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SHARP HAUSDORFF YOUNG INEQUALITY AND INVERSION FORMULA FOR THE COUPLED WINDOWED OFFSET FRACTIONAL FOURIER TRANSFORM

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Abstract. This work presents the Sharp Hausdorff-Young inequality for the Coupled windowed offset fractional Fourier transform. The coupled windowed offset fractional Fourier transform can be regarded as a generalized version of the fractional windowed Fourier transform. Various properties, including an inversion formula, are studied in detail for the coupled windowed offset fractional Fourier transform. Additionally, relationships with both the two-dimensional Fourier transform and the two-dimensional coupled windowed fractional Fourier transform are also studied.

Keywords: coupled windowed offset fractional Fourier transform, fractional Fourier transform, inversion formula

1. Introduction

The fractional Fourier transform (FrFT), initiated by Namias in 1980, has become an essential analytical tool with deep connections to quantum mechanics, optics, signal processing, and various time–frequency methodologies [1,2,3,4]. Its ability to interpolate between the time and frequency domains has motivated extensive research on generalized fractional transforms [5,6]. Among these developments, the coupled fractional Fourier transform (CFrFT) has attracted growing attention due to its wider flexibility and applicability in multidimensional signal analysis. In recent years, several extensions of the CFrFT have been proposed, including windowed and offset formulations, to accommodate localized and shifted features of practical signals [7,8,9,10,11].

A significant contribution in this direction is the formulation of the coupled windowed offset fractional Fourier transform (CWOFrFT), which simultaneously

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incorporates coupling parameters, window functions, and offset operations [12,13]. This general structure unifies several earlier fractional-type transforms, including the windowed FrFT and the two-dimensional Fourier Transform (2-D FT), as special cases. As with any generalized transformation, establishing fundamental analytic inequalities and inversion principles is essential for understanding its behavior and for developing rigorous applications in time–frequency analysis.

Despite significant progress in the development of the generalized fractional Fourier transform, several important theoretical issues remain unresolved. Existing studies on the CWOFrFT have largely focused on its formulation and fundamental properties—such as research [14] addressing only the convolution and correlation forms of the CWOFrFT—while rigorous analytical results, including sharp norm inequalities and explicit reconstruction frameworks, have received insufficient attention. Consequently, the theoretical understanding of the CWOFrFT is not yet as comprehensive as that of the classical Fourier transform or other well-established fractional transforms. To address these limitations, this paper formulates sharp Hausdorff–Young inequalities for the CWOFrFT and delineates its relationships with the two-dimensional Fourier transform and the two-dimensional CWFrFT. These results not only extend the existing theoretical framework but also provide new analytical tools that strengthen the mathematical foundation of coupled windowed fractional transforms and support their future application in multidimensional time-frequency analysis.

One of the core inequalities associated with Fourier-type transforms is the Hausdorff–Young inequality, whose sharp form plays a central role in harmonic analysis and has important implications for boundedness, stability, and signal concentration. While Sharp Hausdorff–Young inequalities have been investigated for the classical FT and the windowed FrFT, analogous results for the more general CWOFrFT remain unexplored. Furthermore, although several basic properties of the CWOFrFT have been established in earlier works, a complete inversion formula that explicitly characterizes reconstruction from its transform has not yet been fully developed [15].

The main objective of this paper is to fill these gaps by establishing a Sharp Hausdorff–Young inequality and deriving a rigorous inversion formula for the CWOFrFT. Our approach relies on exploiting the structural relationship between the CWOFrFT and the 2-D FT, which enables a concise derivation of boundedness results and simplifies the analytic framework [16]. In particular, we show that the sharp form of the Hausdorff–Young inequality for the CWOFrFT follows naturally from the classical Beckner–Babenko inequality applied to an appropriately transformed function. Additionally, the obtained inversion formula provides a direct and explicit reconstruction mechanism, thereby strengthening the theoretical foundation of the CWOFrFT [17].

The paper is structured as follows. Section 2 recalls relevant preliminaries, including the definitions of the FrFT and some existing results related to the transformation. Section 3 presents the definition and properties of the CWOFrFT, as well as its relation with the FT, which will be instrumental in deriving inequalities associated with the CWOFrFT. Section 4 contains a brief conclusion of the work.

2. Preliminaries

First, we recall several preliminary results related to the FrFT and the CFrFT, together with their fundamental properties, which will be required in the subsequent analysis. We begin with the following well-known definition.

Definition 2.1. *For measurable functions on $\mathbb{R}^2$, the $L^r(\mathbb{R}^2)$-norm is defined by:*

$$\|f\|_{L^r(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^r \, d\boldsymbol{x}\right)^{1/r} < \infty, \quad 1 \leq r < \infty. \tag{2.1}$$

Here $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2, d\boldsymbol{x} = dx_1 dx_2$.

In particular, when $r = \infty$ we obtain $L^\infty(\mathbb{R}^2)$-norm:

$$\|f\|_{L^\infty(\mathbb{R}^2)} = \text{ess sup}_{\boldsymbol{x}\in\mathbb{R}^2}|f(\boldsymbol{x})|. \tag{2.2}$$

For a continuous function $f \in L^\infty(\mathbb{R}^2)$, one obtains:

$$\|f\|_{L^\infty(\mathbb{R}^2)} = \sup_{t\in\mathbb{R}^2} |f(\boldsymbol{x})|. \tag{2.3}$$

The usual inner product of $L^2(\mathbb{R}^2)$ is then defined as:

$$\langle f, g\rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})} \, d\boldsymbol{x}. \tag{2.4}$$

Definition 2.2. *For a given function $f \in L^1(\mathbb{R}^2)$, the 2-D FrFT with parameter α is described by the formula [18]:*

$$\mathcal{F}_\alpha\{f\}(\boldsymbol{\zeta}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})K_\alpha(\boldsymbol{\zeta}, \boldsymbol{x}) \, d\boldsymbol{x}, \tag{2.5}$$

where the kernel transform $K_\alpha(\boldsymbol{\zeta}, \boldsymbol{x})$ is defined by:

$$K_\alpha(\boldsymbol{\zeta}, \boldsymbol{x}) = \begin{cases} A_\alpha e^{i(|\boldsymbol{x}|^2+|\boldsymbol{\zeta}|^2)\frac{\cot\alpha}{2} - i\boldsymbol{x}\cdot\boldsymbol{\zeta}\csc\alpha}, & \alpha \neq n\pi, \\ \delta(\boldsymbol{x} - \boldsymbol{\zeta}), & \alpha = 2n\pi, \\ \delta(\boldsymbol{x} + \boldsymbol{\zeta}), & \alpha = (2n+1)\pi, n \in \mathbb{Z}, \end{cases} \tag{2.6}$$

for which Dirac delta function $\delta(\boldsymbol{x} - \boldsymbol{\zeta}) = \delta(x_1 - \zeta_1)\delta(x_2 - \zeta_2)$, and:

$$A_\alpha = \frac{1 - i\cot\alpha}{2\pi}, \quad \overline{A_\alpha} = \frac{1 + i\cot\alpha}{2\pi}. \tag{2.7}$$

The kernel of the FrFT can be shown to possess the following fundamental properties.

$$\overline{K_\alpha(\boldsymbol{\zeta}, \boldsymbol{x})} = K_{-\alpha}(\boldsymbol{\zeta}, \boldsymbol{x}),$$

and:

$$\int_{\mathbb{R}^2} K_\alpha(\boldsymbol{\zeta}, \boldsymbol{x})\overline{K_\alpha(\boldsymbol{\zeta}', \boldsymbol{x})} \, d\boldsymbol{x} = \delta(\boldsymbol{\zeta} - \boldsymbol{\zeta}'),$$

where $\overline{K_\alpha(\boldsymbol{\zeta}, \boldsymbol{x})}$ stands for the conjugate of the kernel function $K_\alpha(\boldsymbol{\zeta}, \boldsymbol{x})$ in the complex sense.

Definition 2.3. *Let $f \in L^1(\mathbb{R}^2)$ and $\mathcal{F}_\alpha\{f\} \in L^1(\mathbb{R}^2)$. The inverse transform of the 2-D FrFT of f defined through the integral below [18]:*

$$\begin{aligned} f(\boldsymbol{x}) &= \mathcal{F}_\alpha^{-1}\big[\mathcal{F}_\alpha\{f\}\big](\boldsymbol{x}), \\ &= \int_{\mathbb{R}^2} \mathcal{F}_\alpha\{f\}(\boldsymbol{\zeta})\overline{K_\alpha(\boldsymbol{\zeta},\boldsymbol{x})}\, d\boldsymbol{\zeta}, \\ &= \int_{\mathbb{R}^2} \mathcal{F}_\alpha\{f\}(\boldsymbol{\zeta})\overline{A_\alpha} e^{-i(|\boldsymbol{x}|^2+|\boldsymbol{\zeta}|^2)\frac{\cot\alpha}{2}+i\boldsymbol{x}\cdot\boldsymbol{\zeta}\csc\alpha}\, d\boldsymbol{\zeta}. \end{aligned} \tag{2.8}$$

Formula (2.8) provides a way to recover a function from its FrFT. The definition of the CFrFT is recalled next.

Definition 2.4 (CFrFT). *The CFrFT of any function $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is defined as [12,19]:*

$$\mathcal{F}_{\alpha,\beta}\{f\}(\boldsymbol{\zeta}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) K_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x}) d\boldsymbol{x}, \tag{2.9}$$

where:

$$K_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x}) = d(\gamma) e^{-i\left(a(\gamma)(|\boldsymbol{x}|^2+|\boldsymbol{\zeta}|^2)-\boldsymbol{x}\cdot M\boldsymbol{\zeta}\right)}. \tag{2.10}$$

In this case,

$$\gamma = \frac{\alpha+\beta}{2}, \quad \delta = \frac{\alpha-\beta}{2}, \quad a(\gamma) = \frac{\cot\gamma}{2}, \quad b(\gamma,\delta) = \frac{\cos\delta}{\sin\gamma},$$
$$c(\gamma,\delta) = \frac{\sin\delta}{\sin\gamma}, \quad d(\gamma) = \frac{ie^{-i\gamma}}{2\pi\sin\gamma}, \quad M = \begin{pmatrix} b(\gamma,\delta) & c(\gamma,\delta) \\ -c(\gamma,\delta) & b(\gamma,\delta) \end{pmatrix},$$

for real numbers α and β such that $\alpha+\beta \notin 2\pi\mathbb{Z}$.

Definition 2.5. *For every $\mathcal{F}_{\alpha,\beta}\{f\}, f \in L^1(\mathbb{R}^2)$, the inverse of the CFrFT is given by [12,19]:*

$$\begin{aligned} f(\boldsymbol{x}) &= \int_{\mathbb{R}^2} \mathcal{F}_{\alpha,\beta}\{f\}(\boldsymbol{\zeta})\overline{K_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x})} d\boldsymbol{\zeta} \\ &= \overline{d(\gamma)} \int_{\mathbb{R}^2} \mathcal{F}_{\alpha,\beta}\{f\}(\boldsymbol{\zeta}) e^{i\left(a(\gamma)(|\boldsymbol{x}|^2+|\boldsymbol{\zeta}|^2)-\boldsymbol{x}\cdot M\boldsymbol{\zeta}\right)} d\boldsymbol{\zeta}. \end{aligned} \tag{2.11}$$

It is straightforward to verify that for $\alpha = \beta = \frac{\pi}{2}$, the CFrFT $\mathcal{F}_{\alpha,\beta}\{f\}$ becomes the 2-D FT $\mathcal{F}\{f\}$ as defined in [3].

The following result will be needed to derive some properties of the CWFrFT in the next section.

Theorem 2.6 (Parseval's formula). *For any $f, g \in L^2(\mathbb{R}^2)$, the subsequent equality remains valid, as reported in [3]:*

$$\langle f, g\rangle_{L^2(\mathbb{R}^2)} = \langle \mathcal{F}_{\alpha,\beta}\{f\}, \mathcal{F}_{\alpha,\beta}\{g\}\rangle_{L^2(\mathbb{R}^2)}. \tag{2.12}$$

and:

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \|\mathcal{F}_{\alpha,\beta}\{f\}\|^2_{L^2(\mathbb{R}^2)}. \tag{2.13}$$

3. The Coupled Windowed Offset Fractional Fourier Transform and Properties

In this section, we are concerned with the construction of the CWOFrFT and we want to prove inversion formula for CWOFrFT as the main results of this paper.

Definition 3.1. *Let $\phi \in L^2(\mathbb{R}^2) \setminus \{0\}$ be a window function. The CWOFrFT of $f \in L^2(\mathbb{R}^2)$, defined as:*

$$\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z}) = \int_{\mathbb{R}^2} f(x)\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}K_{\alpha,\beta}^{m,n}(\boldsymbol{\zeta},\boldsymbol{x})d\boldsymbol{x}$$
$$= d(\gamma)\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-i\left(a(\gamma)(|\boldsymbol{\zeta}|^2+|\boldsymbol{x}|^2+|\boldsymbol{m}|^2)+\boldsymbol{x}\cdot M(\boldsymbol{m}-\boldsymbol{\zeta})+\boldsymbol{\zeta}\cdot(\boldsymbol{n}-\boldsymbol{m}\,a(\gamma))\right)}d\boldsymbol{x}. \tag{3.1}$$

where:

$$\gamma = \frac{\alpha+\beta}{2}, \quad \delta = \frac{\alpha-\beta}{2}, \quad a(\gamma) = \frac{\cot\gamma}{2}, \quad b(\gamma,\delta) = \frac{\cos\delta}{\sin\gamma},$$
$$c(\gamma,\delta) = \frac{\sin\delta}{\sin\gamma}, \quad d(\gamma) = \frac{ie^{-i\gamma}}{2\pi\sin\gamma}, \quad M = \begin{pmatrix} b(\gamma,\delta) & c(\gamma,\delta) \\ -c(\gamma,\delta) & b(\gamma,\delta) \end{pmatrix}.$$

An interesting connection between the WCOFrFT and the CWFrFT is described through by definition:

$$\begin{aligned}
&\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z}) \\
&= d(\gamma)\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-i\left(a(\gamma)(|\boldsymbol{\zeta}|^2+|\boldsymbol{x}|^2+|m|^2)+\boldsymbol{x}\cdot M(\boldsymbol{m}-\boldsymbol{\zeta})+\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))\right)}d\boldsymbol{x}, \\
&= d(\gamma)\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-i\left(a(\gamma)(|\boldsymbol{\zeta}|^2+|\boldsymbol{x}|^2)-\boldsymbol{x}\cdot M\gamma\right)}e^{-ix\cdot Mm}e^{-i\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))}e^{-ia(\gamma)|m|^2}d\boldsymbol{x}, \\
&= d(\gamma)\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-ix\cdot Mm}e^{-i\left(a(\gamma)(|\boldsymbol{\zeta}|^2+|\boldsymbol{x}|^2)-\boldsymbol{x}\cdot M\gamma\right)}d\boldsymbol{x}\,e^{-i\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))}e^{-ia(\gamma)|m|^2}.
\end{aligned} \tag{3.2}$$

Or equivalently,

$$\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z})e^{i\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))}e^{ia(\gamma)|m|^2} = \mathcal{F}_{\alpha,\beta}\{fT_x\phi_m\}(\zeta), \tag{3.3}$$

where:

$$fT_x\phi_m(\boldsymbol{z},\boldsymbol{x}) = f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-ix\cdot Mm}e^{-i(a(\gamma)|\boldsymbol{x}|^2)}. \tag{3.4}$$

From Equation (3.1), we have connection between CWOFrFt and 2-D FT described by:

$$\begin{aligned}
&\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z}) \\
&\qquad = d(\gamma)\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-i\left(a(\gamma)(|\boldsymbol{\zeta}|^2+|\boldsymbol{x}|^2+|m|^2)+\boldsymbol{x}\cdot M(\boldsymbol{m}-\boldsymbol{\zeta})+\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))\right)}d\boldsymbol{x}.
\end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned}
&\frac{1}{d(\gamma)}\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z})e^{i\left(a(\gamma)(|\boldsymbol{\zeta}|^2+|m|^2)+\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))\right)}\\
&\qquad\qquad\qquad = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-ia(\gamma)|\boldsymbol{x}|^2}e^{-i\boldsymbol{x}\cdot Mm}e^{-i\boldsymbol{x}\cdot\boldsymbol{M\zeta}}d\boldsymbol{x},\\
&\qquad\qquad\qquad = \int_{\mathbb{R}^2} h^m_{f,\phi}(\boldsymbol{x},\boldsymbol{z})e^{i\boldsymbol{x}\cdot M\zeta}d\boldsymbol{x},\\
&\qquad\qquad\qquad = \mathcal{F}\{h^m_{f,\phi}\}(-M\zeta), \qquad (3.6)
\end{aligned}$$

where:

$$h^m_{f,\phi}(\boldsymbol{x},\boldsymbol{z}) = f(x)\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-ia(\gamma)|\boldsymbol{x}|^2}e^{-i\boldsymbol{x}\cdot Mm}. \qquad (3.7)$$

3.1. *Orthogonality Relation*

In this section, we will describe the orthogonal relations in CWOFrFT which will be derived in detail as follows.

Theorem 3.2. *Let $\phi_1,\phi_2 \in L^2(\mathbb{R}^2)\setminus\{0\}$ be two window functions and $f_1, f_2 \in L^2(\mathbb{R}^2)$ be arbitrary, then:*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{z})\overline{\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f_2\}(\boldsymbol{\zeta},\boldsymbol{z})}d\boldsymbol{\zeta}d\boldsymbol{z} = \langle f_1,f_2\rangle\langle\phi_1,\phi_2\rangle. \qquad (3.8)$$

Proof. It follows from Equation (3.1) and Equation (3.3), we get:

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{z})\overline{\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f_2\}(\boldsymbol{\zeta},\boldsymbol{z})}d\boldsymbol{\zeta}d\boldsymbol{z},\\
&\qquad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{F}_{\alpha,\beta}\{f_1T_x\phi_m\}(\zeta)e^{-i\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))}e^{-ia(\gamma)|m|^2}\\
&\qquad\quad\times\overline{\mathcal{F}_{\alpha,\beta}\{f_2T_x\phi_m\}(\zeta)e^{-i\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))}e^{-ia(\gamma)|m|^2}}d\boldsymbol{\zeta}d\boldsymbol{z},\\
&\qquad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{F}_{\alpha,\beta}\{f_1T_x\phi_m\}(\zeta)e^{-i\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))}e^{-ia(\gamma)|m|^2}e^{i\boldsymbol{\zeta}\cdot(n-m\,a(\gamma))}e^{ia(\gamma)|m|^2}\\
&\qquad\quad\times\overline{\mathcal{F}_{\alpha,\beta}\{f_2T_x\phi_m\}(\zeta)}d\boldsymbol{\zeta}d\boldsymbol{z},\\
&\qquad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{F}_{\alpha,\beta}\{f_1T_x\phi_m\}(\zeta)\overline{\mathcal{F}_{\alpha,\beta}\{f_2T_x\phi_m\}(\zeta)}d\boldsymbol{\zeta}d\boldsymbol{z},\\
&\qquad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f_1(\boldsymbol{x})\overline{\phi_1(\boldsymbol{x}-\boldsymbol{z})}e^{i\boldsymbol{x}\cdot Mm}\overline{f_2(\boldsymbol{x})\overline{\phi_2(\boldsymbol{x}-\boldsymbol{z})}e^{i\boldsymbol{x}\cdot Mm}}d\boldsymbol{x}d\boldsymbol{z},\\
&\qquad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f_1(\boldsymbol{x})\overline{\phi_1(\boldsymbol{x}-\boldsymbol{z})}\,\overline{f_2(\boldsymbol{x})}\phi_2(\boldsymbol{x}-\boldsymbol{z})d\boldsymbol{x}d\boldsymbol{z}. \qquad (3.9)
\end{aligned}$$

Let $\boldsymbol{u}=\boldsymbol{x}-\boldsymbol{z}, du = dz$. So, we get:

$$\begin{aligned}
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{z})\overline{\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f_2\}(\boldsymbol{\zeta},\boldsymbol{z})}d\boldsymbol{\zeta}d\boldsymbol{z} &= \int_{\mathbb{R}^2}f_1(\boldsymbol{x})\overline{f_2(\boldsymbol{x})}d\boldsymbol{x}\int_{\mathbb{R}^2}\overline{\phi_1(\boldsymbol{u})}\phi_2(\boldsymbol{u})d\boldsymbol{u},\\
&= \langle f_1,f_2\rangle\langle\phi_1,\phi_2\rangle, \qquad (3.10)
\end{aligned}$$

and thus the proof has been completed. □

The theorem above yields the following three consequences.

(1) If $\phi_1 = \phi_2$, then:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{z})\overline{\mathcal{W}_\phi^{(\gamma,m,n)}\{f_2\}(\boldsymbol{\zeta},\boldsymbol{z})}d\boldsymbol{\zeta}d\boldsymbol{z} = \|\phi_1\|^2_{L^2(\mathbb{R}^2)}\langle f_1, f_2\rangle. \tag{3.11}$$

(2) If $f_1 = f_2$, then:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{z})\right|^2 d\boldsymbol{\zeta}d\boldsymbol{z} = \langle\phi_1,\phi_2\rangle\|f_1\|^2_{L^2(\mathbb{R}^2)}. \tag{3.12}$$

(3) If $f_1 = f_2$ and $\phi_1 = \phi_2$, then:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{z})\right|^2 d\boldsymbol{\zeta}d\boldsymbol{z} = \|\phi_1\|^2_{L^2(\mathbb{R}^2)}\|f_1\|^2_{L^2(\mathbb{R}^2)}. \tag{3.13}$$

Let us now we derive the inversion formula for CWOFrFT.

3.2. *Inversion formula*

The inversion theorem for the CWOFrFT will now be derived.

Theorem 3.3 (Inversion formula). *Let $\phi \in L^2(\mathbb{R}^2)\setminus\{0\}$ be a window function. Then for every $f \in L^2(\mathbb{R}^2)$ we have:*

$$f(\boldsymbol{x}) = \frac{1}{\|\phi\|_{L^2(\mathbb{R}^2)}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{t})\phi(\boldsymbol{x}-\boldsymbol{z})\overline{K^{m,n}_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x})}d\boldsymbol{z}d\boldsymbol{\zeta}. \tag{3.14}$$

Proof. From Definition (3.1), and Fubini's theorem, we get:

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{z})\overline{\mathcal{W}_\phi^{(\gamma,m,n)}\{f_2\}(\boldsymbol{\zeta},\boldsymbol{z})}d\boldsymbol{\zeta}d\boldsymbol{z}\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{x})\\
&\quad\times\left(\overline{d(\gamma)\int_{\mathbb{R}^2} f_2(\boldsymbol{x})\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}e^{-i(a(\gamma)(|\boldsymbol{\zeta}|^2+|\boldsymbol{x}|^2+|m|^2)+\boldsymbol{x}\cdot M(\boldsymbol{m}-\boldsymbol{\zeta})+\boldsymbol{\zeta}\cdot(n-m\,a(\gamma)))}d\boldsymbol{x}}\right)d\boldsymbol{z}d\boldsymbol{\zeta},\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{x})\overline{f_2(\boldsymbol{x})}\phi(\boldsymbol{x}-\boldsymbol{z})\overline{K^{m,n}_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x})}d\boldsymbol{x}d\boldsymbol{z}d\boldsymbol{\zeta},\\
&= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{x})\phi(\boldsymbol{x}-\boldsymbol{z})\overline{K^{m,n}_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x})}d\boldsymbol{z}d\boldsymbol{\zeta}\right)\overline{f_2(\boldsymbol{x})}d\boldsymbol{x},\\
&= \left\langle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{x})\phi(\boldsymbol{x}-\boldsymbol{z})\overline{K^{m,n}_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x})}d\boldsymbol{z}d\boldsymbol{\zeta}, f_2\right\rangle.
\end{aligned} \tag{3.15}$$

An application of Theorem (3.2) to identify, we get:

$$\langle f_1, f_2\rangle\|\phi\|_{L^2(\mathbb{R}^2)} = \left\langle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{x})\phi(\boldsymbol{x}-\boldsymbol{z})\overline{K^{m,n}_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x})}d\boldsymbol{z}d\boldsymbol{\zeta}, f_2\right\rangle. \tag{3.16}$$

Or, equivalently:

$$\|\phi\|_{L^2(\mathbb{R}^2)}f_1(\boldsymbol{x}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}_\phi^{(\gamma,m,n)}\{f_1\}(\boldsymbol{\zeta},\boldsymbol{x})\phi(\boldsymbol{x}-\boldsymbol{z})\overline{K^{m,n}_{\alpha,\beta}(\boldsymbol{\zeta},\boldsymbol{x})}d\boldsymbol{z}d\boldsymbol{\zeta}. \tag{3.17}$$

Hence,

$$f(\boldsymbol{x}) = \frac{1}{\|\phi\|_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta}, \boldsymbol{x}) \phi(\boldsymbol{x} - \boldsymbol{z}) \overline{K_{\alpha,\beta}^{m,n}(\boldsymbol{\zeta}, \boldsymbol{x})} d\boldsymbol{z} d\boldsymbol{\zeta}, \tag{3.18}$$

and thus the proof has been completed. □

3.3. *Sharp Hausdorff-Young*

This subsection presents the Sharp Hausdorff–Young inequality formulated within the framework of the CWOFrFT.

Theorem 3.4 (CWOFrFT Sharp Hausdorff-Young). *Let* $p \in [1, 2]$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$*, then we have:*

$$\left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| \mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta}, \boldsymbol{z}) \right|^q d(\boldsymbol{\zeta}) d\boldsymbol{z} \right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}} 2\pi |\csc \gamma|^{\frac{2}{q}-1}} \|\phi\|_{L^p(\mathbb{R}^2)} \|f\|_{L^p(\mathbb{R}^2)} . \tag{3.19}$$

Proof. Based on the Sharp Hausdorff-Young inequality for the 2-D FT, as stated in [19], the stated result follows.

$$\left(\int_{\mathbb{R}^2} |\mathcal{F}\{f\}(\boldsymbol{\zeta})|^q d\boldsymbol{\zeta} \right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}^2} |f(\boldsymbol{t})|^p d\boldsymbol{t} \right)^{\frac{1}{p}} . \tag{3.20}$$

Substituting $h_{f,\phi}^m(\boldsymbol{x}, \boldsymbol{z})$ for $f(\boldsymbol{x})$ defined by Equation (3.7) gives:

$$\left(\int_{\mathbb{R}^2} \left| \mathcal{F}\left\{ h_{f,\phi}^m(\boldsymbol{x}, \boldsymbol{z}) \right\} (\boldsymbol{\zeta}) \right|^q d\boldsymbol{\zeta} \right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}^2} |h_{f,\phi}^m(\boldsymbol{x}, \boldsymbol{z})|^p d\boldsymbol{x} \right)^{\frac{1}{p}} . \tag{3.21}$$

Letting $\boldsymbol{\zeta} = -M\boldsymbol{\zeta}$ in relation Equation (3.21), we get:

$$\left(\int_{\mathbb{R}^2} \left| \mathcal{F}\left\{ h_{f,\phi}^m(\boldsymbol{x}, \boldsymbol{z}) \right\} (-M\boldsymbol{\zeta}) \right|^q d(-M\boldsymbol{\zeta}) \right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}^2} |f(x) \overline{\phi(\boldsymbol{x} - \boldsymbol{z})}|^p d\boldsymbol{x} \right)^{\frac{1}{p}} . \tag{3.22}$$

Or, equivalently:

$$(\csc^2 \boldsymbol{\gamma})^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} \left| \mathcal{F}\left\{ h_{f,\phi}^m(\boldsymbol{x}, \boldsymbol{z}) \right\} (\boldsymbol{\zeta}) \right|^q d(\boldsymbol{\zeta}) \right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}^2} |f(x) \overline{\phi(\boldsymbol{x} - \boldsymbol{z})}|^p d\boldsymbol{x} \right)^{\frac{1}{p}} . \tag{3.23}$$

With the help of Equation (3.6), we get:

$$|\csc^2 \boldsymbol{\gamma}|^{\frac{1}{q}} \left((d^{-1}(\gamma))^q \int_{\mathbb{R}^2} \left| \mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta}, \boldsymbol{z}) \right|^q d(\boldsymbol{\zeta}) \right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}^2} |f(x) \overline{\phi(\boldsymbol{x} - \boldsymbol{z})}|^p d\boldsymbol{x} \right)^{\frac{1}{p}} . \tag{3.24}$$

Hence,

$$\left(\int_{\mathbb{R}^2} \left| \mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta}, \boldsymbol{z}) \right|^q d(\boldsymbol{\zeta}) \right)^{\frac{1}{q}} \leq \frac{1}{2\pi |\csc \boldsymbol{\gamma}|^{\frac{2}{q}-1}} \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}^2} |f(x) \overline{\phi(\boldsymbol{x} - \boldsymbol{z})}|^p d\boldsymbol{x} \right)^{\frac{1}{p}} . \tag{3.25}$$

Equation (3.25) can be rewritten as:

$$\int_{\mathbb{R}^2} \left|\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z})\right|^q d(\boldsymbol{\zeta}) \leq \left(\frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}2\pi|\csc\gamma|^{\frac{2}{q}-1}}\right)^q \left(\int_{\mathbb{R}^2} |f(x)\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}|^p d\boldsymbol{x}\right)^{\frac{q}{p}}. \tag{3.26}$$

Integrate both sides of Equation (3.26) with respect to dz further, we get:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z})\right|^q d(\boldsymbol{\zeta})d\boldsymbol{z} \leq \left(\frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}2\pi|\csc\gamma|^{\frac{2}{q}-1}}\right)^q \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} |f(x)\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}|^p d\boldsymbol{x}\right)^{\frac{q}{p}} d\boldsymbol{z} \tag{3.27}$$

$$\leq \left(\frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}2\pi|\csc\gamma|^{\frac{2}{q}-1}}\right)^q \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |f(x)\overline{\phi(\boldsymbol{x}-\boldsymbol{z})}|^p d\boldsymbol{x}d\boldsymbol{z}\right)^{\frac{q}{p}}. \tag{3.28}$$

By the change of variable $u = x - z, du = dx$, so we get:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z})\right|^q d(\boldsymbol{\zeta})d\boldsymbol{z} \leq \left(\frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}2\pi|\csc\gamma|^{\frac{2}{q}-1}}\right)^q \left(\|\phi\|^p_{L^p(\mathbb{R}^2)}\int_{\mathbb{R}^2} |f(x)|^p d\boldsymbol{x}\right)^{\frac{q}{p}}. \tag{3.29}$$

Hence,

$$\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z})\right|^q d(\boldsymbol{\zeta})d\boldsymbol{z}\right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}2\pi|\csc\gamma|^{\frac{2}{q}-1}} \|\phi\|_{L^p(\mathbb{R}^2)} \|f\|_{L^p(\mathbb{R}^2)}, \tag{3.30}$$

and thus the proof has been completed. □

Remark 3.5. *From Theorem* (3.3) *if we choose* $p = q = 2$*, we get:*

$$\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\mathcal{W}_{\phi}^{(\gamma,m,n)}\{f\}(\boldsymbol{\zeta},\boldsymbol{z})\right|^2 d(\boldsymbol{\zeta})d\boldsymbol{z}\right)^{\frac{1}{2}} \leq \frac{1}{2\pi} \|\phi\|_{L^2(\mathbb{R}^2)} \|f\|_{L^2(\mathbb{R}^2)}. \tag{3.31}$$

4. Conclusion

This paper has presented the CWOFrFT and developed its theoretical foundation through the derivation of its fundamental properties and the establishment of the Sharp Hausdorff–Young inequality. In addition, the relationships between the proposed transform, the 2-D FT, and the two-dimensional CWFrFT have been rigorously established, demonstrating that the CWOFrFT provides a natural extension of existing fractional transform frameworks. The obtained results contribute to the mathematical theory of fractional integral transforms by enriching their analytical structure and offering a broader framework for multidimensional signal representation. These theoretical developments may facilitate further studies on transform-based methods and their applications. Future work will investigate some uncertainty principles like Heissenberg-type dan Logarithmic uncertainty principle, convolution and sampling theorems, and numerical implementations, together with potential applications in image processing, pattern recognition, and other areas of signal analysis.

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