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ON PRIME AND COPRIME EDGE LABELING OF SOME GRAPHS

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Abstract. A coprime edge labeling of a graph is defined as a labeling in which positive integer labels are assigned to each edge of the graph according to certain rules. In this labeling, every pair of edges adjacent to the same vertex must have relatively prime labels, meaning they have no common factors other than one. If the number of labels used is equal to the total number of edges of the graph, this labeling method is designated as a prime edge labeling. In the context of coprime edge labeling, the smallest possible number of labels that allows such a labeling is referred to as the minimum coprime edge number. The present study analyzes prime edge labeling and coprime edge labeling on several special graphs, including tadpoles, combs, disjoint unions of combs, unions of two cycles, disjoint unions of paths, volcano graphs, and regular caterpillars. Furthermore, this study provides precise values for the minimum coprime edge number of the $C_n \cup C_m$ graphs with odd n and m, volcano graphs, and regular caterpillars.

Keywords: Coprime Edge Labeling, Minimum Coprime Edge Number, Prime Edge Labeling

1. Introduction

Graph theory is a branch of discrete mathematics with a wide range of applications, such as frequency assignment [1], shortest path determination [2], message encryption [3], computer networks [4], and others. Graph labeling, a subject within the field of graph theory, entails assigning integers to a graph (edges, vertices, or both) according to specific criteria. Based on its domain, the classification of graph

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labeling is as follows: vertex labeling (vertex domain), edge labeling (edge domain), and total labeling (vertex and edge domain) [5].

Prime labeling is a particular form of vertex labeling introduced by Entringer in 1980 and later popularized by Tout et al. (1982). For a simple graph G, a prime labeling of G is defined as a bijective function $\Phi : V(G) \to \{1, 2, \cdots, |V(G)|\}$ in which the pairs of adjacent vertices are assigned labels that are relatively prime to each other [6]. The other researchers have extended the concept of prime labeling into several variations, including coprime labeling [7], odd prime labeling [8], prime edge labeling [9], coprime edge labeling [10], and others, which can be seen in [11–16].

A coprime edge labeling of a graph $G(p, q)$, where $p = |V(G)|$ and $q = |E(G)|$, is defined as an injection $\Phi : E(G) \to \{1, 2, 3, \cdots, k\}$ subject to the condition that for every $k \geq q$, the labels of all edges adjacent to each vertex $v \in V(G)$ are relatively prime. If $k = q$, the coprime edge labeling is called a prime edge labeling. Graphs that fail to admit a prime edge labeling are likewise called a coprime edge graph. The smallest value of k that allows such labeling is known as the minimum coprime edge number, written as $\mathfrak{p}r_{\varepsilon}(G)$ [10]. Cycle graphs and particular tree graphs admit prime edge labeling (see [9]). Janani and Ramachandran [10] showed that the complete graph (K_p) satisfies $\mathfrak{p}r_{\varepsilon}(K_p) \leq p_{[\pi(p)+q-p]}$ for $p = 4$.

From an application perspective, coprime edge labeling can be interpreted as an assignment mechanism for communication channels, routing identifiers, or resource codes in which adjacent connections must avoid common divisibility patterns. Such assignments may reduce local interference, enhance distinguishability among neighboring links, and support efficient indexing or coding schemes in distributed systems and database architectures.

Despite recent progress, coprime edge labeling remains largely unexplored for many graph families possessing heterogeneous structures. Determining exact minimum coprime edge numbers and identifying classes of graphs that admit prime edge labelings continue to constitute important problems in graph labeling theory.

This paper investigates prime edge labeling and coprime edge labeling on several special graph families, namely tadpole graphs, comb graphs, disjoint unions of comb graphs, disjoint unions of paths, unions of two cycles, volcano graphs, and regular caterpillar graphs. These graph classes were selected because they exhibit diverse structural characteristics involving combinations of cycles, paths, and star-like components, thereby producing distinct adjacency configurations and coprime constraints.

Beyond their theoretical significance, these graph families may also serve as simplified models of practical network structures. Tadpole and volcano graphs resemble communication networks consisting of cyclic backbones connected to peripheral branches, while comb and regular caterpillar graphs share structural similarities with hierarchical data structures, indexing systems, and database organizations. Disjoint unions of paths and cycles can represent modular architectures or independent subnetworks. Consequently, studying coprime edge labeling on these graphs contributes not only to graph labeling theory but also provides insights into constructing network configurations with locally distinguishable identifiers and minimal adjacency conflicts.

Graphs containing vertices of high degree generally impose stronger coprime restrictions because a larger number of adjacent edges must receive pairwise relatively prime labels. Conversely, disjoint union graphs facilitate the investigation of labeling behavior under scaling processes. In addition, the graph $C_n \cup C_m$ is particularly interesting due to parity considerations that influence the minimum coprime edge number. Regular caterpillar graphs provide a structured extension of tree graphs, while the unresolved case of $C_{m,n}$ for $m \geq 3$ offers a meaningful direction for further study.

The main contributions of this paper are threefold. First, explicit prime edge labeling constructions are presented for tadpole graphs, comb graphs, disjoint unions of comb graphs, and disjoint unions of paths. Second, exact values of the minimum coprime edge number are established for $C_n \cup C_m$ when both n and m are odd, volcano graphs, and regular caterpillar graphs $C_{2,n}$. Finally, an open problem concerning coprime edge labeling of regular caterpillar graphs $C_{m,n}$ with $m \geq 3$ is proposed for future investigation.

2. Preliminaries

The following section discusses some definitions and relevant lemmas to this paper.

Definition 2.1. *[17] A tadpole graph $T_{m,n}$ for integer $m \geq 3$ and $n \geq 2$ is a type of graph of order and size $m+n$, constructed by subjecting a vertex of a cycle graph C_m to an identification process with a vertex of a path graph P_{n+1}. Specifically, the identification is made between $v_1 \in V(C_m)$ and $u_0 \in V(P_{n+1})$ where u_0 is one of the vertices in the path graph that has degree one.*

Definition 2.2. *[18] A comb graph Cb_n of order $2n+1$ is a graph obtained from $P_2 \bigodot_{u_0 v_1} C_{n,1}$ by attaching one vertex of P_2 to a vertex of degree two in the regular caterpillar graph $C_{n,1}$.*

Definition 2.3. *[19] A volcano graph V_n represents a graph of order and size $n+3$ derived from the identification of a cycle graph C_3 and a star graph S_n specifically by attaching the star graph's central vertex (the vertex of the highest degree) to one vertex of the cycle graph C_3.*

Definition 2.4. *[20] A regular caterpillar graph $C_{m,n}$ of order $mn+m$ refers to a graph formulated out of $P_m \triangleright S_n$ whereby the vertex of degree n from each copy within the star graph S_n is affixed to every vertex within the path graph P_m.*

In proving that two vertex labels are relatively prime, we use a particular lemma as outlined below.

Lemma 2.5. *[21] If a and b are consecutive, then a and b are relatively prime.*

Lemma 2.6. *[22] In the instance that a is odd, there exists a positive integer r with no odd factors except one such that* $\gcd(a, a+r) = 1$.

3. Result and Discussion

This section presents prime edge labeling and coprime edge labeling on several special graphs, including tadpoles, combs, disjoint unions of combs, unions of two cycles, disjoint unions of paths, volcano graphs, and regular caterpillars.

3.1. *The Tadpole Graph*

A tadpole graph $T_{m,n}$ features the vertex and edge sets:

$$V(T_{m,n}) = \{v_\mu \mid \mu = 1, 2, \cdots, m\} \cup \{u_\psi \mid \psi = 1, 2, \cdots, n\},$$
$$E(T_{m,n}) = \{v_1 v_m,\, v_\mu v_{\mu+1} \mid \mu = 1, 2, \cdots m-1\} \cup \{v_1 u_1\} \cup \{u_\psi u_\psi \mid \psi = 1, 2, \cdots, n-1\}.$$

Theorem 3.1. *The tadpole graph ($T_{m,n}$) is a prime edge graph.*

Proof. To show that the tadpole graph ($T_{m,n}$) qualifies as a prime edge graph, we construct a prime edge labeling on the graph. The function $\Phi : E(T_{m,n}) \to \{1, 2, \cdots, m+n\}$ is defined below.

$$\begin{aligned}
\Phi(v_\mu v_{\mu+1}) &= \mu, \quad \mu = 1, 2, \cdots, m-1,\\
\Phi(v_m v_1) &= m,\\
\Phi(v_1 u_1) &= m+1,\\
\Phi(u_\psi u_{\psi+1}) &= m + \psi + 1, \quad \psi = 1, 2, \cdots, n-1.
\end{aligned}$$

Based on Lemma 2.5, for each $\mu = 1, 2, \cdots, m-2$, the edge labels $v_\mu v_{\mu+1}$ and $v_{\mu+1} v_{\mu+2}$ are relatively prime since they are consecutive integers. We observe that $\Phi(v_1 v_2) = 1$, which is relatively prime to all integers; hence, $\gcd(\Phi(v_1 v_2), \Phi(v_m v_1)) = \gcd(\Phi(v_1 v_2), \Phi(v_1 u_1)) = 1$. The edge labels $v_m v_1$ and $v_1 u_1$ are also relatively prime because they are consecutive integers. Similarly, the edge labels $u_\psi u_{\psi+1}$ and $u_{\psi+1} u_{\psi+2}$ are consecutive integers for $\psi = 1, 2, \cdots, n-2$, so that $\gcd(u_\psi u_{\psi+1}, u_{\psi+1} u_{\psi+2}) = 1$. Based on this analysis, it can be concluded that the function Φ satisfies the prime edge labeling condition. Therefore, the tadpole graph admits a prime edge labeling. □

An example illustrating Theorem 3.1 is shown in Figure 1.

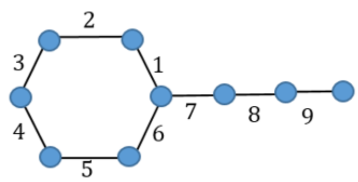

Figure. 1. Prime Edge Labeling on the Tadpole $T_{5,3}$.

3.2. *The Comb Graph*

The vertex and edge sets of a comb graph are defined as follows:

$V(\mathrm{Cb}_n) = \{v_0\} \cup \{u_\mu \mid \mu = 1, 2, \cdots, n\} \cup \{v_\psi \mid \psi = 1, 2, \cdots, n\}$,
$E(\mathrm{Cb}_n) = \{v_0 v_1\} \ \cup \ \{v_\psi v_\psi \mid \psi = 1, 2, \cdots, n-1\} \cup \{u_\mu v_\psi \mid \mu = \psi; (\mu, \psi) = 1, 2, \cdots, n\}$.

Theorem 3.2. *The comb graph* (Cb_n) *is a prime edge graph.*

Proof. Similarly, to show that the comb graph (Cb_n) is a prime edge graph, we construct a prime edge labeling on the graph. Suppose the function of $\Phi : E(Cb_n) \to \{1, 2, \cdots, 2n\}$ is defined as follows.

$$\Phi(v_\psi v_{\psi+1}) = 2\psi + 1, \quad \psi = 0, 1, \cdots, n-1,$$
$$\Phi(v_\psi u_\mu) = 2\mu, \quad \mu = \psi.$$

It is evident that $|\Phi(v_\psi v_{\psi+1}) - \Phi(v_{\psi+1} v_{\psi+2})| = |(2\psi + 1) - (2\psi + 3)| = 2$. Thus, applying the Lemma 2.6 concept, $\gcd(\Phi(v_\psi v_{\psi+1}), \Phi(v_{\psi+1} v_{\psi+2})) = 1$. Given the observed data, there is clear evidence that every pair of edges adjacent to the same vertex has relatively prime labels. This proves that the comb graph is indeed a prime edge graph. □

Figure 2 illustrates an example of a prime edge labeling of the comb graph Cb_4.

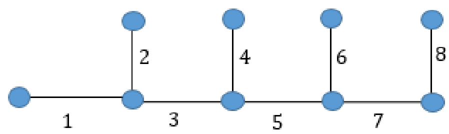

Figure. 2. Prime Edge Labeling on the Comb Cb_4.

Furthermore, we investigate the prime edge labeling of m disjoint comb graphs of order $2n + 1$, denoted by mCb_n.

Theorem 3.3. *The disjoint union of comb graph* mCb_n *is a prime edge graph.*

Proof. Define the function $\Phi : E(mCb_n) \to \{1, 2, \cdots, 2nm\}$ as follows. For $\tau = 1, 2, \cdots, m$,

$$\Phi(v_0^\tau v_1^\tau) = (2n + 1)(\tau - 1) - \tau + 2,$$
$$\Phi(v_\psi^\tau v_{\psi+1}^\tau) = (2n + 1)(\tau - 1) + 2(\psi - 1) - \tau + 4, \quad \psi = 1, 2, \cdots, n-1,$$
$$\Phi(u_\mu^\tau v_\psi^\tau) = (2n + 1)(\tau - 1) + 2(\psi - 1) - \tau + 3, \quad \mu = \psi.$$

It can be easily verified that all edges' labels adjacent to the same vertex are relatively prime. Hence, the disjoint union of comb graphs is a prime edge graph. □

3.3. *The Graph mP_n*

The graph mP_n is formed by the disjoint union of m paths, each of order n. The vertex and edge sets of the disjoint union of path graphs mP_n are defined as follows:

$$V(mP_n) = \{\, v_\psi^\mu \mid \mu = 1, 2, \cdots, m;\ \psi = 1, 2, \cdots, n \,\},$$
$$E(mP_n) = \{\, v_\psi^\mu v_{\psi+1}^\mu \mid \mu = 1, 2, 3, \ldots, m;\ \psi = 1, 2, 3, \ldots, n-1 \,\}.$$

Theorem 3.4. *The disjoint union of path graphs mP_n is a prime edge graph.*

Proof. To show that the disjoint union of path graphs mP_n classifies as a prime edge graph, we construct a prime edge labeling on the graph. Outline the function of $\Phi : E(mP_n) \to \{1, 2, \cdots, mn - m\}$ down below.

$$\Phi(v_\psi^\mu v_{\psi+1}^\mu) = (n-1)(\mu-1) + \psi, \quad \mu = 1, 2, 3, \cdots, m;\ \psi = 1, 2, 3, \cdots, n-1.$$

Since each pair of adjacent edges shares consecutive integer labels, all adjacent edges must be relatively prime. Clearly, the disjoint union of path graphs is a prime edge graph. □

3.4. *The Graph $C_n \cup C_m$*

The vertex and edge sets of the disjoint union of cycle graphs $(C_n \cup C_m)$ are defined as follows:

$$V(C_n \cup C_m) = \{\, v_\mu^1 \mid \mu = 1, 2, \cdots, n \,\} \cup \{\, v_\psi^2 \mid \psi = 1, 2, \cdots, m \,\},$$
$$E(C_n \cup C_m) = \{\, v_1^1 v_n^1,\ v_\mu^1 v_{\mu+1}^1 \mid \mu = 1, 2, \cdots, n-1 \,\} \cup \{\, v_1^2 v_m^2,\ v_\psi^2 v_{\psi+1}^2 \mid \psi = 1, 2, \cdots, m-1 \,\}.$$

The vertex and edge notation of the graph $C_n \cup C_m$ can be found in Figure 3.

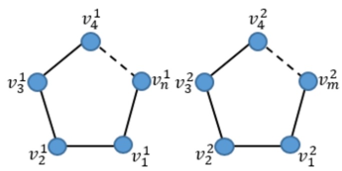

Figure. 3. The Vertex and Edge notation of the Graph $C_n \cup C_m$.

Theorem 3.5. *For all odd integers n and m, we have $\mathfrak{p}r_\varepsilon(C_n \cup C_m) = n + m + 1$.*

Proof. We begin the proof of Theorem 3.5 by showing that $\mathfrak{p}r_\varepsilon(C_n \cup C_m) \geq n + m + 1$. We observe that the cycles C_n and C_m require $\frac{n+1}{2}$ and $\frac{m+1}{2}$ odd labels, respectively. Consequently, the graph $C_n \cup C_m$ requires $\frac{m+n+2}{2}$ odd labels and $\frac{m+n-2}{2}$ even labels. Since the set $\{1, 2, \cdots, m + n + 1\}$ contains $\frac{m+n+2}{2}$ odd integers, we conclude that $\mathfrak{p}r_\varepsilon(C_n \cup C_m) \geq n + m + 1$.

Next, we establish that $\mathfrak{pr}_\varepsilon(C_n \cup C_m) \leq n+m+1$ by defining a bijection $\Phi : E(C_n \cup C_m) \to \{1, 2, \cdots, m+n+1\}$ as follows.

$$\Phi(v^1_\mu v^1_{\mu+1}) = \begin{cases} 1, & \mu = 1, \\ \mu+1, & \mu = 2, 3, \cdots, n-1, \end{cases}$$

$$\Phi(v^1_1 v^1_n) = n+1,$$

$$\Phi(v^2_\psi v^2_{\psi+1}) = \begin{cases} 2, & \psi = 1, \\ n+\psi, & \psi = 2, 3, \cdots, m-1, \end{cases}$$

$$\Phi(v^2_1 v^2_m) = m+n+1.$$

Since 1 is relatively prime to every integer, we have $\gcd(v^1_1 v^1_2, v^1_2 v^1_3) = \gcd(v^1_1 v^1_2, v^1_1 v^1_n) = 1$. By Lemma 2.5, the labels $v^1_\mu v^1_{\mu+1}$ and $v^1_{\mu+1} v^1_{\mu+2}$ are relatively prime for each $\mu = 1, 2, \cdots, n-2$, as they are consecutive integers. Moreover, $n+2$ and $m+n+1$ are odd; since 2 is relatively prime to every odd integer, $\gcd(v^2_1 v^2_2, v^2_2 v^2_3) = \gcd(v^2_1 v^2_2, v^2_1 v^2_m) = 1$. Likewise, the labels $v^2_\psi v^2_{\psi+1}$ and $v^2_{\psi+1} v^1_{\psi+2}$ are consecutive for each $\psi = 1, 2, \cdots, m-2$; hence, by Lemma 2.5, their greatest common divisor equals 1. Thus, Φ satisfies the coprime edge labeling condition, implying $\mathfrak{pr}_\varepsilon(C_n \cup C_m) \leq n+m+1$. Together with the previously established lower bound, we obtain $\mathfrak{pr}_\varepsilon(C_n \cup C_m) = n+m+1$. □

Theorem 3.6. *The graph $C_n \cup C_m$ is a prime edge graph under all conditions not covered in Theorem 3.5.*

Proof. Similarly, the function of $\Phi : E(C_n \cup C_m) \to \{1, 2, \cdots, m+n\}$ defined in the following.

$$\Phi(v^1_\mu v^1_{\mu+1}) = \begin{cases} 1, & \mu = 1, n \textit{ odd, } m \textit{ even,} \\ 2, & \mu = 1 \textit{ and otherwise,} \\ \mu+1, & \mu = 2, 3, \cdots, n-1, \end{cases}$$

$$\Phi(v^1_1 v^1_n) = n+1,$$

$$\Phi(v^2_\psi v^2_{\psi+1}) = \begin{cases} 2, & \psi = 1, n \textit{ odd, } m \textit{ even,} \\ 1, & \psi = 1 \textit{ and otherwise,} \\ n+\psi, & \psi = 2, 3, \cdots, n-1, \end{cases}$$

$$\Phi(v^2_1 v^2_m) = m+n.$$

Using a similar argument, we conclude that Φ satisfies the prime edge labeling condition. Therefore, the graph $C_n \cup C_m$ admits a prime edge labeling for all cases other than those stated in Theorem 3.5. □

3.5. *The Volcano Graph*

In this subsection, we discuss the coprime edge labeling on the volcano graph (V_n). The vertex set of a comb graph is $V(V_n) = \{u_\mu | \mu = 1, 2, 3\} \cup \{v_\psi | \psi = 1, 2, \cdots, n\}$ and its edge set is $E(V_n) = \{u_1u_2, u_2u_3, u_3u_1\} \cup \{u_1v_\psi | \psi = 1, 2, \cdots, n\}$.

Theorem 3.7. *For every $n \geq 2$, $\mathfrak{p}r_\varepsilon(V_n) = p_{n+1}$ where p_{n+1} denotes the first $n+1$ prime numbers.*

Proof. Next, we will show that $\mathfrak{p}r_\varepsilon(V_n) \leq p_{n+1}$ by constructing an edge labeling on the graph V_n. Define the function $\Phi : E(V_n) \to \{1, 2, \cdots, p_{n+1}\}$ as follows. It is shown that $\mathfrak{p}r_\varepsilon(V_n) \geq p_{n+1}$. The vertex u_1 is adjacent to $n+2$ edges. Suppose label 1 is assigned to one of these edges; then, $n+1$ prime labels are required to label the remaining edges. It is clear that $\mathfrak{p}r_\varepsilon(V_n) \geq p_{n+1}$.

Next, we will show that $\mathfrak{p}r_\varepsilon(V_n) \leq p_{n+1}$ by constructing an edge labeling on the graph V_n. The function of $\Phi : E(V_n) \to \{1, 2, \cdots, p_{n+1}\}$ is defined in the following.

$$\Phi(u_\mu u_{\mu+1}) = \mu + 2, \ \ \mu = 1, 2,$$
$$\Phi(u_3u_1) = 5,$$
$$\Phi(u_1v_\psi) = \begin{cases} \psi, & \psi = 1, 2, \\ p_{\ \psi+1}, & \psi = 3, 4, \cdots, n. \end{cases}$$

It is clear that the function Φ satisfies the coprime edge labeling rule, hence $\mathfrak{p}r_\varepsilon(V_n) \leq p_{n+1}$. Since $\mathfrak{p}r_\varepsilon(V_n) \geq p_{n+1}$ and $\mathfrak{p}r_\varepsilon(V_n) \leq p_{n+1}$, then it follows that $\mathfrak{p}r_\varepsilon(V_n) = p_{n+1}$. □

An example of coprime edge labeling on the volcano graph is illustrated in the Figure 4.

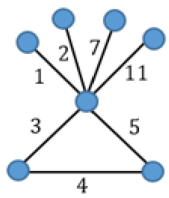

Figure. 4. The Coprime Edge Labeling on the Graph V_4.

3.6. *The Regular Caterpillar Graph*

The vertex and edge sets of the regular caterpillar graph $C_{(2,n)}$ are calculated below.

$V(C_{(2,n)}) = \{\, w_\mu,\, v_\mu,\, u_\mu \mid \mu = 1, 2, \cdots, n \,\}$,
$E(C_{(2,n)}) = \{\, w_\mu w_{\mu+1} \mid \mu = 1, 2, \cdots, n-1 \,\} \cup \{\, w_\mu v_\psi,\, w_\mu u_\psi \mid \mu = \psi; \mu = 1, 2, \cdots, n \,\}$.

Theorem 3.8. *For every $n \geq 2$,$\mathfrak{p}r_\varepsilon(C_{2,n}) = 4n - 3$.*

Proof. The first step in proving that $\mathfrak{pr}_\varepsilon(C_{2,n}) = 4n-3$ is to show that $\mathfrak{pr}_\varepsilon(C_{2,n}) \geq 4n-3$. It is known that each vertex of w_i is adjacent to three edges. Hence, one edge is labeled with an even number and two edges are labeled with odd numbers. To minimize the labels, even labels are not assigned to the edge w_iw_{i+1} because this edge is adjacent with four edges. As a result, the graph $C_{2,n}$ requires $2n-1$ odd labels. Given that there are $2n-1$ odd labels available within the set of $\{1, 2, \cdots, 4n-3\}$, it follows that $\mathfrak{pr}_\varepsilon(C_{2,n}) \geq 4n-3$.

Next, we show that $\mathfrak{pr}_\varepsilon(C_{2,n}) \leq 4n-3$ by constructing a coprime edge labeling function on the graph $C_{2,n}$. The function $\Phi : E(C_{2,n}) \to \{1, 2, \cdots, 4n-3\}$ is defined below.

For $\mu = 1, 2, \cdots, n$

$$\Phi(w_\mu u_\mu) = \begin{cases} 2, & \mu = 1, \\ 4\mu - 4, & \mu = 2, 3, \cdots, n. \end{cases}$$

$$\Phi(w_\mu v_\mu) = \begin{cases} 2\mu + 1, & \mu = 1, 2, \\ 4\mu - 5, & \mu \equiv 2 \pmod 3, \\ 4\mu - 3, & \mu \not\equiv 2 \pmod 3). \end{cases}$$

For $\mu = 1, 2, \cdots, n-1$

$$\Phi(w_\mu w_{\mu+1}) = \begin{cases} 6\mu - 5, & \mu = 1, 2, \\ 4\mu + 1, & \mu \equiv 1 \pmod 3, \\ 4\mu - 1, & \mu \not\equiv 1 \pmod 3. \end{cases}$$

It is evident that every pair of edges adjacent to the same vertex has relatively prime labels, hence $\mathfrak{pr}_\varepsilon(C_{2,n}) \leq 4n-3$. Hence, $\mathfrak{pr}_\varepsilon(C_{2,n}) = 4n-3$. $\square$

Figure 5 below illustrates an example of coprime edge labeling on the graph $C_{2,5}$.

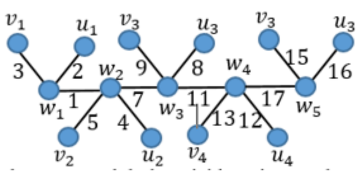

Figure. 5. The Coprime Edge Labeling on the Graph $C_{2,5}$

In this paper, we have not yet determined a coprime edge labeling for the regular caterpillar graph $C_{m,n}$ for $m > 2$.

Question 3.9. Does the regular caterpillar graph $C_{m,n}$, for $m \geq 3$, admit a coprime edge labeling?

4. Conclusion

This paper studies prime edge labeling and coprime edge labeling on several classes of special graphs, namely tadpoles, combs, disjoint unions of combs, unions of two cycles, disjoint unions of paths, volcano graphs, and regular caterpillars. For each graph family, explicit labeling constructions were provided to verify the coprime condition on every pair of adjacent edges. We proved that tadpoles, combs, disjoint unions of combs, and disjoint unions of paths admit prime edge labelings. For the union of two cycles $C_n \cup C_m$, we determined the exact value of the minimum coprime edge number when both n and m are odd, namely $\mathfrak{pr}_\varepsilon(C_n \cup C_m) = n + m + 1$. The result was established by deriving a parity-based lower bound and constructing a bijective labeling that attains this bound. For all remaining parity cases, we showed that $C_n \cup C_m$ admits a prime edge labeling, completing the characterization of this graph family. For volcano graphs V_n, we determined that $\mathfrak{pr}_\varepsilon(V_n) = p_{n+1}$, where p_{n+1} denotes the $(n+1)$-th prime number. In addition, we obtained the exact minimum coprime edge number for the regular caterpillar graph $C_{2,n}$, namely $\mathfrak{pr}_\varepsilon(C_{2,n}) = 4n - 3$.

These findings broaden the class of graphs for which prime edge labeling and minimum coprime edge numbers are completely determined. The constructions presented in this paper rely primarily on parity distribution and consecutive integer properties, which may serve as a general strategy for investigating more complex graph structures. An open problem remains for regular caterpillar graphs $C_{m,n}$ with $m \geq 3$. Determining whether such graphs admit a coprime edge labeling constitutes an interesting direction for future research.

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