(x), \nu_A(x)\}) \mid x \in X\}, \\ &= \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}, \\ &= A. \end{aligned}$$

Hence, it is proven that $A \cup A = A$.

(3) Similarly, based on the definition of intersection in PFS, we have

$$\begin{aligned} A \cap A &= \{(x, \min\{\mu_A(x), \mu_A(x)\}, \min\{\eta_A(x), \eta_A(x)\}, \max\{\nu_A(x), \nu_A(x)\}) \mid x \in X\}, \\ &= \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}, \\ &= A. \end{aligned}$$

Therefore, it is proven that $A \cap A = A$. □

However, the properties

$$A \cup A^c = \hat{1} \text{ and } A \cap A^c = \hat{0}$$

are not satisfied.

Let $X = \{p, q, r\}$ and $A \in PFS(X)$ be defined as

$$A = \{(p, 0.3, 0.2, 0.4), (q, 0.5, 0.2, 0.3), (r, 0, 0.3, 0.7)\},$$

then its complement is given by

$$A^c = \{(p, 0.4, 0.2, 0.3), (q, 0.3, 0.2, 0.5), (r, 0.7, 0.3, 0)\}.$$

Hence,

$$A \cup A^c = \{(p, 0.4, 0.2, 0.3), (q, 0.5, 0.2, 0.3), (r, 0.7, 0.3, 0)\} \neq \hat{1},$$

and

$$A \cap A^c = \{(p, 0.3, 0.2, 0.4), (q, 0.3, 0.2, 0.5), (r, 0, 0.3, 0.7)\} \neq \hat{0}.$$

Moreover, the properties

$$A \cup \hat{0} = A \text{ and } A \cap \hat{1} = A$$

are also not satisfied.

Let

$$A = \{(p, 0.3, 0.2, 0.4), (q, 0.5, 0.2, 0.3), (r, 0, 0.3, 0.7)\},$$

then

$$A \cup \hat{0} = \{(p, 0.3, 0, 0.4), (q, 0.5, 0, 0.3), (r, 0, 0, 0.7)\} \neq A,$$

and

$$A \cap \hat{1} = \{(p, 0.3, 0, 0.4), (q, 0.5, 0, 0.3), (r, 0, 0, 0.7)\} \neq A.$$

From these examples, the following theorem can be generalized.

Theorem 3.4. *Let X be a non-empty set and $A \in PFS(X)$. Then, the following property holds:*

$$A \cup \hat{0} = A \cap \hat{1}.$$

Proof. Let $A \in PFS(X)$ be arbitrary, such that

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x))|x \in X\}.$$

Then,

$$\begin{aligned} A \cup \hat{0} &= \{(x, \max\{\mu_A(x), 0\}, \min\{\eta_A(x), 0\}, \min\{\nu_A(x), 1\})|x \in X\}, \\ &= \{(x, \mu_A(x), 0, \nu_A(x))|x \in X\}, \end{aligned}$$

and

$$\begin{aligned} A \cap \hat{1} &= \{(x, \min\{\mu_A(x), 1\}, \min\{\eta_A(x), 0\}, \max\{\nu_A(x), 0\})|x \in X\}, \\ &= \{(x, \mu_A(x), 0, \nu_A(x))|x \in X\}. \end{aligned}$$

Hence, it is proven that $A \cup \hat{0} = A \cap \hat{1}$. □

Theorem 3.5. *Let X be a non-empty set and $A \in PFS(X)$. Then, the following properties hold:*

(1) $A \cap \hat{0} = \hat{0}$,
(2) $A \cup \hat{1} = \hat{1}$,
(3) $(\hat{0})^c = \hat{1}$,
(4) $(\hat{1})^c = \hat{0}$.

Proof. Let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x))|x \in X\}$.

(1) Based on the definition of intersection in PFS, it is obtained that:

$$\begin{aligned} A \cap \hat{0} &= \{(x, \min\{\mu_A(x), 0\}, \min\{\eta_A(x), 0\}, \max\{\nu_A(x), 1\})|x \in X\}, \\ &= \{(x, 0, 0, 1)|x \in X\} = \hat{0}. \end{aligned}$$

(2) Based on the definition of union in PFS, it is obtained that:

$$\begin{aligned} A \cup \hat{1} &= \{(x, \max\{\mu_A(x), 1\}, \min\{\eta_A(x), 0\}, \min\{\nu_A(x), 0\})|x \in X\}, \\ &= \{(x, 1, 0, 0)|x \in X\} = \hat{1}. \end{aligned}$$

(3) Based on the definition of complement in PFS, it follows that:

$$\begin{aligned} (\hat{0})^c &= \{(x, \nu_{\hat{0}}(x), \eta_{\hat{0}}(x), \mu_{\hat{0}}(x))|x \in X\}, \\ &= \{(x, 1, 0, 0)|x \in X\} = \hat{1}. \end{aligned}$$

(4) Similarly, based on the definition of complement in PFS, it follows that:

$$\begin{aligned}(\hat{1})^c &= \{(x, \nu_{\hat{1}}(x), \eta_{\hat{1}}(x), \mu_{\hat{1}}(x))|x \in X\},\\ &= \{(x, 0, 0, 1)|x \in X\} = \hat{0}.\end{aligned}$$ □

Statements 1 and 2 in Theorem 3.5 are called the dominance laws, Statement 3 is called the zero complement law and Statement 4 is called the one complement law.

However, the properties

$$A \cup (A \cap B) = A \text{ and } A \cap (A \cup B) = A$$

are not satisfied.

Let $X = \{p, q, r\}$ and $A, B \in PFS(X)$ be defined as:

$$\begin{aligned}A &= \{(p, 0.3, 0.2, 0.4), (q, 0.5, 0.4, 0.1), (r, 0.4, 0.4, 0.1)\},\\ B &= \{(p, 0.4, 0.2, 0.2), (q, 0.5, 0.2, 0.1), (r, 0.3, 0.5, 0.2)\}.\end{aligned}$$

Then,

$$A \cap B = \{(p, 0.3, 0.2, 0.4), (q, 0.5, 0.2, 0.1), (r, 0.3, 0.4, 0.2)\},$$

and hence,

$$A \cup (A \cap B) = \{(p, 0.3, 0.2, 0.4), (q, 0.5, 0.2, 0.1), (r, 0.4, 0.4, 0.1)\} \neq A.$$

Similarly,

$$A \cap (A \cup B) = \{(p, 0.3, 0.2, 0.4), (q, 0.5, 0.2, 0.1), (r, 0.4, 0.4, 0.1)\} \neq A.$$

From these examples, the following theorem can be generalized.

Theorem 3.6. *Let X be a non-empty set and $A, B \in PFS(X)$. Then, the following property holds:*

$$A \cup (A \cap B) = A \cap (A \cup B).$$

Proof. Let $A, B \in PFS(X)$ be arbitrary, where:

$$\begin{aligned}A &= \{(x, \mu_A(x), \eta_A(x), \nu_A(x))|x \in X\},\\ B &= \{(x, \mu_B(x), \eta_B(x), \nu_B(x))|x \in X\}.\end{aligned}$$

Then,

$$\begin{aligned}A \cup (A \cap B) &= A \cup \{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \max\{\nu_A(x), \nu_B(x)\})|x \in X\},\\ &= \{(x, \max(\mu_A(x), \min\{\mu_A(x), \mu_B(x)\}), \min(\eta_A(x), \min\{\eta_A(x), \eta_B(x)\}),\\ &\quad \min(\nu_A(x), \max\{\nu_A(x), \nu_B(x)\}))|x \in X\},\\ &= \{(x, \mu_A(x), \min\{\eta_A(x), \eta_B(x)\}, \nu_A(x))|x \in X\}.\end{aligned}$$

Similarly,

$$\begin{aligned}A \cap (A \cup B) &= A \cap \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\eta_A(x), \eta_B(x)\}, \min\{\nu_A(x), \nu_B(x)\})|x \in X\},\\ &= \{(x, \min(\mu_A(x), \max\{\mu_A(x), \mu_B(x)\}), \min(\eta_A(x), \min\{\eta_A(x), \eta_B(x)\}),\\ &\quad \max(\nu_A(x), \min\{\nu_A(x), \nu_B(x)\}))|x \in X\},\\ &= \{(x, \mu_A(x), \min\{\eta_A(x), \eta_B(x)\}, \nu_A(x))|x \in X\}.\end{aligned}$$

Therefore, it is proven that:

$$A \cup (A \cap B) = A \cap (A \cup B). \quad \square$$

4. Conclusion

In the framework of Picture Fuzzy Sets (PFS), several classical algebraic properties are preserved. The subset relation satisfies the transitive property, while the union and intersection operations adhere to the commutative, associative, distributive, and De Morgan's laws. Furthermore, the intersection of subsets equals the smaller set, and the empty set $\hat{0}$ is a subset of every PFS. The complement, union, and intersection operations in PFS also satisfy the idempotent property. Additionally, the dominance and zero/one complement laws hold true in PFS.

However, not all classical algebraic properties are retained. The union of subset sets is not necessarily equal to their superset, and not every PFS is a subset of the universal set $\hat{1}$. Moreover, the complement and absorption laws do not strictly hold under union and intersection operations, although both yield equivalent results within the PFS structure. This finding confirms that the algebraic structure of PFS possesses unique characteristics that require modification of operations or the development of a new algebraic framework for more consistent application in complex uncertainty modeling.

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