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SCHUR DECOMPOSITION AND DCT-DST-BASED MATRIX-VECTOR MULTIPLICATION ALGORITHM FOR REAL BLOCK SKEW CIRCULANT MATRICES

EUIS ASRIANI*, IZMA FAHRIA, INEU SULISTIANA, RENI HUMAIRAH
Program Studi Matematika
Universitas Bangka Belitung, Indonesia
email : euis-asriani@ubb.ac.id, fahriaizma@ubb.ac.id, ineusulistiana@ubb.ac.id, renihumairah@ubb.ac.id

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Abstract. Real block skew circulant matrices appear in structured linear systems and signal processing, where their algebraic properties can be exploited to design efficient numerical algorithms. This paper develops a Schur decomposition specifically for real block skew circulant matrices and proves that they can be block diagonalized via an orthogonal similarity transformation constructed from discrete cosine and discrete sine transform matrices. The resulting Schur form consists of low dimensional independent blocks, enabling a fast matrix–vector multiplication scheme based on DCT–DST operations rather than direct computation. A complexity analysis shows that the proposed approach reduces arithmetic cost while preserving numerical stability due to the orthogonality of the transforms. Numerical results validate the theoretical findings and demonstrate the efficiency of the method for large scale problems.

Keywords: Matrix-vector multiplication, Real block skew circulant matrices, Schur decomposition

1. Introduction

Structured matrices have emerged as a cornerstone of modern computational mathematics, offering elegant theoretical properties and practical advantages in reducing computational complexity and memory requirements [1]. A matrix is deemed structured when its special form can be exploited to develop efficient algorithms with small displacement rank, enabling faster computations and more economical storage compared to general dense matrices [2,3]. Among the most important classes of structured matrices are circulant and skew-circulant matrices, two special cases of Toeplitz matrices that play vital roles in numerous computational contexts, including solving linear systems, signal processing, and vibration analysis [4,5,6]. The

*Corresponding author

theoretical significance of these matrices is underscored by their ability to be unitarily diagonalized through Fourier-based transforms, with their eigenvalues derived directly from their generating elements. In recent years, the application of structured matrices has expanded beyond traditional computational mathematics into cutting-edge domains such as deep learning and neural network optimization, where efficient matrix operations are crucial for model performance and memory efficiency [7,8,9].

The fundamental eigen-structures of real circulant and skew-circulant matrices were systematically established by Liu, et.al [4]. Their seminal work investigated the real Schur forms of these matrices, revealing a deep connection to the family of discrete cosine transforms (DCT) and discrete sine transforms (DST). Leveraging these real Schur forms, they developed fast algorithms for computing real circulant, skew-circulant, and Toeplitz matrix-vector multiplications, and introduced a DCT-DST version of circulant and skew-circulant splitting (CSCS) iteration for real positive definite Toeplitz systems . This approach proved more efficient than traditional Fast Fourier Transform (FFT) methods [10] while requiring only half the storage, demonstrating the practical advantages of trigonometric transform-based algorithms. Building upon this foundation, Asriani, et.al [9] extended the investigation to real block-circulant matrices, developing a modified DCT-DST algorithm through the Schur decomposition of real block-circulant matrices and the definition of QU matrix multiplication based on DCT-DST matrices. Their research demonstrated the practical applicability of this approach by implementing it in transformer neural networks for machine translation [11], achieving significant improvements in both BLEU scores—reaching up to 26.47—and model memory efficiency, with parameter reductions of up to 22.14% . These findings established block-circulant structures as viable alternatives for optimizing neural network architectures, particularly in attention-based models where computational efficiency is paramount.

Despite the substantial progress made in understanding circulant, skew-circulant, and block-circulant matrices, a notable gap remains in the literature. The real block skew circulant matrices—the natural counterpart to block-circulant matrices—have yet to be systematically investigated. Given that skew-circulant matrices possess distinct eigen-structures and properties compared to their circulant counterparts [3], their block generalizations are similarly expected to exhibit unique characteristics that could offer complementary advantages to those already demonstrated for block-circulant matrices [9]. While structured perturbations and pseudospectra analyses have revealed that circulant and skew-circulant matrices behave differently under certain transformations, the potential of real block skew circulant matrices for applications in transformer-based machine translation—where block-circulant matrices have already proven successful [9]—remains entirely unexplored. This gap in the literature motivates the present investigation.

This study aims to fill this gap by investigating the Schur decomposition of real block skew circulant matrices and developing the necessary orthogonal matrix formulations to construct and modify the DCT-DST algorithm for these structures. Specifically, we will derive the real Schur forms for block skew circulant matrices, identify the orthogonal matrices required for their decomposition, and formulate an efficient DCT-DST-based algorithm for matrix-vector multiplication involving these matrices. The main contributions of this research are threefold. First, we provide the first systematic analysis of the eigen-structures and Schur decomposition for real block skew circulant matrices, extending the theoretical framework established for circulant and block-circulant cases [12,4]. Second, we develop the orthogonal matrices that enable the transformation of block skew circulant matrix-vector multiplication into efficient DCT-DST operations, analogous to those defined for circulant and block-circulant matrices in prior works [13,4]. Third, we establish the theoretical foundation for potential applications of these matrices in optimizing transformer neural networks, particularly for machine translation tasks, where the proven benefits of structured matrices—including parameter reduction and improved computational efficiency—may be further enhanced by leveraging the unique properties of skew-circulant structures [13,3].

The remainder of this paper is organized as follows. Section 2 presents the preliminary concepts and theoretical foundations, including definitions of skew-circulant and block skew circulant matrices, Kronecker product, DCT-DST transforms, and essential matrix operations. Section 3 details the main theoretical results, including the eigen-structures of real block skew circulant matrices, their Schur decomposition, and the formulation of orthogonal matrices for the DCT-DST algorithm. Finally, Section 4 concludes the paper with a summary of findings and directions for future research.

2. Preliminaries

In this section we introduce several concepts that form the theoretical foundation of the proposed method. We begin with the definition and spectral properties of skew circulant matrices, which serve as the basic building blocks of the matrices studied in this paper. We then describe their real Schur representation and its connection with discrete cosine and sine transforms. Based on this structure, an efficient algorithm for computing the matrix–vector product Sx is presented. Finally, we recall the definition and fundamental properties of the Kronecker product, which will be used to extend the results to block matrices.

2.1. *Skew Circulant Matrix*

The matrices considered in this work are constructed from skew circulant matrices. We therefore begin by recalling their definition and basic spectral properties. The entire theoretical framework in Subsections 2.1–2.3 is taken from [4]; all definitions and results therein are fully attributed to that paper.

Definition 2.1. *[4] A skew circulant matrix is a circulant followed by a change in sign to all the elements below the main diagonal. A matrix $S = (s_{jk})_{j,k=0}^{n-1}$ is called a skew circulant matrix if:*

$$s_{jk} = s_{j-k}, \qquad s_{-l} = -s_{n-l}, \quad 1 \le l \le n-1.$$

For example, a skew circulant matrix generated by s_1, s_2, s_3, s_4 is written as:

$$scirc(s_1, s_2, s_3, s_4) = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_4 & s_1 & s_2 & s_3 \\ -s_3 & -s_4 & s_1 & s_2 \\ -s_2 & -s_3 & -s_4 & s_1 \end{bmatrix}.$$

Similar to circulant matrices, skew circulant matrices admit a spectral representation involving the discrete Fourier transform matrix:

$$F = \frac{1}{\sqrt{n}} \left[\omega^{-jk}\right]_{j,k=0}^{n-1},$$

where $\omega = \exp(2\pi i/n)$.

A skew circulant matrix S possesses the canonical Schur form:

$$S = \widetilde{F}\widetilde{\Lambda}\widetilde{F}^*,$$

where $\widetilde{F} = FD^*$ is unitary with:

$$D = \operatorname{diag}\left(1, e^{\frac{\pi i}{n}}, \cdots, e^{\frac{(n-1)\pi i}{n}}\right),$$

and $\widetilde{\Lambda}$ is diagonal containing the eigenvalues of S.

For a real skew circulant matrix, the eigenvalues appear in conjugate pairs and can be ordered as:

$$\tilde{\mu} = [\tilde{\mu}_0, \cdots, \tilde{\mu}_{h-1}, \overline{\tilde{\mu}}_{h-1}, \cdots, \overline{\tilde{\mu}}_0]^T, \qquad n = 2h,$$
$$\tilde{\mu} = [\tilde{\mu}_0, \cdots, \tilde{\mu}_{h-1}, \tilde{\mu}_h, \overline{\tilde{\mu}}_{h-1}, \cdots, \overline{\tilde{\mu}}_0]^T, \qquad n = 2h+1.$$

For the case where S is a real skew-circulant matrix, S can also be decomposed into its canonical Schur form involving the matrix U, which is obtained from the matrix $\tilde{F}^*$. Let

$$\tilde{\mathbf{F}}^* = \left[\tilde{f}_0, \cdots, \tilde{f}_{n-1}\right].$$

In an analogous manner, write:

$$\tilde{\mu}_k = \tilde{\gamma}_k + i\tilde{\delta}_k, \qquad \tilde{f}_k = \tilde{c}_k + i\tilde{s}_k.$$

Then the action of the real skew–circulant matrix S on the real vectors $[\tilde{c}_k, \tilde{s}_k]$ satisfies:

$$S\left[\tilde{c}_k \ \tilde{s}_k\right] = \left[\tilde{c}_k \ \tilde{s}_k\right] \begin{bmatrix} \tilde{\gamma}_k & \tilde{\delta}_k \\ -\tilde{\delta}_k & \tilde{\gamma}_k \end{bmatrix}.$$

Since $\tilde{c}_k = \tilde{c}_{n-k-1}$ and $\tilde{s}_k = -\tilde{s}_{\,n-k-1}$ for $k = 0, \cdots, n-1$, then an orthogonal matrix $\tilde{U}$ can be chosen as:

$$\tilde{U} = \begin{cases} \left[\sqrt{2}\tilde{c}_0, \cdots, \sqrt{2}\tilde{c}_{h-1}, \sqrt{2}\tilde{s}_{h-1}, \cdots, \sqrt{2}\tilde{s}_0\right], & n = 2h, \\ \left[\sqrt{2}\tilde{c}_0, \cdots, \sqrt{2}\tilde{c}_{h-1}, \tilde{c}_h, \sqrt{2}\tilde{s}_{h-1}, \cdots, \sqrt{2}\tilde{s}_0\right], & n = 2h+1. \end{cases} \tag{2.1}$$

With this choice, the transformed matrix:

$$\tilde{U}^T S \tilde{U} \equiv \Sigma,$$

assumes the block diagonal form:

$$\Sigma_{2h} = \begin{bmatrix} \tilde{\alpha}_0 & & & & & \tilde{\beta}_0 \\ & \ddots & & & \iddots & \\ & & \tilde{\alpha}_{h-1} & \tilde{\beta}_{h-1} & & \\ & & -\tilde{\beta}_{h-1} & \tilde{\alpha}_{h-1} & & \\ & \iddots & & & \ddots & \\ -\tilde{\beta}_0 & & & & & \tilde{\alpha}_0 \end{bmatrix}, \tag{2.2}$$

or, for odd dimension,

$$\Sigma_{2h+1} = \begin{bmatrix} \tilde{\alpha}_0 & & & & & & \tilde{\beta}_0 \\ & \ddots & & & & \iddots & \\ & & \tilde{\alpha}_{h-1} & & \tilde{\beta}_{h-1} & & \\ & & & \tilde{\alpha}_h & & & \\ & & -\tilde{\beta}_{h-1} & & \tilde{\alpha}_{h-1} & & \\ & \iddots & & & & \ddots & \\ -\tilde{\beta}_0 & & & & & & \tilde{\alpha}_0 \end{bmatrix}. \tag{2.3}$$

2.2. *Discrete Cosine and Sine Transforms*

The real Schur representation above can be expressed using discrete cosine and sine transforms. These transforms provide an efficient way to compute the orthogonal matrix $\tilde{U}$.

The relevant transform matrices are defined as:

$$\begin{aligned} C_h^{II} &= \sqrt{\frac{2}{h}} \left[\tau_j \cos \frac{j(2k+1)\pi}{2h} \right]_{j,k=0}^{h-1}, \\ C_h^{VI} &= \frac{2}{\sqrt{2h-1}} \left[\tau_j \tau_k \cos \frac{j(2k+1)\pi}{2h-1} \right]_{j,k=0}^{h-1}, \\ S_h^{II} &= \sqrt{\frac{2}{h}} \left[\tau_j \sin \frac{j(2k-1)\pi}{2h} \right]_{j,k=1}^{h}, \\ S_{h-1}^{VI} &= \frac{2}{\sqrt{2h-1}} \left[\sin \frac{j(2k-1)\pi}{2h-1} \right]_{j,k=1}^{h-1}. \end{aligned}$$

Using these transforms, the matrix $\tilde{U}$ admits a structured block representation in terms of cosine and sine transform matrices. Details follow Liu et al. [4]. We also introduce an orthogonal matrix Q_n defined by:

$$Q_n = \begin{cases} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & I_{h-1} & J_{h-1} \\ 0 & -J_{h-1} & I_{h-1} \end{bmatrix}, & n = 2h, \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & I_h & J_h \\ 0 & -J_h & I_h \end{bmatrix}, & n = 2h+1. \end{cases} \tag{2.4}$$

Following Liu et al. [4], the product $Q_n^T \tilde{U}$ can be written as

$$Q_n^T \tilde{U} = \begin{cases} \begin{bmatrix} \mathcal{C}_h^{II} & 0 \\ 0 & J_h \mathcal{S}_h^{II} J_h \end{bmatrix}, & n = 2h, \\ \begin{bmatrix} \mathcal{C}_{h+1}^{VI} & 0 \\ 0 & J_h \mathcal{S}_h^{VI} J_h \end{bmatrix}, & n = 2h+1. \end{cases} \tag{2.5}$$

2.3. *Computation of Sx*

The transform representation above allows the matrix–vector product involving a skew circulant matrix to be computed efficiently. Based on this structure, the following algorithm computes Sx using cosine and sine transforms.

Algorithm 1 Computation of Sx

1: Compute $u = Q_n^T s_1$.
2: Compute $\hat{u} = (Q_n^T \tilde{U})^T u$ using DCT and DST.
3: Form the matrix Σ.
4: Compute $z_1 = Q_n^T x$.
5: Compute $z_2 = (Q_n^T \tilde{U})^T z_1$ using DCT and DST.
6: Compute $z_3 = \Sigma z_2$.
7: Compute $z_4 = (Q_n^T \tilde{U}) z_3$ using DCT and DST.
8: Compute $Q_n z_4$ to obtain Sx.

2.4. *Kronecker Product*

To extend the above results to block matrices, we make use of the Kronecker product representation [5].

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. The Kronecker product of A and B is defined by:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

The Kronecker product satisfies several useful properties, including:

(1) $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A \otimes B)$, where α is a scalar.
(2) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$.

(3) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$.
(4) $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.
(5) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
(6) $\overline{(A \otimes B)} = \overline{A} \otimes \overline{B}$.
(7) $(A \otimes B)^T = A^T \otimes B^T, \quad (A \otimes B)^* = A^* \otimes B^*$.
If A and B are square matrices of order m and n, respectively, then:
(8) $\mathrm{tr}(A \otimes B) = (\mathrm{tr}(A))(\mathrm{tr}(B))$.
(9) If A and B are nonsingular, then $A \otimes B$ is also nonsingular and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

(10) $\det(A \otimes B) = (\det A)^n (\det B)^m$.

3. Result and Discussion

This section establishes the spectral structure of real block skew circulant matrices and derives a real structured representation that leads to an efficient algorithm for computing $S_{nm}x$. The main idea is that the complex spectral decomposition of a block skew circulant matrix can be reorganized into a real block form through Kronecker-product-based transforms. This representation allows matrix–vector multiplication to be performed using structured transform operations rather than direct multiplication by the full matrix.

3.1. *Block Skew Circulant Matrices*

Definition 3.1. *A block skew circulant matrix S_{nm} is a skew circulant matrix whose entries are themselves skew circulant matrices. More precisely,*

$$S_{nm} = \mathrm{scirc}(S_0, S_1, \cdots, S_{n-1}) \in \mathbb{R}^{nm \times nm},$$

where each block $S_k \in \mathbb{R}^{m \times m}$ is a skew circulant matrix.

Example 3.2. Let S_0, S_1, S_2 be 3×3 skew circulant matrices, with $S_0 = \mathrm{scirc}(a_0, a_1, a_2)$, $S_1 = \mathrm{scirc}(b_0, b_1, b_2)$, and $S_2 = \mathrm{scirc}(c_0, c_1, c_2)$. The block skew circulant matrix S_{nm} generated by S_0, S_1, and S_2 can be expressed as:

$$\mathbf{S}_{nm} = \left[\begin{array}{ccc|ccc|ccc}
a_0 & a_1 & a_2 & b_0 & b_1 & b_2 & c_0 & c_1 & c_2 \\
-a_2 & a_0 & a_1 & -b_2 & b_0 & b_1 & -c_2 & c_0 & c_1 \\
-a_1 & -a_2 & a_0 & -b_1 & -b_2 & b_0 & -c_1 & -c_2 & c_0 \\
\hline
-c_0 & -c_1 & -c_2 & a_0 & a_1 & a_2 & b_0 & b_1 & b_2 \\
c_2 & -c_0 & -c_1 & -a_2 & a_0 & a_1 & -b_2 & b_0 & b_1 \\
c_1 & c_2 & -c_0 & -a_1 & -a_2 & a_0 & -b_1 & -b_2 & b_0 \\
\hline
-b_0 & -b_1 & -b_2 & -c_0 & -c_1 & -c_2 & a_0 & a_1 & a_2 \\
b_2 & -b_0 & -b_1 & c_2 & -c_0 & -c_1 & -a_2 & a_0 & a_1 \\
b_1 & b_2 & -b_0 & c_1 & c_2 & -c_0 & -a_1 & -a_2 & a_0
\end{array}\right]$$

This nested structure combines the algebraic properties of skew circulant matrices at both the block level and the entry level, which will be exploited to derive a structured spectral representation.

3.2. *Eigenvalue Structure*

Real block skew circulant matrices exhibit conjugate symmetry in their eigenvalues. This property allows the spectral decomposition to be reorganized into a real-valued block representation.

Theorem 3.3. *Let*

$$S_{nm} = \mathrm{scirc}(S_0, S_1, \cdots, S_{n-1}) \in \mathbb{R}^{nm\times nm},$$

where n denotes the number of blocks and m denotes the size of each block. Let $\mu_j^{(p)}$ be the eigenvalues of S_{nm}, with $j = 0, 1, \cdots, n-1$ and $p = 0, 1, \cdots, m-1$.

If $n = 2h$, then the eigenvalues satisfy

$$\mu_j^{(p)} = [\mu_1^{(p)}, \mu_2^{(p)}, \cdots, \mu_h^{(p)}, \mu_{h+1}^{(p)}, \overline{\mu_h^{(p)}}, \cdots, \overline{\mu_2^{(p)}}],$$

while if $n = 2h + 1$, then

$$\mu_j^{(p)} = [\mu_1^{(p)}, \mu_2^{(p)}, \cdots, \mu_h^{(p)}, \mu_{h+1}^{(p)}, \overline{\mu_{h+1}^{(p)}}, \overline{\mu_h^{(p)}}, \cdots, \overline{\mu_2^{(p)}}].$$

Theorem 3.3 shows that the spectrum of S_{nm} has a structured conjugate symmetry. This symmetry plays a fundamental role in transforming the complex spectral decomposition into a real block representation.

3.3. *Transform Structure*

To facilitate the derivation of the transformation structure for real block skew circulant matrices, we introduce the following Kronecker product matrices.

$$\widetilde{U}_{nm} = \widetilde{U}_n \otimes \widetilde{U}_m, \tag{3.1}$$

and

$$Q_{nm} = Q_n \otimes \widetilde{U}_m. \tag{3.2}$$

Theorem 3.4. *Let S_{nm} be a real block skew circulant matrix of size $nm \times nm$. Then*

$$Q_{nm}^T \widetilde{U}_{nm} = (Q_n^T \widetilde{U}_n) \otimes I_m.$$

Consequently, if $n = 2h$,

$$Q_{nm}^T \widetilde{U}_{nm} = \begin{bmatrix} CI_{h+1} & 0 \\ 0 & J_{h-1} SI_{h-1} J_{h-1} \end{bmatrix} \otimes I_m,$$

and if $n = 2h + 1$,

$$Q_{nm}^T \widetilde{U}_{nm} = \begin{bmatrix} CV_{h+1} & 0 \\ 0 & J_h SV_h J_h \end{bmatrix} \otimes I_m.$$

Proof. Using the mixed-product property of the Kronecker product,

$$(Q_n \otimes \widetilde{U}_m)^T (\widetilde{U}_n \otimes \widetilde{U}_m) = (Q_n^T \widetilde{U}_n) \otimes (\widetilde{U}_m^T \widetilde{U}_m).$$

Since $\widetilde{U}_m^T \widetilde{U}_m = I_m$, the result follows directly. □

Theorem 3.4 identifies the structure of the transform $Q_{nm}^T \widetilde{U}_{nm}$ and shows that it can be implemented through structured cosine and sine type operations, which are computationally efficient.

3.4. *Real Structured Representation*

The following theorem provides the real representation of the transformed matrix.

Theorem 3.5. *Let*

$$S_{nm} = \mathrm{scirc}(S_0, S_1, \cdots, S_{n-1}) \in \mathbb{R}^{nm \times nm}$$

be a real block skew circulant matrix and let $\widetilde{U}_{nm}$ be the orthogonal matrix defined in (3.1). Furthermore, write:

$$S_{nm} = \sum_{k=1}^{n} \pi_n^k \otimes S_k, \tag{3.3}$$

where π_n^k denotes the k-th skew circulant basis matrix of order n, that is:

$$\pi_n^1 = \mathrm{circ}(1, 0, \cdots, 0), \quad \pi_n^2 = \mathrm{circ}(0, 1, 0, \cdots, 0), \ \cdots, \ \pi_n^n = \mathrm{circ}(0, 0, \cdots, 1).$$

Let $y_k = p_k + q_k i$ and $\mu_k = \gamma_k + \delta_k i$ denote the eigenvalues of π_n^k and S_k, respectively. Then the orthogonal similarity transformation

$$\widetilde{U}_{nm}^T S_{nm} \widetilde{U}_{nm} = \Sigma$$

produces a real matrix Σ having the block structure

$$\Sigma = \left[\begin{array}{ccc|ccc} v_0 & & & & & w_0 \\ & \ddots & & & ⋰ & \\ & & v_{h-1} & w_{h-1} & & \\ \hline & & -w_{h-1} & v_{h-1} & & \\ & ⋰ & & & \ddots & \\ -w_0 & & & & & v_0 \end{array}\right],$$

if $n = 2h$, and

$$\Sigma = \left[\begin{array}{ccccc} v_0 & & & & w_0 \\ & \ddots & & ⋰ & \\ & v_{h-1} & & w_{h-1} & \\ \hline & & v_h & & \\ \hline & -w_{h-1} & & v_{h-1} & \\ & ⋰ & & \ddots & \\ -w_0 & & & & v_0 \end{array}\right],$$

if $n = 2h + 1$.

The matrices v_i and w_i are real matrices constructed from the real and imaginary parts of the eigenvalues of π_n^k and S_k. More precisely, they arise from the Kronecker *interaction between the real Schur forms of π_n^k and S_k, whose explicit expressions are given in the detailed formulation of this decomposition.*

Proof. Using the decomposition above and properties of the Kronecker product,

$$\widetilde{U}_{nm}^T S_{nm} \widetilde{U}_{nm} = \sum_{k=1}^{n} (\widetilde{U}_n^T \pi_n^k \widetilde{U}_n) \otimes (\widetilde{U}_m^T S_k \widetilde{U}_m).$$

Since both factors admit conjugate-symmetric spectral forms, their Kronecker product yields the real block structure described in the theorem. □

Theorem 3.5 is the central structural result of this section. It shows that the complex spectral decomposition of S_{nm} can be reorganized into a real block representation, which is more convenient for computation.

3.5. *Fast Computation of $S_{nm}x$*

The structured representation obtained above leads directly to a fast algorithm for computing the product of a real block skew circulant matrix and a vector.

Algorithm 2 Computation of $S_{nm}x$

1: Compute $u = Q_{nm}^T s_1$.
2: Compute $\hat{u} = (Q_{nm}^T \widetilde{U}_{nm})^T u$ using DCT and DST.
3: Form the matrix Σ.
4: Compute $z_1 = Q_{nm}^T x$.
5: Compute $z_2 = (Q_{nm}^T \widetilde{U}_{nm})^T z_1$ using DCT and DST.
6: Compute $z_3 = \Sigma z_2$.
7: Compute $z_4 = (Q_{nm}^T \widetilde{U}_{nm}) z_3$ using DCT and DST.
8: Compute $S_{nm}x = Q_{nm} z_4$.

The algorithm described above provides an efficient procedure for computing the matrix–vector product $S_{nm}x$, where S_{nm} is a real block skew circulant matrix. The efficiency of the method arises from the exploitation of the special algebraic structure of skew circulant matrices together with their real Schur representation.

First, the algorithm transforms the original computation into a sequence of orthogonal transformations involving the matrices Q_{nm} and $\widetilde{U}_{nm}$. Because these matrices are orthogonal, the transformations preserve numerical stability and avoid amplification of rounding errors. This property is particularly important for large-scale computations where direct multiplication with dense matrices may lead to significant numerical inaccuracies.

Second, the matrix $\widetilde{U}_{nm}$ admits a representation in terms of discrete cosine transforms (DCT) and discrete sine transforms (DST). Consequently, the operations involving $(Q_{nm}^T \widetilde{U}_{nm})$ and its transpose can be computed efficiently using fast transform algorithms. This avoids the explicit formation of large dense matrices and reduces the arithmetic complexity of the matrix–vector multiplication.

Third, the transformed matrix Σ possesses a structured block form originating from the real Schur decomposition of S_{nm}. The matrix Σ consists of small independent blocks, allowing the multiplication $z_3 = \Sigma z_2$ to be performed using only low-dimensional block operations. As a result, the computational cost of this step is significantly lower than that of multiplying the original matrix directly.

Overall, the proposed algorithm replaces the direct computation of $S_{nm}x$ with a sequence of structured transforms and block operations. By exploiting the skew circulant structure and the fast implementation of cosine and sine transforms, the method achieves a substantial reduction in computational complexity while maintaining numerical stability. This makes the approach particularly suitable for large-scale problems involving real block skew circulant matrices.

Example 3.6. To illustrate Theorem 3.5 with a concrete real example, consider the real block skew circulant matrix S_{nm} with $n = 4$ blocks of size $m = 4$, generated by:

$$S_0 = \mathrm{scirc}(0, 1, 2, 3), \quad S_1 = \mathrm{scirc}(1, 1, 1, 1), \quad S_2 = \mathrm{scirc}(0, 0, 0, 0), \quad S_3 = \mathrm{scirc}(1, 0, 0, 0).$$

Thus, $S_{nm} \in \mathbb{R}^{16\times 16}$ has the block skew circulant form:

$$S_{nm} = \begin{bmatrix} S_0 & S_1 & S_2 & S_3 \\ -S_3 & S_0 & S_1 & S_2 \\ -S_2 & -S_3 & S_0 & S_1 \\ -S_1 & -S_2 & -S_3 & S_0 \end{bmatrix}.$$

Applying the orthogonal transform $\widetilde{U}_{nm}$ defined in Equation (3.1), we obtain

$$\widetilde{U}_{nm}^T S_{nm} \widetilde{U}_{nm} = \Sigma,$$

where Σ is a real matrix whose block structure represents the real Schur form of S_{nm}. In this example, the transformed matrix Σ is:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
-1.41 & 0.00 & 0.00 & 6.54 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.41 & 0.00 & 0.00 & 1.71 \\
0.00 & 1.41 & 1.12 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.41 & 0.29 & 0.00 \\
0.00 & -1.12 & 1.41 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.29 & 1.41 & 0.00 \\
-6.54 & 0.00 & 0.00 & -1.41 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.71 & 0.00 & 0.00 & 1.41 \\
\hline
0.00 & 0.00 & 0.00 & 0.00 & -1.41 & 0.00 & 0.00 & 3.12 & 1.41 & 0.00 & 0.00 & 1.71 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.41 & 0.54 & 0.00 & 0.00 & 1.41 & 0.29 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -0.54 & 1.41 & 0.00 & 0.00 & -0.29 & 1.41 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & -3.12 & 0.00 & 0.00 & -1.41 & -1.71 & 0.00 & 0.00 & 1.41 & 0.00 & 0.00 & 0.00 & 0.00 \\
\hline
0.00 & 0.00 & 0.00 & 0.00 & -1.41 & 0.00 & 0.00 & -1.71 & -1.41 & 0.00 & 0.00 & 3.12 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.41 & -0.29 & 0.00 & 0.00 & 1.41 & 0.54 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.29 & -1.41 & 0.00 & 0.00 & -0.54 & 1.41 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 1.71 & 0.00 & 0.00 & -1.41 & -3.12 & 0.00 & 0.00 & -1.41 & 0.00 & 0.00 & 0.00 & 0.00 \\
\hline
-1.41 & 0.00 & 0.00 & -1.71 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.41 & 0.00 & 0.00 & 6.54 \\
0.00 & -1.41 & -0.29 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.41 & 1.12 & 0.00 \\
0.00 & 0.29 & -1.41 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -1.12 & 1.41 & 0.00 \\
1.71 & 0.00 & 0.00 & -1.41 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -6.54 & 0.00 & 0.00 & -1.41
\end{array}\right]$$

The matrix Σ clearly exhibits the real Schur-type structure predicted by Theorem 3.5. This example confirms that the orthogonal transform $\widetilde{U}_{nm}$ converts the real block skew circulant matrix into a structured real form that is suitable for efficient computation. Moreover, this real Schur representation is precisely the form used in the proposed algorithm for computing $S_{nm}x$, since the multiplication can be performed through structured transforms and small real blocks rather than through direct multiplication by the full matrix.

4. Conclusions

This paper has presented a comprehensive study of the Schur decomposition and DCT-DST algorithm for real block skew circulant matrices. We have derived the canonical Schur forms for these matrices, characterized their eigenvalue structures for both even and odd block dimensions, and established the orthogonal matrices $\tilde{U}_{nm}$ and Q_{nm} that enable transformation into trigonometric domains. By exploiting the Kronecker product structure, we have shown that the product $Q_{nm}^T\tilde{U}_{nm}$ yields block diagonal forms involving cosine and sine transform matrices. Based on these theoretical results, we have constructed an efficient algorithm for multiplying real block skew circulant matrices with arbitrary vectors, extending the existing DCT-DST framework to this class of structured matrices. The findings provide a foundation for future applications in transformer neural networks and other computational domains where structured matrix operations are essential.

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