

ON EDGE IRREGULAR LABELING WITH MINIMUM DISTINCT INTEGER LABELS FOR SOME STANDARD GRAPHS

HUSNUL KHOTIMAH^{a*}, FITRIANA HASNANI^b, MERYTA FEBRILIAN FATIMAH^c

^a Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Sriwijaya, Indralaya, Indonesia,

^b Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Bandar Lampung, Indonesia,

^c Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Sulawesi Barat, Majene, Indonesia,

email : husnul@unsri.ac.id, fitriana.hasnani@fmipa.unila.ac.id, merytaff@unsulbar.ac.id

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Abstract. *In this paper, we introduce a new variant of edge irregular labeling on graphs. For a simple graph G , let $f : E(G) \rightarrow S \subseteq \mathbb{Z}_{\geq 0}$ be an edge labeling, where S is a finite set of nonnegative integers, and define the weight of each vertex as the sum of the labels of all incident edges. In the case where all vertex weights are pairwise distinct, then the labeling is called edge irregular. We define the minimum distinct edge-label number of G , denoted by $md(G)$, as the minimum cardinality of a label set S for which G admits such a labeling. This parameter may be viewed as a distinct irregularity-type invariant, since it minimizes the number of distinct labels used rather than the largest label value. We establish exact values of $md(G)$ for three standard graph families: paths, cycles, and stars. The obtained results demonstrate that this parameter exhibits different behavior across these graph classes and is strongly influenced by local degree structure and edge-incidence patterns.*

Keywords: Edge irregular labeling, irregularity strength, minimum distinct edge-label number

1. Introduction

Graph theory has become a central area of discrete mathematics, offering a natural framework for describing relationships and interaction patterns across a wide range of applications, including computer science, communication networks, chemistry, and transportation systems. Within graph theory, graph labeling has developed into an active research area concerned with assigning numbers or algebraic objects to the vertices, edges, or both, under certain prescribed conditions. Over the years,

*Corresponding author

many labeling variants have been introduced, such as graceful, harmonious, cordial, and irregular labelings.

One of the classical directions in this area is *irregular labeling*. The study of irregular assignments was initiated by Chartrand *et al.* in [1], who introduced the *irregularity strength* of a graph. In this setting, positive integer labels are assigned to edges so that all vertex weights, defined as the sums of labels on incident edges, are pairwise distinct, and the objective is to minimize the largest label used. This concept has inspired a substantial body of work on exact values and bounds for many graph families.

A closely related development is the *total edge irregularity strength*, introduced by Bača *et al.* in [2], in which labels are assigned to both vertices and edges, and the weight of an edge is taken to be the sum of the labels of its two incident vertices and the edge itself. Since its introduction, this parameter has continued to attract attention and has been studied for various graph classes [3,4,5,6,7,8,9]. Other related directions have also connected irregular labeling with modular arithmetic and group-theoretic structures [10,11,12,13,14,15,16]. In particular, the *group irregularity strength*, introduced for connected graphs in [17] and later extended to disconnected graphs in [18], considers edge labels taken from an Abelian group and measures the minimum group order that guarantees pairwise distinct vertex weights.

These developments show that irregular labeling problems can vary significantly depending on the labeled objects, the induced weights, and the algebraic structure of the label set. Motivated by these related developments, in this paper we consider an edge-labeling variant in which the induced weights are taken at vertices, as in the classical irregularity strength, but the quantity to be minimized is different from the usual maximum label. Instead of minimizing the largest label value, we study the minimum number of *distinct nonnegative integer labels* required to produce pairwise distinct vertex weights.

More precisely, we study edge labelings of a graph using a finite set of non-negative integers, under the condition that the sums of labels on edges incident to distinct vertices are all different. The main parameter studied here measures the minimum cardinality of such a label set. To the best of our knowledge, this particular viewpoint, namely, minimizing the number of distinct integer edge labels rather than the maximum label or the order of an underlying group, has not been explicitly studied as a separate graph parameter in the existing literature.

As an initial study of this parameter, we focus on several standard graph families, namely paths, stars, and cycles. These classes are chosen because they are among the most fundamental graph families in graph labeling research and, at the same time, exhibit different structural characteristics that influence the resulting vertex weights. The main contributions of this paper are to formally introduce the *minimum distinct edge-label number* and to determine its values for these graph families.

2. Some Concepts

2.1. Graph terminology and notation

We adopt the usual graph-theoretic notation and terminology; see, for example, [19]. Unless otherwise stated, every graph considered in this paper is assumed to be finite, simple, and connected. For a graph G , its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. We write $|V(G)|$ and $|E(G)|$ for the order and size of G . Two vertices $u, v \in V(G)$ are adjacent whenever $uv \in E(G)$. The degree of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is defined as the number of edges incident to v .

We now recall the graph families that will be used throughout the paper.

Definition 2.1.

(1) For every integer $n \geq 2$, the graph P_n is the graph with:

$$V(P_n) = \{v_1, v_2, \dots, v_n\}, \text{ and} \\ E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}.$$

(2) For every integer $n \geq 3$, the cycle graph C_n is the graph with:

$$V(C_n) = \{v_1, v_2, \dots, v_n\}, \\ E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}.$$

(3) For $n \geq 1$, the star graph $K_{1,n}$ is the graph with:

$$V(K_{1,n}) = \{u, v_1, v_2, \dots, v_n\}, \\ E(K_{1,n}) = \{uv_i : 1 \leq i \leq n\}.$$

2.2. Additional graph-theoretic terminology

For later use, we recall several standard graph-theoretic terms. A *loop* is an edge whose two endpoints coincide. In a graph containing loops, each loop contributes 2 to the degree of its incident vertex.

A *trail* is a walk in which each edge appears at most once. Such a trail is called *open* when its starting and ending vertices are different, and *closed* when these two vertices are the same.

An *Euler trail* is a trail that traverses every edge of the graph exactly one time. A connected graph is said to be *Eulerian* whenever it contains a closed Euler trail.

A *matching* in a graph is a collection of edges that are mutually nonincident, meaning that no two edges in the collection have a common endpoint. A matching is called a *perfect matching* if every vertex of the graph is incident with one of its edges.

Given a graph G , a *spanning subgraph* is a subgraph whose vertex set is exactly $V(G)$. A *Hamilton cycle* is a cycle that visits each vertex of the graph exactly once.

2.3. Edge irregular labelings and the parameter $md(G)$

Motivated by the notion of edge irregularity strength introduced in [20], we define the following edge-labeling variant.

Definition 2.2. Let G be a graph and let $S \subseteq \mathbb{Z}_{\geq 0}$ be a finite set of nonnegative integers. An edge-labeling of G is a function

$$f : E(G) \rightarrow S.$$

For each vertex $v \in V(G)$, the weight of v under f is defined by

$$w(v) = \sum_{uv \in E(G)} f(uv).$$

In this paper, the labeling f is called an edge irregular labeling of G if

$$w(u) \neq w(v)$$

for every two distinct vertices $u, v \in V(G)$.

Definition 2.3. The minimum distinct edge-label number of G , denoted by $md(G)$, is defined as the smallest size of a finite set $S \subseteq \mathbb{Z}_{\geq 0}$ for which G admits an edge irregular labeling $f : E(G) \rightarrow S$.

Remark 2.4. Unlike the classical irregularity strength, the parameter $md(G)$ minimizes the number of distinct labels used, rather than the maximum label value.

3. Results

We begin with a general lower bound that will be useful in the subsequent results.

Proposition 3.1. Let G be a graph and let $f : E(G) \rightarrow S$ be an edge irregular labeling of G , where $S \subseteq \mathbb{Z}_{\geq 0}$ and $|S| = k$. For each integer $d \geq 1$, let:

$$n_d = |\{v \in V(G) : \deg_G(v) = d\}|.$$

Then,

$$n_d \leq \binom{d+k-1}{k-1}.$$

Proof. Let

$$S = \{a_1, a_2, \dots, a_k\},$$

where the a_i are distinct nonnegative integers. Fix $d \geq 1$, and let $v \in V(G)$ be a vertex with $\deg_G(v) = d$. Since v is incident with exactly d edges, its weight can be written as:

$$w(v) = x_1 a_1 + x_2 a_2 + \dots + x_k a_k,$$

where each $x_i \in \mathbb{Z}_{\geq 0}$ and

$$x_1 + x_2 + \dots + x_k = d.$$

Thus, each possible weight of a vertex of degree d corresponds to a nonnegative integer solution of:

$$x_1 + x_2 + \dots + x_k = d.$$

The number of such solutions is:

$$\binom{d+k-1}{k-1}.$$

Hence, there are at most $\binom{d+k-1}{k-1}$ possible weights for vertices of degree d .

On the other hand, since f is an edge irregular labeling, any two distinct vertices must have distinct weights. In particular, all vertices of degree d must have pairwise distinct weights. Therefore, the number n_d of vertices of degree d cannot exceed the number of possible weights for degree- d vertices. Consequently,

$$n_d \leq \binom{d+k-1}{k-1}. \quad \square$$

In particular, the above bound can be applied to specific graph families by considering the multiplicities of their vertex degrees.

3.1. The star graph $K_{1,n}$

We first determine the value of $md(G)$ for the star graph.

Theorem 3.2. For $n \geq 2$,

$$md(K_{1,n}) = n.$$

Proof. In the star graph $K_{1,n}$, there are exactly n vertices of degree 1.

Let $f : E(K_{1,n}) \rightarrow S$ be an edge irregular labeling with $|S| = k$. Applying Proposition 3.1 with $d = 1$, we obtain:

$$n \leq \binom{1+k-1}{k-1} = \binom{k}{k-1} = k.$$

Hence,

$$md(K_{1,n}) \geq n.$$

To prove the reverse inequality, let:

$$V(K_{1,n}) = \{u, v_1, v_2, \dots, v_n\}$$

and

$$E(K_{1,n}) = \{uv_i : 1 \leq i \leq n\},$$

where u is the central vertex.

Define: $S = \{1, 2, \dots, n\}$ and let $f : E(K_{1,n}) \rightarrow S$ be given by: $f(uv_i) = i$ for $1 \leq i \leq n$. Then, $w(v_i) = i$ for $1 \leq i \leq n$, and:

$$w(u) = \sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

Since $n \geq 2$, we have:

$$\frac{n(n+1)}{2} > n,$$

so $w(u)$ is distinct from each $w(v_i)$. Therefore, all vertex weights are pairwise distinct, and f is an edge irregular labeling using exactly n distinct labels. Thus,

$$md(K_{1,n}) \leq n.$$

Consequently, $md(K_{1,n}) = n$. $\square$

Remark 3.3. *The graph $K_{1,1}$ does not admit an edge irregular labeling, since both vertices have the same weight under any edge-labeling.*

Example 3.4. To illustrate Theorem 3.2, consider the star graph $K_{1,4}$. Let u be the central vertex and let v_1, v_2, v_3, v_4 be its pendant vertices. Define an edge-labeling

$$f(uv_i) = i, \text{ for } 1 \leq i \leq 4.$$

Thus, the labels used are 1, 2, 3, and 4. The induced vertex weights are:

$$w(v_1) = 1, \quad w(v_2) = 2, \quad w(v_3) = 3, \quad w(v_4) = 4, \text{ and}$$

$$w(u) = 1 + 2 + 3 + 4 = 10.$$

Hence, all vertex weights are pairwise distinct, so f is an edge irregular labeling of $K_{1,4}$. This example also illustrates that four distinct edge labels are necessary, in accordance with $md(K_{1,4}) = 4$. The labeling is illustrated in Figure 1.

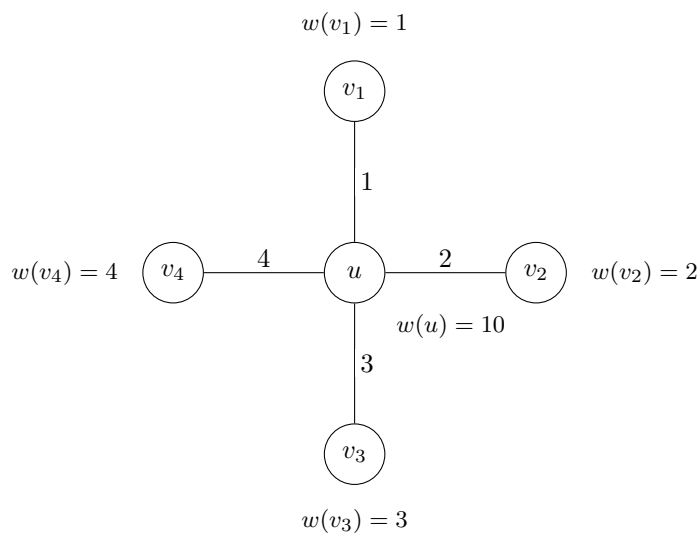


Figure 1. An edge irregular labeling of $K_{1,4}$.

3.2. The path graph P_n

We now determine the exact value of $md(P_n)$. To do so, we first introduce an auxiliary graph.

For $k \geq 2$, define H_k to be the graph whose vertex set is $\{x_1, x_2, \dots, x_k\}$, in which every two distinct vertices are adjacent and each vertex is incident with exactly one loop. In other words, H_k is obtained from the complete graph K_k by attaching one loop to every vertex. Hence,

$$|E(H_k)| = \binom{k+1}{2},$$

and every vertex of H_k has degree $k+1$.

As an illustration, Figure 2 shows the graph H_4 .

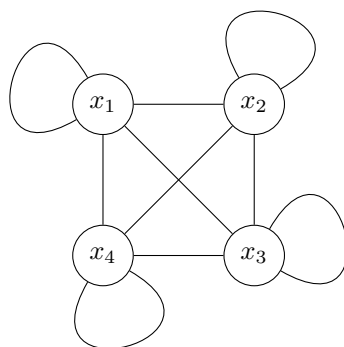


Figure. 2. The auxiliary graph H_4 , obtained from K_4 by attaching one loop to each vertex.

Its loopless part is the complete graph K_4 , and exactly one loop is attached to each vertex. Thus,

$$V(H_4) = \{x_1, x_2, x_3, x_4\}$$

and

$$E(H_4) = \{x_i x_j : 1 \leq i < j \leq 4\} \cup \{x_i x_i : 1 \leq i \leq 4\}.$$

Consequently, H_4 has

$$|E(H_4)| = \binom{5}{2} = 10$$

edges, and each vertex has degree 5, where its incident loop contributes 2 to the degree.

Lemma 3.5. *For $k \geq 2$, the graph H_k contains an open trail of length:*

$$L_k = \begin{cases} \binom{k+1}{2} - 1, & \text{if } k \text{ is odd,} \\ \binom{k+1}{2} - \frac{k-2}{2}, & \text{if } k \text{ is even.} \end{cases}$$

Moreover, for every integer ℓ with $1 \leq \ell \leq L_k$, the graph H_k contains an open trail of length ℓ .

Proof. If k is odd, then every vertex of H_k has even degree. Hence H_k is Eulerian. By deleting any non-loop edge, the resulting graph remains connected and has exactly two odd vertices, so it admits an open Euler trail of length $\binom{k+1}{2} - 1$.

If k is even, then every vertex of H_k has odd degree. Delete a matching of size $\frac{k-2}{2}$ from the underlying complete graph K_k . The resulting graph remains connected and has exactly two odd vertices, hence it admits an open Euler trail of length $\binom{k+1}{2} - \frac{k-2}{2}$. This proves the existence of an open trail of length L_k . The second statement follows immediately, since any initial segment of such a trail is again an open trail. $\square$

Lemma 3.6. *Let $S_k = \{1, 3, 3^2, \dots, 3^{k-1}\}$. Then the elements of S_k are pairwise distinct, all sums $a+b$ with $a, b \in S_k$ (not necessarily distinct) are pairwise distinct, and no element of S_k is equal to any such sum.*

Proof. The claim follows from the uniqueness of base-3 representation. Indeed, each element of S_k is a single power of 3, whereas each sum $a+b$ has either two nonzero base-3 digits or one digit equal to 2. Thus distinct unordered pairs $\{a, b\}$ yield distinct sums, and no such sum equals a single power of 3. $\square$

Theorem 3.7. *For $n = 2$, the path P_2 does not admit an edge irregular labeling. For $n \geq 3$,*

$$md(P_n) = \min\{k \geq 2 : n \leq N_k\},$$

where:

$$N_k = \begin{cases} \binom{k+1}{2} + 1, & \text{if } k \text{ is odd,} \\ \binom{k+1}{2} - \frac{k-2}{2} + 2, & \text{if } k \text{ is even.} \end{cases}$$

Proof. The graph P_2 has only one edge, so both vertices always receive the same weight. Hence P_2 does not admit an edge irregular labeling.

Now let $n \geq 3$, and suppose that $f : E(P_n) \rightarrow S \subseteq \mathbb{Z}_{\geq 0}$ is an edge irregular labeling with $|S| = k$. Write:

$$E(P_n) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\}, \quad f(e_i) = a_i.$$

Then,

$$w(v_1) = a_1, \quad w(v_n) = a_{n-1},$$

and:

$$w(v_i) = a_{i-1} + a_i, \quad 2 \leq i \leq n-1.$$

Since the labeling is edge irregular, the internal weights:

$$a_1 + a_2, a_2 + a_3, \dots, a_{n-2} + a_{n-1}$$

are pairwise distinct. Therefore, the unordered pairs:

$$\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{n-2}, a_{n-1}\}$$

are distinct, and thus correspond to distinct edges of H_k forming a trail. Since $w(v_1) \neq w(v_n)$, we have $a_1 \neq a_{n-1}$, so this trail is open. By Lemma 3.5,

$$n-2 \leq L_k,$$

which is equivalent to $n \leq N_k$. Hence,

$$md(P_n) \geq \min\{k \geq 2 : n \leq N_k\}.$$

Conversely, let $k = \min\{t \geq 2 : n \leq N_t\}$. Then $n-2 \leq L_k$. By Lemma 3.5, the graph H_k contains an open trail of length $n-2$. Let $x_{i_1}, x_{i_2}, \dots, x_{i_{n-1}}$ be the corresponding sequence of vertices, so that $x_{i_1}x_{i_2}, x_{i_2}x_{i_3}, \dots, x_{i_{n-2}}x_{i_{n-1}}$ are distinct edges of H_k .

Assign the numerical labels in $S_k = \{1, 3, 3^2, \dots, 3^{k-1}\}$ to $x_1, x_2, \dots, x_k$, and define

$$f(e_j) = x_{i_j}, \quad 1 \leq j \leq n-1,$$

interpreting each x_{i_j} as its assigned numerical value.

Since the trail is open, $f(e_1) \neq f(e_{n-1})$, so the endpoint weights are distinct. The internal weights correspond to distinct unordered pairs of consecutive labels, hence they are pairwise distinct by Lemma 3.6. Moreover, by Lemma 3.6, no endpoint weight coincides with an internal weight. Thus all vertex weights are pairwise distinct, and f is an edge irregular labeling of P_n using exactly k distinct labels. Therefore,

$$md(P_n) \leq k = \min\{t \geq 2 : n \leq N_t\}.$$

The result follows. $\square$

Example 3.8. To illustrate Theorem 3.7, consider the path:

$$P_5 = v_1v_2v_3v_4v_5.$$

Let $e_i = v_iv_{i+1}$, $1 \leq i \leq 4$, and define an edge-labeling:

$$f(e_1) = 1, \quad f(e_2) = 1, \quad f(e_3) = 3, \quad f(e_4) = 3.$$

Thus, the labeling uses the set $S = \{1, 3\}$. The induced vertex weights are:

$$\begin{aligned} w(v_1) &= f(e_1) = 1, \\ w(v_2) &= f(e_1) + f(e_2) = 2, \\ w(v_3) &= f(e_2) + f(e_3) = 4, \\ w(v_4) &= f(e_3) + f(e_4) = 6, \\ w(v_5) &= f(e_4) = 3. \end{aligned}$$

Hence, the vertex weights are 1, 2, 4, 6, and 3, which are pairwise distinct. Therefore, f is an edge irregular labeling of P_5 using two distinct edge labels. Since P_5 cannot admit an edge irregular labeling using only one distinct label, we have $md(P_5) = 2$. The labeling is illustrated in Figure 3.

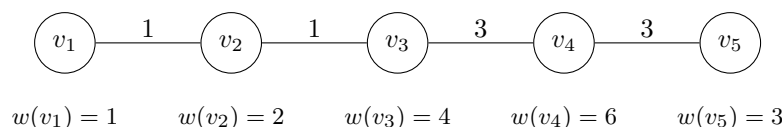


Figure. 3. An edge irregular labeling of P_5 using the label set $\{1, 3\}$.

3.3. The cycle graph C_n

We next determine the exact value of $md(C_n)$. As in the case of paths, we use the auxiliary graph H_k introduced in the previous subsection.

For $k \geq 2$, define:

$$M_k = \begin{cases} \binom{k+1}{2}, & \text{if } k \text{ is odd,} \\ \binom{k+1}{2} - \frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

Lemma 3.9. *For $k \geq 2$, the graph H_k contains a closed trail of length ℓ for every integer ℓ with:*

$$3 \leq \ell \leq M_k.$$

Proof. We use the well-known fact that K_k admits a decomposition into $\frac{k-1}{2}$ Hamilton cycles when k is odd, and that $K_k \setminus M$, where M is a perfect matching, admits a decomposition into $\frac{k-2}{2}$ Hamilton cycles when k is even (see, e.g., [21]). We first consider the case when k is odd. Then the loopless part of H_k is the complete graph K_k . Hence, for every integer q with:

$$1 \leq q \leq \frac{k-1}{2},$$

the union of any q of these Hamilton cycles is a connected even subgraph of K_k with qk edges. By adding r loops, where $0 \leq r \leq k$, we obtain a connected even subgraph of H_k with:

$$qk + r$$

edges. Since every connected even graph admits a closed Euler trail, all such lengths are realizable.

Furthermore, for any integer ℓ such that $3 \leq \ell \leq k$, the complete graph K_k includes a cycle of length ℓ . Thus every length from 3 to k is also realizable. Combining these observations, we obtain all lengths from 3 up to:

$$\frac{k-1}{2} \cdot k + k = \binom{k+1}{2} = M_k.$$

Now suppose that k is even. Choose a perfect matching M in K_k . Then the graph $K_k \setminus M$ is connected and $(k-2)$ -regular, and hence all its vertices have even degree. Moreover, $K_k \setminus M$ admits a decomposition into $\frac{k-2}{2}$ Hamilton cycles. Therefore, for every integer q with:

$$1 \leq q \leq \frac{k-2}{2},$$

the union of any q of these Hamilton cycles is a connected even subgraph with qk edges. Again, by adding r loops, where $0 \leq r \leq k$, we obtain a connected even subgraph of H_k with $qk + r$ edges.

It remains to realize the lengths $3 \leq \ell \leq k$. Label the vertices of $K_k \setminus M$ by $0, 1, \dots, k-1$, where the deleted perfect matching is:

$$M = \{\{i, i + k/2\} : 0 \leq i \leq k/2 - 1\},$$

with indices taken modulo k . If $\ell \neq k/2 + 1$, then:

$$0, 1, 2, \dots, \ell - 1, 0$$

forms a cycle of length ℓ , since consecutive vertices differ by 1 modulo k , and the closing edge joins vertices whose difference is $\ell - 1 \neq k/2$. If $\ell = k/2 + 1$, then:

$$0, 1, 2, \dots, k/2 - 1, k/2 + 1, 0$$

forms a cycle of length ℓ . Therefore, $K_k \setminus M$ admits a cycle of length ℓ for every integer ℓ satisfying $3 \leq \ell \leq k$.

Hence, in the even case, every length from 3 up to $\frac{k-2}{2} \cdot k + k = \binom{k+1}{2} - \frac{k}{2} = M_k$, is realizable as a closed trail in H_k . $\square$

Theorem 3.10. For $n \geq 3$,

$$md(C_n) = \min\{k \geq 2 : n \leq M_k\},$$

where:

$$M_k = \begin{cases} \binom{k+1}{2}, & \text{if } k \text{ is odd,} \\ \binom{k+1}{2} - \frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{e_n = v_n v_1\}$. Suppose that $f : E(C_n) \rightarrow S \subseteq \mathbb{Z}_{\geq 0}$ is an edge irregular labeling with $|S| = k$, and write:

$$f(e_i) = a_i \quad (1 \leq i \leq n).$$

Then the vertex weights are: $w(v_1) = a_n + a_1$, and: $w(v_i) = a_{i-1} + a_i$, $2 \leq i \leq n$.

Since the labeling is edge irregular, the n weights:

$$a_1 + a_2, a_2 + a_3, \dots, a_{n-1} + a_n, a_n + a_1$$

are pairwise distinct. Therefore, the unordered pairs:

$$\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_1\}$$

are distinct, and hence correspond to distinct edges of H_k forming a closed trail of length n . By Lemma 3.9, $n \leq M_k$. Hence,

$$md(C_n) \geq \min\{k \geq 2 : n \leq M_k\}.$$

Conversely, let $k = \min\{t \geq 2 : n \leq M_t\}$. Then $n \leq M_k$. By Lemma 3.9, the graph H_k contains a closed trail of length n . Let $x_{i_1}, x_{i_2}, \dots, x_{i_n}, x_{i_1}$ be the corresponding cyclic sequence of vertices, so that $x_{i_1}x_{i_2}, x_{i_2}x_{i_3}, \dots, x_{i_{n-1}}x_{i_n}, x_{i_n}x_{i_1}$ are distinct edges of H_k .

Assign the numerical labels in $S_k = \{1, 3, 3^2, \dots, 3^{k-1}\}$ to $x_1, x_2, \dots, x_k$, and define: $f(e_j) = x_{i_j}$, $1 \leq j \leq n$, interpreting each x_{i_j} as its assigned numerical value.

Each vertex weight of C_n is the sum of two consecutive labels in the cyclic sequence. Since the corresponding unordered pairs are distinct, Lemma 3.6 implies that these sums are pairwise distinct. Therefore, f is an edge irregular labeling of C_n using exactly k distinct labels. Hence,

$$md(C_n) \leq k = \min\{t \geq 2 : n \leq M_t\}.$$

The result follows. $\square$

Example 3.11. To illustrate Theorem 3.10, consider the cycle $C_4 = v_1v_2v_3v_4v_1$. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, $e_4 = v_4v_1$, and define an edge-labeling f by

$$f(e_1) = 1, \quad f(e_2) = 1, \quad f(e_3) = 3, \quad f(e_4) = 9.$$

Thus, the labeling uses the label set $S = \{1, 3, 9\}$. The induced vertex weights are:

$$w(v_1) = f(e_4) + f(e_1) = 9 + 1 = 10,$$

$$w(v_2) = f(e_1) + f(e_2) = 1 + 1 = 2,$$

$$w(v_3) = f(e_2) + f(e_3) = 1 + 3 = 4,$$

$$w(v_4) = f(e_3) + f(e_4) = 3 + 9 = 12.$$

Hence, the vertex weights 10, 2, 4, and 12 are pairwise distinct. Therefore, f is an edge irregular labeling of C_4 using three distinct edge labels. Moreover, $M_2 = 2 < 4$, while $M_3 = 6 \geq 4$. It follows from Theorem 3.10 that: $md(C_4) = 3$. The labeling is illustrated in Figure 4.

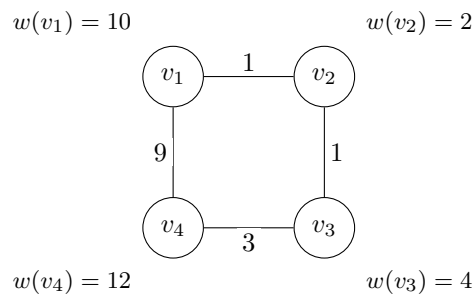


Figure 4. An edge irregular labeling of C_4 using the label set $\{1, 3, 9\}$.

4. Conclusion

In this work, we introduced the *minimum distinct edge-label number* $md(G)$, a new irregularity-type parameter that measures the minimum number of distinct nonnegative integer edge labels required to produce pairwise distinct vertex weights. Unlike the classical irregularity strength and related variants, the focus here is on minimizing the number of distinct labels rather than the largest label value.

We determined the exact values of $md(G)$ for three standard graph families, namely the families of paths P_n , cycles C_n , and stars $K_{1,n}$. Illustrative edge irregular labelings were also provided to demonstrate the obtained results for representative members of these graph families. These results show that the parameter is influenced not only by the order of the graph, but also by its local structural properties, especially edge incidence and vertex degrees.

This work provides a first step toward a broader study of $md(G)$. Future work may consider other graph families, such as fan graphs, wheel graphs, and friendship graphs, as well as general bounds and structural properties of this parameter.

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